Small dilatation pseudo-Anosov mapping classes coming from the simplest hyperbolic braid.

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Abstract

In this paper we study the minimum dilatation pseudo-Anosov mapping classes coming from fibrations over the circle of a single 3-manifold, namely the mapping torus for the "simplest pseudo-Anosov braid". The dilatations that arise include the minimum dilatations for orientable mapping classes for genus $g = 2, 3, 4, 5, 8$ as well as Lanneau and Thiffeault's conjectural minima for orientable mapping classes, when$g = 2, 4 (\text{mod } 6)$. Our examples also show that the minimum dilatation for orientable mapping classes is strictly greater than the minimum dilatation for non-orientable ones when $g = 4, 6, 8$.

1 Introduction

Let S_g be a closed oriented surface of genus $g \geq 1$, and let Mod_g be the *mapping class group*, that is, the group of orientation preserving automorphisms of S_g up to isotopy. A mapping class $\phi \in \text{Mod}_g$ is called *pseudo-Anosov* if S_g has a pair of ϕ-invariant, transversally measured, singular foliations on which ϕ acts by stretching along one and contracting along the other by a constant $\delta(\phi) > 1$. The constant $\delta(\phi)$ is called the *(geometric) dilatation* of ϕ. A mapping class is pseudo-Anosov if it is neither periodic nor reducible [Thur88] [FLP79] [CB88].

A pseudo-Anosov mapping class ϕ is defined to be *orientable* if its invariant foliations are orientable. Let $\delta_{\text{hom}}(\phi)$ be the spectral radius of the action of ϕ on the first homology of S. Then

$$\delta_{\text{hom}}(\phi) \leq \delta(\phi),$$

with equality if and only if ϕ is orientable (see, for example, [LT09] p. 5).

The dilatations $\delta(\phi)$ satisfy reciprocal monic integer polynomials of degree bounded from above by $6g - 6$ [Thur88]. If ϕ is orientable the degree is bounded by $2g$. For fixed g, it follows that $\delta(\phi)$ achieves a minimum $\delta_g > 1$ in Mod_g (cf. [AY80] [Iva90]). Let δ_g^+ be the minimum dilatation for orientable pseudo-Anosov elements of Mod_g.

In this paper, we address the question:

Question 1.1 *What is the behavior of δ_g and δ_g^+ as functions of g?*

So far, exact values of δ_g have only been found for $g \leq 2$. For $g = 1$, $\text{Mod}_1 = \text{SL}(2; \mathbb{Z})$, and we have

$$\delta_1 = \delta_1^+ = \frac{3 + \sqrt{5}}{2}.$$

For $g = 2$, Cho and Ham [CH08] show that δ_2 is the largest real root of

$$x^4 - x^3 - x^2 - x + 1 = 0$$

or approximately 1.72208.

In the orientable case more is known due to recent results of Lanneau and Thiffeault [LT09]. Given $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ with $0 < a < b$, let

$$LT_{(a,b)}(t) = t^{2b} - t^b(1 + x^a + x^{-a}) + 1,$$

and let $\lambda_{(a,b)}$ be the largest real root of $LT_{(a,b)}(t)$.

Theorem 1.2 (Lanneau-Thiffeault [LT09] Thm. 1.2, Thm. 1.3) *For $g = 2, 3, 4, 6, 8$,*

$$\lambda_{(1,g)} \leq \delta_g^+$$

with equality when $g = 2, 3, 4$.

For $g = 2$, the value of δ_2^+ was first determined by Zhirov [Zhi95]. For $g = 5$, Lanneau and Thiffeault show that δ_5^+ equals Lehmer's number, which is realized by an example of Leininger [Lein04]. Lanneau and Thiffeault also find a lower bound for δ_7^+, but so far no one has found an example with that dilatation.

Based on their calculations, Lanneau and Thiffeault ask: *is $\delta_g^+ = \lambda_g$ for even g?* We call the affirmative answer to their question the LT-conjecture.

In our first result, we improve on the following previously known best bounds for minimum dilatation of infinite families

$$(\delta_g)^g \leq (\delta_g^+)^g \leq 2 + \sqrt{3}$$

(see [Min06] [HK06]).

Theorem 1.3 *If $g = 0, 1, 3, 4 (\text{mod } 6)$, $g \geq 3$, then*

$$\delta_g \leq \lambda_{(3,g+1)},$$

and if $g = 2, 5 (\text{mod } 6)$ and $g \geq 5$, then

$$\delta_g \leq \lambda_{(1,g+1)}.$$

For the orientable case, our results complement those of Lanneau and Thiffeault for $g = 2, 4 (\text{mod } 6)$.

Theorem 1.4 *If $g = 1, 3 (\text{mod } 6)$,*

$$\delta_g^+ \leq \lambda_{(3,g+1)};$$

if $g = 2, 4 (\text{mod } 6)$,

$$\delta_g^+ \leq \lambda_{(1,g)};$$

and if $g = 5 (\text{mod } 6)$,

$$\delta_g^+ \leq \lambda_{(1,g+1)}.$$

Putting Theorem 1.4 together with Lanneau and Thiffeault's lower bound for $g = 8$, we prove the LT-conjecture for $g = 8$.

