Initial-Boundary Layer Associated with the Nonlinear Darcy-Brinkman System

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Abstract

We study the interaction of initial layer and boundary layer in the nonlinear Darcy-Brinkman system in the vanishing Darcy number limit. In particular, we show the existence of a function of corner layer type (so called initial-boundary layer) in the solution of the nonlinear Darcy-Brinkman system. An approximate solution is constructed by the method of multiple scale expansion in space and in time. We establish the optimal convergence rates in various Sobolev norms via energy method.

Keywords: initial layer, boundary layer, initial-boundary layer, vanishing Darcy number limit, Darcy-Brinkman system, multiple scale expansion

1. Introduction

In this article we investigate a singular perturbation problem in fluid dynamics, which is governed by the following incompressible nonlinear Darcy-Brinkman

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system in a 2-D periodic channel $\Omega = [0,1] \times [0,1]$

$$\begin{cases} \epsilon \left(\dfrac{\partial \mathbf{v}^\epsilon}{\partial t} + (\mathbf{v}^\epsilon \cdot \nabla)\mathbf{v}^\epsilon \right) + \mathbf{v}^\epsilon - \epsilon \Delta \mathbf{v}^\epsilon + \nabla p^\epsilon = \mathbf{F}, \\ \operatorname{div} \mathbf{v}^\epsilon = 0, \\ \mathbf{v}^\epsilon|_{z=0,1} = 0, \quad \mathbf{v}^\epsilon \text{ periodic in } x\text{-direction}, \\ \mathbf{v}^\epsilon|_{t=0} = \mathbf{v}_0. \end{cases} \tag{1.1}$$

where we use (x,z)-coordinates, suppressing the y variable, so that the z variable is always in the direction normal to the boundary. Here ϵ is a dimensionless parameter which is small in our problem. $\mathbf{v}^\epsilon = (v_1^\epsilon, v_2^\epsilon)$ is the velocity field, p^ϵ is the pressure, and $\mathbf{F}$ is the given external forcing which can be time dependent. We also assume the zeroth order compatibility condition $\mathbf{v}_0|_{z=0,1} = 0$.

The Darcy-Brinkman equation (1.1) can be viewed as an appropriately non-dimensionalized version of the following volume-averaged Navier-Stokes equation (equation (2.13) in [28])

$$\rho_l \frac{\partial}{\partial t}\mathbf{u} + \rho_l \mathbf{u} \cdot \nabla \left(\frac{\mathbf{u}}{\chi} \right) = -\chi \nabla [p_l]^l + \eta \Delta \mathbf{u} + \chi \rho_l \mathbf{g} - \frac{\eta \chi}{\Pi(\chi)} \mathbf{u}, \tag{1.2}$$

where ρ_l is the density of the fluid, $\mathbf{u}$ the Darcy velocity, χ the porosity (liquid volume fraction), $[p_l]^l$ the average value of the liquid pressure, η the viscosity, and $\Pi(\chi)$ the permeability defined in Darcy's law. For a given velocity scale V and a given length scale L, if one chooses Darcy's pressure scale $P = \frac{\eta L V}{\Pi_0}$ (Π_0 is the scale of permeability), one can non-dimensionalize (1.2) to obtain (equation (2.16) in [28])

$$DaRe \left[\frac{\partial}{\partial t}\mathbf{u} + \mathbf{u} \cdot \nabla \left(\frac{\mathbf{u}}{\chi} \right) \right] = -\chi \left[\nabla [p_l]^l + \frac{\rho_l g \Pi_0}{\eta V} \mathbf{e}_z + \frac{\Pi_0}{\Pi(\chi)} \mathbf{u} \right] + Da \Delta \mathbf{u}, \tag{1.3}$$

where $Da = \frac{\Pi_0}{L^2}$ is the Darcy number, $Re = \frac{\rho_l V L}{\eta}$ the Reynolds number, and $\mathbf{e}_z$ a unit vertical vector. In the formal limit of $Da \to 0$ (small permeability), the Darcy-Brinkman equation reduces to the classical Darcy equation

$$\mathbf{u} = -\frac{\Pi(\chi)}{\Pi_0} \left(\nabla [p_l]^l + \frac{\rho_l g \Pi_0}{\eta V} \mathbf{e}_z \right). \tag{1.4}$$