Corollary 1.5 *For $g = 8$, we have*

$$\delta_8^+ = \lambda_{(1,8)}.$$

For large g, it is known that δ_g and δ_g^+ converges to 1. Furthermore, we have

$$\log(\delta_g) \asymp \frac{1}{g} \qquad \text{and} \qquad \log(\delta_g^+) \asymp \frac{1}{g} \tag{1}$$

(see [Pen91] [McM1] [Min06] [HK06]). The LT-conjecture together with (1) leads to the natural question:

Question 1.6 (e.g., [McM1], p.551, [Farb06], Problem 7.1) *Do the sequences*

$$(\delta_g)^g \qquad \textit{and} \qquad (\delta_g^+)^g$$

converge as g grows? What is the limit?

The examples in this paper show the following.

Proposition 1.7 *If the limit exists, then*

$$\limsup_{g\to\infty}(\delta_g)^g \leq \frac{3+\sqrt{5}}{2}.$$

If the LT-conjecture is true, then δ_{2m}^+ is a monotone strictly decreasing sequence (see Proposition 4.6) that converges to $\frac{3+\sqrt{5}}{2}$. Thus, the LT-conjecture implies equality in Proposition 1.7.

Lanneau and Thiffeault show that $\delta_5^+ \leq \delta_6^+$, and hence δ_g^+ is not strictly monotone decreasing (cf. [Farb06] Question 7.2). Theorem 1.4 shows the stronger statement.

Proposition 1.8 *If the LT-conjecture is true, then $\delta_g^+ \leq \delta_{g+1}^+$, whenever $g = 5 (\mathrm{mod}\ 6)$.*

Another example concerns the question of whether the inequality $\delta_g \leq \delta_g^+$ is strict for any or all g. Table 1 shows the following.

Proposition 1.9 *For $g = 4, 6, 8$ we have*

$$\delta_g < \delta_g^+.$$

If the LT conjecture is true, then Theorem 1.3 and Proposition 4.6 imply that the phenomena revealed in Proposition 1.9 repeats itself periodically.

Proposition 1.10 *If the LT-conjecture is true, then for all even $g \geq 4$ we have*

$$\delta_g < \delta_g^+.$$

We prove Theorem 1.3 and Theorem 1.4 using a family of mapping classes $\phi_{(a,b)}$ that come from a fibered face of a single 3-manifold M. This is interesting in light of the *Universal Finiteness Theorem* due to Farb, Leininger and Margalit [FLM09]. For any pseudo-Anosov mapping class $\phi \in \mathrm{Mod}_g$, let $M(\phi)$ be the mapping torus of ϕ after removing tubular neighborhoods of suspensions of the singularities. Let

$$\mathcal{T}_P = \{M(\phi) \ : \ \lambda(\phi) \leq P^g\}.$$

Then $\mathcal{T}_P$ is a finite set for all P ([FLM09] Thm. 1.1). The asymptotic equations (1) imply that

$$\mathcal{T} = \{M(\phi) \ : \ \phi \in \mathrm{Mod}_g, \phi \text{ pseudo-Anosov}, \lambda(\phi) = \delta_g\}$$

and

$$\mathcal{T}^+ = \{M(\phi) \ : \ \phi \in \text{Mod}_g, \phi \text{ pseudo-Anosov}, \lambda(\phi) = \delta_g^+\}$$

are finite. For our examples, M is the complement of a two component link L, known as 6^2_2 in Rolfsen's table [Rolf76]. (See also [KT08] for another example of a single manifold producing small dilatations.)

The following is a table of the minimal dilatations that arise in this paper's examples for genus 1 through 12. All numbers in the table are truncated to 5 decimals. An asterisk $*$ marks the numbers that have been verified to equal δ_g^+ (resp. δ_g). For singularity-type, we use the convention that $(a_1, \dots, a_k)$ means that the singularities of the invariant foliations have degrees $a_1, \dots, a_k$ (see Lanneau and Thiffeault's notation[LT09], p.3). The singularity-types for our examples are derived from the formula given in Proposition 3.5.

g	orientable	degrees of singularities	unconstrained	degrees of singularities
1	2.61803*	no sing.	2.61803*	no sing.
2	1.72208*	(4)	1.72208*	(4)
3	1.40127*	$(2, 2, 2, 2)$	1.40127	(2,2,2,2)
4	1.28064*	(10,2)	1.26123	(3,3,3,3)
5	1.17628*	(16)	1.17628	(16)
6	-	-	1.1617	(5,5,5,5)
7	1.13694	(6,6,6,6)	1.13694	(6,6,6,6)
8	1.12876*	(22,6)	1.1135	(25,1,1,1)
9	1.1054	(8.8.8.8)	1.1054	(8,8,8,8)
10	1.10149	(28,8)	1.09466	(9,9,9,9)
11	1.08377	(34,2,2,2)	1.08377	(34,2,2,2)
12	-	-	1.07874	(11,11,11,11)

Table 1: Minimal orientable and unconstrained dilatations coming from M

For $g = 1, 2, 3, 4, 5$, our orientable examples agree both in dilatation and in singularity-type with previously found minimizing examples. Thus, for example, we have shown that

$$M \in \mathcal{T} \cap \mathcal{T}^+.$$

For $g = 8$, it agrees with the singularity-type anticipated by Lanneau and Thiffeault (see [LT09]). For $g = 6k$, we do not get any orientable examples out of fibrations of M, and for $g = 7$, our minimal example gives a strictly larger dilatation than Lanneau and Thiffeault's lower bound.

Section 2 contains a brief review of Thurston norms, Alexander norms, and the Teichmüller polynomial. These are the basic tools used in this paper. In Section 3 we describe our family of examples, and in Section 4 we prove Theorem 1.3 and Theorem 1.4.

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2 Background and tools

We give a brief review of fibrations of a hyperbolic 3-manifold M and their invariants, emphasizing tools that we will use in the rest of the paper. For more details see, for example, [Thur86] [FLP79] [McM1] [McM02].

The theory of fibered faces of the Thurston norm ball and Teichmuller polynomials provides a way to map out the possible fibrations of a given hyperbolic manifold and the dilatations of their monodromies in a single picture. Assume M is a compact hyperbolic 3-manifold with boundary. Given an embedded surface S on M, let $\chi_-(S)$ be the sum of $|\chi(S_i)|$, where S_i are the irreducible components of F with negative Euler characteristic. The Thurston norm of ϕ is defined to be

$$||\psi||_T = \min \chi_-(S)$$

where the minimum is taken over oriented embedded surfaces $(S, \partial S) \subset (M, \partial M)$ such that the class of S in $\mathrm{H}_2(M, \partial M; \mathbb{Z})$ is dual to ψ

Elements of $\mathrm{H}^1(M; \mathbb{Z})$ are canonically associated with epimorphisms

$$\pi_1(M; \mathbb{Z}) \to \mathbb{Z}$$

which factor through epimorphisms

$$\mathrm{H}_1(M; \mathbb{Z}) \to \mathbb{Z}.$$

Thus, we have a lattice $\Lambda_M \subset \mathbb{R}^{b_1(M)}$ equal to any of the following naturally identified objects.

$$\mathrm{H}^1(M; \mathbb{Z}) = \mathrm{Hom}(\pi_1(M) \to \mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M; \mathbb{Z}), \mathbb{Z}).$$

If $\psi \in \Lambda_M$ is induced by a fibration

$$M \to S^1$$

we say that ψ is *fibered*. In this case,

$$||\psi||_T = \chi_-(S),$$

where S is the fiber of ψ. The *monodromy* of ψ is the mapping class $\phi : S \to S$, such that M is the mapping torus of ϕ, and ψ is the natural projection to S^1. Since M is hyperbolic, ϕ is automatically pseudo-Anosov.

Let Σ be the unit sphere in $\mathbb{R}^{b_1(M)}$ with respect to the extended Thurston norm. Then Σ is a polyhedron and Λ_M projects to a dense subset of Σ, called the *rational points* of Σ. The fibered elements of Λ_M project to the (open) faces of Σ. If a face of Σ contains the projection of a fibered element, then all elements that project to the same face must be fibered. A face of Σ containing the projection of a fibered element is thus called a *fibered face* of Σ.

Let $\psi \in \Lambda_M$ be a fibered element, and let ψ_0 be the element of Λ_M that lies closest to the origin along the ray containing ψ. Then

$$\psi = r(\psi_0),$$

for some positive integer r, and the fibration associated to ψ is obtained by taking the fibration associated to ψ_0 and composing with the r-cyclic covering of S^1. It follows that ψ_0 has connected fibers, while the fibers of ψ have r-connected components. The dilatation of ϕ is given by

$$\delta(\phi) = \delta(\phi_0)^{1/r},$$

while $||\psi||_T = r||\psi_0||_T$.

Theorem 2.1 ([Fri82], Theorem E) *There is a continuous function $\mathcal{Y}(\psi)$ defined on the entire fibered cone in $\mathbb{R}^{b_1(M)}$, so that if ψ is fibered with monodromy ϕ, then*

$$\mathcal{Y}(\psi) = \frac{1}{\log(\delta(\phi))}.$$

The function $\mathcal{Y}$ is homogeneous of degree one, and is a concave function tending to zero along the boundary of the cone.