If the variation of the porosity χ is not large, one can simply take χ as a constant. Introducing the seepage velocity $\mathbf{v} = \frac{\mathbf{u}}{\chi}$ and utilizing the Carman-Kozeny permeability function [5]

$$\Pi(\chi) = \Pi_0 \frac{\chi^3}{(1-\chi)^2},$$

the Darcy-Brinkman system (1.3) becomes

$$\frac{\chi^2}{(1-\chi)^2} DaRe \left[\frac{\partial}{\partial t}\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\right] = -\nabla p + \mathbf{F} - \mathbf{v} + \frac{\chi^2}{(1-\chi)^2} Da\Delta\mathbf{v}, \quad (1.5)$$

where $p = \frac{\chi^2}{(1-\chi)^2}[p_l]^l$ and $\mathbf{F} = -\frac{\chi^2}{(1-\chi)^2}\frac{\rho_l g \Pi_0}{\eta V}\mathbf{e}_z$. For the convenience of mathematical analysis, we denote $\epsilon = \frac{\chi^2}{(1-\chi)^2}Da$, set $Re = 1$, and assume more generic forcing $\mathbf{F}$. This leads to the Darcy-Brinkman model (1.1). The case of general Re can be treated in exactly the same fashion.

We note that in many applications ϵ is a small parameter due to either small permeability or small porosity. Formally taking $\epsilon = 0$ in system (1.1), we arrive at the following Darcy equation

$$\begin{cases} \mathbf{v^0} + \nabla p^0 = \mathbf{F}, \\ \operatorname{div} \mathbf{v^0} = 0, \\ \mathbf{v^0}\cdot\mathbf{n}|_{z=0,1} = 0, \quad \mathbf{v^0} \text{ periodic in } x\text{-direction.} \end{cases} \quad (1.6)$$

Note that there is no initial condition for problem (1.6), and the time dependence of $\mathbf{v}^0$ is through the external forcing $\mathbf{F}$. Moreover, one can only impose no penetration boundary condition for the Darcy equation, whereas the velocity field of equation (1.1) must satisfy both the no-slip, and no-penetration boundary conditions. We observe that $\mathbf{v}^0$ can also be viewed as the Helmholtz projection of $\mathbf{F}$, cf. [52] for details.

Our aim in this article is to study the convergence of the nonlinear Darcy-Brinkman system (1.1) to the Darcy equation (1.6) in the vanishing Darcy number limit ($\epsilon \to 0$). This is a singular perturbation problem involving both an initial layer (multiple time scales) and a boundary layer (and hence multiple spatial scales). On the one hand, this is similar to the classical boundary layer problem for incompressible viscous fluids at small viscosity that we recall [49, 43, 54, 59, 60, 26]. Indeed, following the original work of Prandtl [45], we can derive a Prandtl type equation for this model which indicates the existence of a boundary layer in the velocity field of a width proportional to $\sqrt{\epsilon}$ and with no boundary layer in the pressure field (to the leading order). On the other hand, the problem involves an initial layer as well. In this connection, a similar problem has been studied by the second author in the context of Rayleigh-Bénard convection [63, 64, 65], see also [44]. As a consequence, the interaction of the boundary layer and initial layer introduces another singular structure of corner layer type (initial-boundary layer), which is new in the present literature to the best of our knowledge.

There is an abundant literature on boundary layer associated with incompressible flows and the related question of vanishing viscosity (see for instance [2, 7, 47, 48, 10, 40, 14, 22, 46, 19, 20, 8, 27, 71, 31, 3, 4, 32, 29, 68, 1, 69, 21, 57, 58, 53, 55, 56, 62, 23, 9, 24, 18, 25, 18, 25, 38, 6, 15, 33, 62, 39, 36, 35, 12, 13, 67, 61] among many others). We will refrain from surveying the literature here, but emphasize that the boundary layer problem associated with the Navier-Stokes equation is still open and that there is a need to develop tools and methods to tackle it.

The definitions of all of our function spaces reflect the fact that we are working in a domain that is periodic in the horizontal direction (periodic channel). Thus, for instance, $H^m = H^m_{\overline{per}}(\Omega)$, m a nonnegative integer, is the Sobolev space consisting of all functions on Ω whose weak derivatives up to order m are square integrable and whose weak derivatives up to order $m-1$ are periodic in the horizontal direction, with the usual norm. Similarly, $H^1_{0,per}(\Omega)$ is the subspace of functions in $H^1_{\overline{per}}(\Omega)$ that vanish on $z = 0, 1$. We will use the classical function spaces of fluid mechanics,

$$
\begin{aligned}
H &= H(\Omega) = \left\{ \mathbf{v} \in (L^2_{\overline{per}}(\Omega))^2 \colon \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } z = 0, 1 \right\}, \\
V &= V(\Omega) = \left\{ \mathbf{v} \in (H^1_{0,per}(\Omega))^2 \colon \operatorname{div} \mathbf{v} = 0 \right\},
\end{aligned}
$$

where $\mathbf{n}$ denotes the unit outer normal to $\partial\Omega$. We put the L^2-norm on H and the H^1-norm on V. Because of the Poincaré's inequality, we can equivalently use $\|u\|_V = \|\nabla u\|_{L^2}$. We follow the convention that $\|\cdot\|$ is the L^2-norm.