The Alexander polynomial Δ_M of M is a polynomial in $\mathbb{Z}[G]$, where $G = \mathrm{H}_1(M;\mathbb{Z})$. Each element $\psi \in \Lambda_M$ determines an epimorphism of $\mathbb{Z}$-modules:

$$\rho : \mathbb{Z}[G] \to \mathbb{Z}[t, t^{-1}],$$

where we identify $\mathbb{Z}[t]$ with the group ring over $\mathbb{Z} = \mathrm{H}_1(S^1;\mathbb{Z})$. This defines a specialization

$$\Delta_{(M,\psi)} = \rho(\Delta_M) \in \mathbb{Z}[t].$$

The polynomial $\Delta_{(M,\psi)}$ is the characteristic polynomial for the monodromy ϕ of ψ acting on $\mathrm{H}_1(S;\mathbb{Z})$, where S is the fiber of ψ. Thus, the degree of the Alexander polynomial specialized to a particular ψ is the rank of $\mathrm{H}_1(S;\mathbb{Z})$. This is called the Alexander norm of ψ. The *homological dilatation* of ϕ, that is, the spectral radius of the action of ϕ on $\mathrm{H}_1(S;Z)$, is the maximum among norms of roots of $\Delta_{(M,\phi)}$. We denote the homological dilatation by $\delta_{\mathrm{hom}}(\phi)$.

The *Teichmuller polynomial* Θ associated to a fibered face of Σ_M is analogous to the Alexander polynomial. It is a polynomial defined on λ_M such that for each ψ in the cone over the fibered face, the geometric dilatation $\delta(\phi)$ of the monodromy is the largest real root of Θ specialized to ψ [McM1].

The following result will be used in Section 3.

Theorem 2.2 ([McM02] Thm. 7.1) *Let M be the complement in S^3 of a tubular neighborhood of a link L. If L has 8 or fewer crossings, then the Thurston and Alexander norms agree on $\mathrm{H}^1(M;Z)$.*

3 The mapping torus for the simplest pseudo-Anosov braid

We now look at a particular 3-manifold, and study properties of its fibrations. Let $M = S^3 \setminus N(L)$, where L is the link drawn in two ways in Figure 1, and $N(L)$ is a tubular neighborhood. As seen

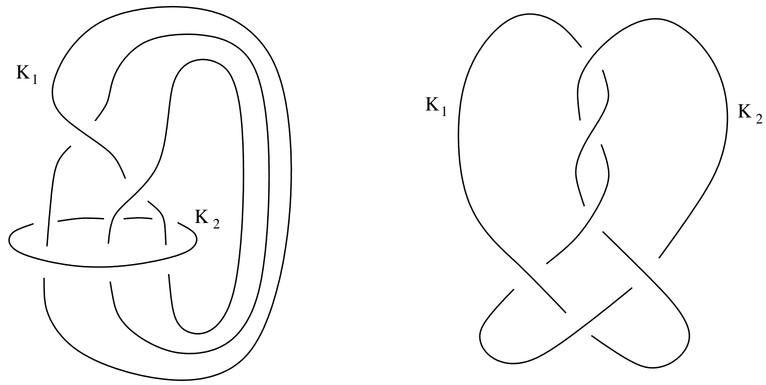

Figure 1: Two diagrams for the link 6^2_2.

from the left diagram in Figure 1, M fibers over the circle with fiber a four times punctured sphere S. Let $\psi \in \Lambda_M$ be the induced element. Let K_1 be the component of L passing through S, and let K_2 be component of L that is one of the four boundary components of L.

The monodromy ϕ of ψ is the composition of two Dehn twists determined by 180 degree rotations as drawn in FIgure 2, and has dilatation

$$\lambda(\phi) = \frac{3+\sqrt{5}}{2}.$$

Its lift to a torus realizes δ_1, and its dilatation is smallest possible for mapping classes defined on S making β the so-called "simplest pseudo-Anosov braid".

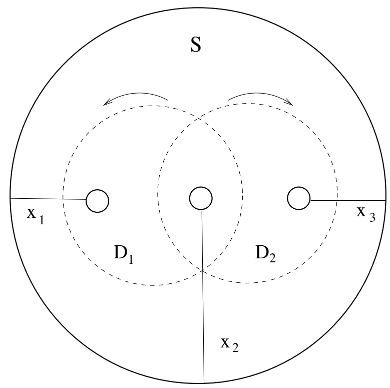

Figure 2: Braid monodromy associated to $\beta = \sigma_1\sigma_2^{-1}$.

The Thurston norm and the Alexander norm both are given by

$$||(a,b)|| = \max\{2|a|, 2|b|\}, \tag{2}$$

where $(a,b) \in \mathrm{H}^1(M;\mathbb{Z})$ denotes the class that evaluates to a on the meridian μ_1 of K_1 and b on the meridian μ_2 of K_2 (see also [McM1] §11).

The lattice points Λ_M in the fibered cone (of points projecting to the fibered face) defined by $\psi = (0,1)$ is the set

$$\Psi = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \ : \ b > 0, -b < a < b\}$$

as shown in Figure 3. For the rest of this paper, we will only be concerned with elements of Ψ with connected fibers, that is,

$$\overline{\Psi} = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \ : \ b > 0, -b < a < b, \gcd(a,b) = 1\}.$$

The elements of $\overline{\Psi}$ are in one-to-one correspondence with the rational points on the fibered face defined by ψ, which can be thought of as the projectivization of Ψ.

The Alexander polynomial for L is given by

$$\Delta_L(x,u) = u^2 - u(1 - x - x^{-1}) + 1 \tag{3}$$

(see Rolfsen's table [Rolf76]), and the Teichmuller polynomial is given by

$$\Theta_L(x,u) = u^2 - u(1 + x + x^{-1}) + 1 \tag{4}$$

(see [McM1] 11.I).

Specialization to the element $(a,b) \in \mathrm{H}^1(M;\mathbb{Z})$ discussed in Section 2 is the same as plugging (t^a, t^b) into the equations for the Alexander and Teichmuller polynomials.

Proposition 3.1 *If $(a,b) \in \overline{\Psi}$, then the associated monodromy $\phi_{(a,b)}$ is pseudo-Anosov and its homological dilatation is the maximum norm among roots of the polynomial*

$$\Delta_L(t^a, t^b) = t^{2b} - t^b(1 - t^a - t^{-a}) + 1,$$

and the geometric dilatation is the largest real root $\lambda_{(a,b)}$ of

$$\Theta_L(t^a, t^b) = t^{2b} - t^b(1 + t^a + t^{-a}) + 1.$$

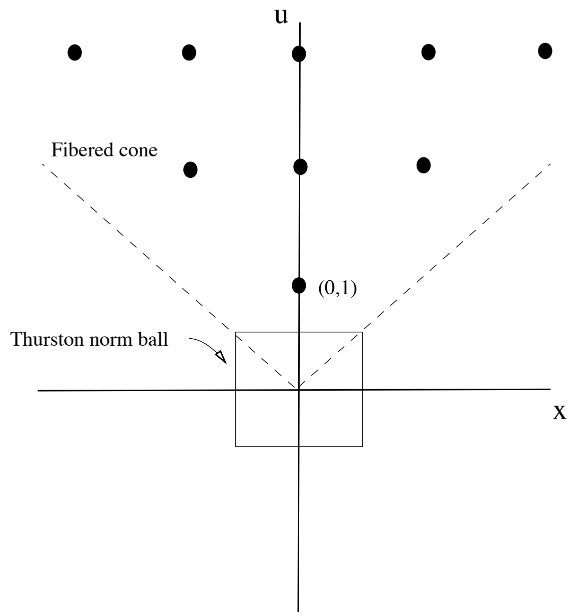

Figure 3: Fibered cone Ψ containing $\psi = (0,1)$.

Corollary 3.2 *If $(a,b) \in \overline{\Psi}$, then the associated monodromy $\phi_{(a,b)}$ is orientable if a is odd and b is even.*

Proof. If a is odd and b is even, then the roots of $\Theta_L(t^a, t^b)$ are the negatives of the roots of $\Delta_L(t^a, t^b)$. This implies that the geometric and homological dilatations of $\phi_{(a,b)}$ are equal, and therefore $\phi_{(a,b)}$ is orientable. □

Later in this section, we will show the converse. In preparation, we consider how the monodromy behaves near the boundary of $S_{(a,b)}$ (Corollary 3.8).

Proposition 3.3 *Let $\phi_{(a,b)} : S_{(a,b)} \to S_{(a,b)}$ be the monodromy associated to (a,b). The boundary components of $S_{(a,b)}$ consists of $\gcd(3,a)$ components coming from $T(K_1)$ and $\gcd(3,b)$ coming from $T(K_2)$. Thus, the total number of boundary components of $S_{(a,b)}$ is given by*

$$\begin{cases} 2 & \textit{if } \gcd(3,ab) = 1 \\ 4 & \textit{if } 3 \textit{ divides } ab \end{cases}$$

Proof. The number of components in $T(K_i) \cap S_{(a,b)}$ is the index of the image of $\pi_1(T(K_i))$ in $\mathbb{Z}$ under the composition of maps

$$\pi_1(T(K_i)) \to \pi_1(M) \to \mathbb{Z}$$

induced by inclusion and $\psi_{(a,b)}$.

For $i = 1,2$, let ℓ_i be the longitude of K_i that is contractible in $S^3 \setminus K_i$. Then, for $T(K_1)$ we have

$$\psi_{(a,b)}(\mu_1) = a \quad \text{and} \quad \psi_{(a,b)}(\ell_1) = 3\psi_{(a,b)}(\mu_2) = 3b,$$

so the number of boundary components contributed by $T(K_1)$ is

$$\gcd(a, 3b) = \gcd(3, a),$$

since we are assuming that $\gcd(a,b) = 1$. The contribution of $T(K_2)$ is computed similarly. □