For system (1.1), we work with weak solutions. The following proposition can be proved in a similar fashion as the classical theory of Navier-Stokes equation, cf. [51, 52].

Proposition 1.1. *Let $\mathbf{F}, \mathbf{v}_0$ be given in $L^2(0, T; V')$ and H respectively. Then there exists a unique weak solution $\mathbf{v}^\epsilon$ of (1.1), and $\mathbf{v}^\epsilon \in C([0, T], H) \bigcap L^2(0, T; V)$.*

In addition, if we assume $\mathbf{v}_0 \in V$ and $\mathbf{F} \in L^2(0, T; H)$, then $\mathbf{v}^\epsilon$ is in $C([0, T]; V) \bigcap L^2(0, T; H^2(\Omega))$, and $\frac{\partial \mathbf{v}^\epsilon}{\partial t}$ is in $L^2(0, T; H)$.

The well-posedness of Darcy equation (1.6) can be found in [34] and references therein. For later use, we assume regular data $\mathbf{F} \in C^1([0, T]; H^m(\Omega))$ with $m \geq 5$. It follows that $\mathbf{v}^0 \in C^1([0, T]; H^m(\Omega))$.

The main result in this paper is summarized in the following theorem.

Theorem 1.2. *Assume* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ *with* $m \geq 5$. *Then there exists an approximate solution* $\mathbf{v}^{app}$ *defined in* (4.1) *as the sum of the solution to Darcy equation, an explicit initial layer, an explicit boundary layer and an initial-boundary layer, so that the following optimal convergence rates hold*

$$||\mathbf{v}^\epsilon - \mathbf{v}^{app}||_{L^\infty(0,T,L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{1.7a}$$

$$||\mathbf{v}^\epsilon - \mathbf{v}^{app}||_{L^\infty(0,T,H^1(\Omega))} \leq C\epsilon^{\frac{1}{4}}, \tag{1.7b}$$

$$||\mathbf{v}^\epsilon - \mathbf{v}^{app}||_{L^\infty(0,T,L^\infty(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{1.7c}$$

$$||\nabla(p^\epsilon - p^0)||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}} \tag{1.7d}$$

where C *is a generic constant independent of* ϵ.

The convergence rate estimates in the theorem reveals the structure of the solution $\mathbf{v}^\epsilon$ in terms of asymptotic expansion: (1.7a) indicates the existence of an initial layer; (1.7b) shows the presence of both a boundary layer and an initial boundary layer; (1.7c) verifies the existence of an initial-boundary layer. The approach for proving the theorem is of Prandtl type, cf. [66, 17] and references therein. Namely, we first assume an asymptotic expansion of $\mathbf{v}^\epsilon$ which comprises the solution $\mathbf{v}^0$, the initial layer, boundary layer and initial-boundary layer, and derive the Prandtl-type equation satisfied by each layer; then we can construct an approximate solution based on the asymptotic analysis and deduce the equation for the approximate solution; eventually the energy method is applied to the equation satisfied by the difference of $\mathbf{v}^\epsilon$ and the approximate solution. To obtain the optimal convergence rates, we also considered the $O(\sqrt{\epsilon})$ expansion. The key to our success here is a mild nonlinear term in the sense that the convection term $\mathbf{v}^\epsilon \cdot \nabla \mathbf{v}^\epsilon$ has a small coefficient ϵ. Because of that, the Prandtl type equations for the boundary layer and initial-boundary layer (to the leading order) are all linear though the Darcy-Brinkman model (1.1) itself is nonlinear. This is similar to the case of the boundary layer for the incompressible Navier-Stokes flows with non-characteristic boundary conditions [60, 59] as well as secondly boundary layer associated with the Navier-Stokes equations under Navier type slip boundary conditions. The main difficulty for us is the existence of initial-boundary layer which necessitates the simultaneous treatment of multiple scales in space and in time.