Proposition 3.4 *The genus of $S_{(a,b)}$, for $(a,b) \in \overline{\Psi}$ is given by*

$$\begin{aligned} g(S_{(a,b)}) &= |b| + \left(1 - \frac{\gcd(3,a)+\gcd(3,b)}{2}\right) \\ &= \begin{cases} |b| & \text{if } 3 \text{ does not divide } ab \\ |b|-1 & \text{if } 3|a \text{ or } 3|b. \end{cases} \end{aligned}$$

Proof. From (2) we have

$$2|b| = \chi_-(S_{(a,b)}) = 2g - 2 + \gcd(3,a) + \gcd(3,b),$$

□

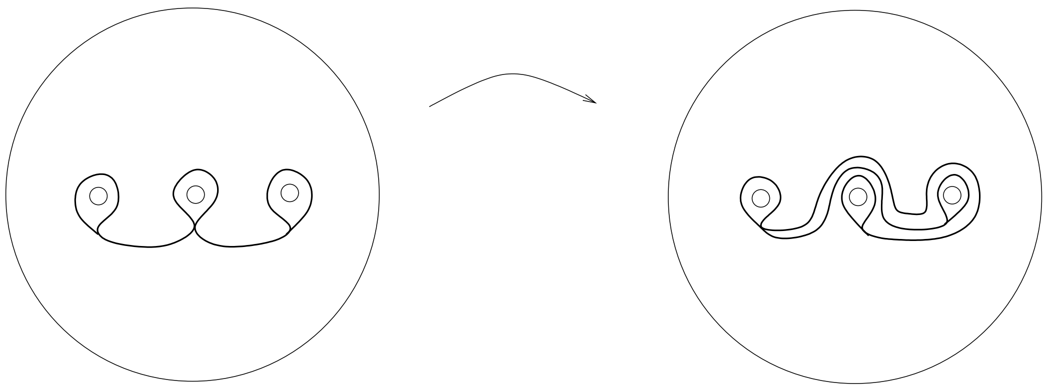

Figure 4: Train track for $\phi : S \to S$.

Proposition 3.5 *Let $(a,b) \in \overline{\Psi}$, and let $\mathcal{F}$ be a $\phi_{(a,b)}$-invariant foliation. Then $\mathcal{F}$*

(i) has no interior singularities, and at the m_1 boundary components coming from $T(K_1)$,

(ii) is $(3b/\gcd(3,a))$-pronged at the $\gcd(3,a)$ boundary components coming from $T(K_1)$), and

(iii) is $(b/\gcd(3,b))$-pronged at the $\gcd(3,b)$ boundary components coming from $T(K_2)$.

Proof. Let $\mathcal{L}$ be the lamination of M defined by suspending $\mathcal{F}$ over M considered as the mapping torus of ϕ. From the train track for ϕ (Figure 4), one sees that each of the boundary components of S are one-pronged, and that there are no other singularities. It follows that $\mathcal{L}$ has no singularities outside a neighborhood of the K_i, and near each K_i the leaves of $\mathcal{L}$ come together at a simple closed curve $\gamma_i \in \mathrm{H}_1(T(K_i)$. Write

$$\gamma_i = r_i\mu_i + s_i\ell_i$$

for $i = 1, 2$.

For $(a,b) \in \overline{P}si$, the number of intersections of γ_i with $S_{(a,b)}$ is the image of γ_i under the epimorphism

$$\psi_{(a,b)} : \pi_1(M) \to \mathbb{Z}$$

defining the fibration. Figure 4 shows that $s_1 = 1$ and $r_2 = 1$. Using the identities

$$\begin{aligned} s_1 &= 1 & \lambda_1 &= 3\mu_2, \\ r_2 &= 1 & \lambda_2 &= 3\mu_1, \end{aligned}$$

we have

$$\begin{aligned}\psi_{(a,b)}(\gamma_1) &= r_1\psi_n(\mu_1) + 3\psi_n(\mu_2) = r_1 a + 3b \\ \psi_{(a,b)}(\gamma_2) &= \psi_n(\mu_2) + 3s_2\psi_n(\mu_1) = 3s_2 a + b.\end{aligned}$$

This implies that $\phi_{(a,b)}$ is $(r_1a + 3b)/m_1$-pronged at m_1 boundary components and $(3s_2a + b)/m_2$-pronged at m_2 boundary components. We find r_1 and s_2 by looking at some particular examples.

In general, if $f : \Sigma \to \Sigma$ is pseudo-Anosov on a compact oriented surface Σ with genus g and and $n_1, \dots, n_k$ are the number of prongs at the singularities and boundary components, then by the Poincaré-Hopf theorem

$$\sum_{i=1}^{k}(n_i - 2) = 4g - 4. \tag{5}$$

For $(a, b) = (1, n)$, n not divisible by 3, we have two singularities with number of prongs given by:

$$\begin{aligned}\psi_n(\gamma_1) &= r_1 + 3n \\ \psi_n(\gamma_2) &= 3s_2 + n.\end{aligned}$$

Plugging into (5) gives

$$r_1 + 3s_2 = 0.$$

Let $s = s_2$. The mapping class $\phi_{(1,2)}$ is the unique genus 2 pseudo-Anosov mapping class with dilatation equal to λ_2 [CH08][LT09], and has one 6-pronged singularity (see, for example, [HK06]). Thus, $s = 0$ and we have

$$\gamma_1 = \ell_1 = 3\mu_2$$

and

$$\gamma_2 = \mu_2.$$

The claim follows. □

Corollary 3.6 *The map $\phi_{(a,b)}$ has singularities with number of prongs given by:*

$$\begin{cases} (3b, b) & \text{if } \gcd(3, ab) = 1 \\ (3b, b/3, b/3, b/3) & \text{if } \gcd(3, b) = 3 \\ (b, b, b, b) & \text{if } \gcd(3, a) = 3 \end{cases}$$

Remark 3.7 *The degrees of the singularities is obtained by subtracting two from the number of prongs.*

Corollary 3.8 *If b is odd, then $\phi_{(a,b)}$ is not orientable.*

Proof. By Corollary 3.6, the number of prongs at each boundary component is odd if b is odd. Thus, $\phi_{(a,b)}$ is not locally orientable near the boundary components. □

Corollary 3.9 *For $(a, b) \in \overline{\Psi}$, $\phi_{(a,b)}$ is 1-pronged at one or more boundary components of $S_{(a,b)}$ if and only if $(a, b) \in \{(0, 1), (\pm 1, 3), (\pm 2, 3)\}$.*

Corollary 3.10 *If $(a, b) \notin \{(0, 1), (\pm 1, 3), (\pm 2, 3)\}$, then $\phi_{(a,b)}$ extends to the closure of $S_{(a,b)}$ over the boundary components to a mapping class $\overline{\phi}_{(a,b)}$ with the same dilatation as $\phi_{(a,b)}$.*

Proposition 3.11 *Table 2 below describes the pairs $(a,b) \in \overline{\Psi}$ that give rise to an orientable (or non-orientable) genus g pseudo-Anosov mapping class. (Here $g \geq 4$.)*

$g \pmod 6$	*orientable*	*non-orientable*
0	*no example*	$b = g+1,\ a = 0 \pmod 3$
1	$b = g+1,\ a = 3 \pmod 6$	$b = g,\ a = 1,2 \pmod 3$
2	$b = g,\ a = 1,5 \pmod 6$	$b = g+1,\ a = 1,2 \pmod 3$
3	$b = g+1,\ a = 3 \pmod 6$	*no example*
4	$b = g,\ a = 1,5 \pmod 6$	$b = g+1, a = 0 \pmod 3$
5	$b = g+1,\ a = 1,5 \pmod 6$	$b = g,\ a = 1,2 \pmod 3$

Table 2: Fibrations of M according to genus.

4 Dilatations on the fibered face.

Let $\overline{\Psi}$ be the fibered cone discussed in Section 3. Let

$$\begin{aligned} d_g &= \min\{\delta(\psi) : \psi \in \overline{\Psi}, \text{genus of } \psi \text{ is } g\} \\ d_g^+ &= \min\{\delta(\psi) : \psi \in \overline{\Psi}, \text{genus of } \psi \text{ is } g, \text{ the monodromy of } \psi \text{ is orientable}\} \end{aligned}$$

In this section, we finish the proofs of Theorem 1.3 and Theorem 1.4 and their consequences by determining d_g and d_g^+.

Proposition 4.1 *If $|a| < |a'|$, and $(a,b) \in \Psi$, then*

$$\lambda_{(a,b)} < \lambda_{(a',b)}.$$

Proof. The function $\mathcal{Y}((a,b))$ (see Theorem 2.1) is concave and symmetric with respect to reflection around the b-axis. Thus for fixed b, $\delta(\phi_{(a,b)}) < \delta(\phi_{a',b})$ as long as a is closer to the b-axis than a'. □

Corollary 4.2

$$\lim_{g\to\infty} (d_g)^g = \lim_{g\to\infty} (d_g^+)^g = \frac{3+\sqrt{5}}{2}.$$

Proposition 4.3 *For $b = 3$,*

$$\lambda_{(1,3)} = \lambda_{(3,4)}$$

and for any $b \geq 4$,

$$\lambda_{(1,b)} > \lambda_{(3,b+1)}.$$

Proof. Let $\lambda = \lambda_{(3,b+1)}$. Again, we would like to show that $LT_{(1,b)}(\lambda) < 0$. Consider the function

$$f(x) = -x^{b+4} + x^{b+2} + x^6 - x^2 - x + 1.$$

Then for $x > 1$,

$$\begin{aligned} f'(x) &= -(b+4)x^{b+3} + (b+2)x^{b+1} + 6x^5 - 2x \\ &= x[-(b+4)x^{b+2} + (b+2)x^b + 6x^4 - 2] \\ &< x[-(b+4)x^{b+2} + (b+3)x^b] \\ &< 0. \end{aligned}$$

Since $f(1) = 0$, this implies that $f(x) < 0$, for $x > 1$. Using this fact, we see that

$$\begin{aligned} LT_{(1,b)}(\lambda) &= LT_{(1,b)}(\lambda) - LT_{(3,b+1)}(\lambda) \\ &= \lambda^{2b} - \lambda^{2b+2} + (\lambda^{b+4} - \lambda^{b+1}) + (\lambda^{b-2} - \lambda^{b-1}) + (\lambda^{b+1} - \lambda^{b}) \\ &= -\lambda^{2b+2} + \lambda^{2b} + \lambda^{b+4} - \lambda^{b} - \lambda^{b-1} + \lambda^{b-2} \\ &= \lambda^{b-2}[-\lambda^{b+4} + \lambda^{b+2} + \lambda^{6} - \lambda^{2} - \lambda + 1] \\ &< 0. \end{aligned}$$

□

Remark 4.4 *Since $\lambda_{(1,3)} = \lambda_{(3,4)}$, it follows that the dilatation δ_3^+ is realized by both an orientable and a non-orientable mapping class in Ψ.*

Lemma 4.5 *For $n \geq 2$, Then*

$$\lim_{n\to\infty} (\lambda_{(a,n)})^n = \frac{3+\sqrt{5}}{2},$$

for any fixed a.

Proof. The projections of the lattice points $(a, n) \in \Lambda_M$ on the fibered face of ψ converge to $(0, 1/2)$. □

Putting together Proposition 4.1 and Proposition 4.3, we have the following.

Proposition 4.6 *The sequences $\lambda_{(1,b)}$ and $\lambda_{(3,b)}$ satisfy:*

$$\lambda_{(1,b)} > \lambda_{(3,b+1)} > \lambda_{(1,b+1)}.$$

Proposition 4.7 *The following table describes the pairs $(a, b) \in \overline{\Psi}$ that give rise to minimal dilatation examples for each genus g. Here unconstrained means not constrained to be orientable.*

g mod 6	*constrained*	*unconstrained*
0	*no example*	$(3, g+1)$
1	$(3, g+1)$	$(3, g+1)$
2	$(1, g)$	$(1, g+1)$
3	$(3, g+1)$	$(3, g+1)$
4	$(1, g)$	$(3, g+1)$
5	$(1, g+1)$	$(1, g+1)$

Table 3: Pairs (a, b) giving smallest dilatations.

Proposition 4.7 and Corollary 3.10 complete the proofs of Theorem 1.3 and Theorem 1.4. A pictorial view of the elements of Ψ giving the least dilatations for each genus up to 12 is given in Figure 5.

Remark 4.8 *For $g = 5$, δ_5^+ is realized by a mapping class associated to the E_{10} Coxeter diagram (see [Lein04]). It is also the monodromy of the $(-2, 3, 7)$-pretzel link [Hir04]. One can verify directly that the monodromy $\phi_{1,6}$ is the same as this mapping class by noting that the $(-2, 3, 7)$-pretzel link can be obtained from the braid $\sigma_1\sigma_2^{-1}$ by composing the braid with two full twists and taking the braid closure.*

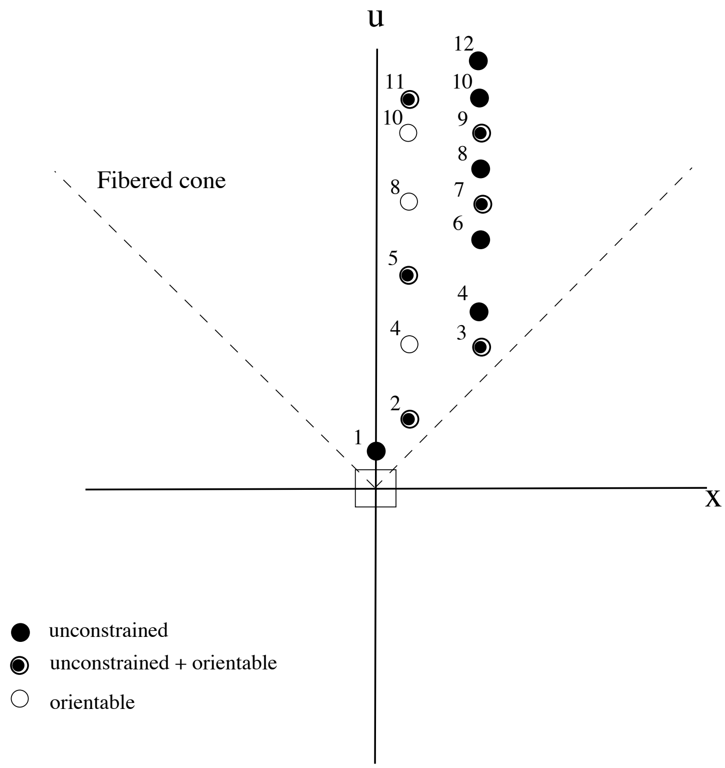

Figure 5: Minima for d and d^+ in genus $g = 1, \ldots, 12$.

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