The paper is organized as follows. An example is given in section 2 to illustrate the phenomenon inducing the existence of an initial-boundary layer. In section 3, we give a detailed asymptotic analysis of the singular perturbation problem (1.1), and derive the Prandtl type equations for each layer. In section 4, the approximate solution is constructed, and the convergence result is proved via energy method.

High order expansion is given in the fifth section. We point out that though theorem 1.2 is proved for the case of 2D, it will be clear that the whole argument below applies to the 3D case provided the solution to the Darcy equation is sufficiently smooth.

2. An Example

The interaction of initial layer and boundary layer can be well illustrated through the following simple example. We consider here a special case of shear flow in the half plane $z > 0$. Assuming in system (1.1) the data take the form of

$$\mathbf{F} = \big(e^{-z}, 0\big), \quad \mathbf{v}_0 = (v_0(z), 0),$$

we seek a solution of the form

$$\mathbf{v}^\epsilon = (v_1^\epsilon(t, z), 0), \quad p^\epsilon = 0.$$

Then the system (1.1) reduces to a scalar parabolic equation on the half line $z > 0$

$$\begin{cases} \epsilon \dfrac{\partial v_1^\epsilon}{\partial t} + v_1^\epsilon - \epsilon \dfrac{\partial^2 v_1^\epsilon}{\partial z^2} = e^{-z}, \\ v_1^\epsilon(0, z) = v_0(z), \\ v_1^\epsilon(t, 0) = 0, \\ v_1^\epsilon \to 0, \text{as } z \to \infty. \end{cases} \tag{2.1}$$

The explicit solution to equation (2.1) can be found by using Green's function, see for instance [42]. One can then try to explore the structure of the solution by the method of asymptotic expansion of integrals. Here we take an alternative approach. One first solves the ODE

$$\begin{cases} u^\epsilon - \epsilon \dfrac{\partial^2 u^\epsilon}{\partial z^2} = e^{-z}, \\ u^\epsilon(t, 0) = 0, \\ u^\epsilon \to 0, \text{as } z \to \infty. \end{cases} \tag{2.2}$$

And the solution is found to be

$$u^\epsilon = \frac{1}{1-\epsilon}(e^{-z} - e^{-\frac{z}{\sqrt{\epsilon}}}).$$

Note that u^ϵ contains a boundary layer type component $e^{-\frac{z}{\sqrt{\epsilon}}}$. Defining $w^\epsilon = v_1^\epsilon - u^\epsilon$, one sees $w^\epsilon = w_I^\epsilon + w_C^\epsilon$ satisfies

$$\begin{cases} \epsilon\dfrac{\partial w^\epsilon}{\partial t} + w^\epsilon - \epsilon\dfrac{\partial^2 w^\epsilon}{\partial z^2} = 0, \\ w^\epsilon(0,z) = f_R(z) + f_B(z), \\ w^\epsilon(t,0) = 0, \\ w^\epsilon \to 0, \text{as } z \to \infty, \end{cases} \tag{2.3}$$

with

$$\begin{array}{l} f_R(z) = v_0(z) - \frac{1}{1-\epsilon}e^{-z}, \\ f_B(z) = \frac{1}{1-\epsilon}e^{-\frac{z}{\sqrt{\epsilon}}}. \end{array} \tag{2.4}$$

The regular function $f_R(z)$ contributes to an initial layer type solution $w_I^\epsilon = e^{-\frac{t}{\epsilon}}r(t,z)$ with $r(t,z)$ satisfying the heat equation on the positive half line

$$\begin{cases} \dfrac{\partial r}{\partial t} - \dfrac{\partial^2 r}{\partial z^2} = 0, \\ r(0,z) = f_R(z), \\ r(t,0) = 0, \\ r \to 0, \text{as } Z \to \infty. \end{cases} \tag{2.5}$$

On the other hand, the boundary layer type initial data f_B develops an initial-boundary layer type solution, since, if one defines $\tau = \frac{t}{\epsilon}, Z = \frac{z}{\sqrt{\epsilon}}$, one finds $w_C^\epsilon = \frac{1}{1-\epsilon}w(\tau, Z)$ satisfying the equation

$$\begin{cases} \dfrac{\partial w}{\partial \tau} + w - \dfrac{\partial^2 w}{\partial Z^2} = 0, \\ w(0,Z) = e^{-Z}, \\ w(t,0) = 0, \\ w \to 0, \text{as } z \to \infty. \end{cases} \tag{2.6}$$

To sum up, one finds

$$v_1^\epsilon = \frac{1}{1-\epsilon}e^{-z} - \frac{1}{1-\epsilon}e^{-\frac{z}{\sqrt{\epsilon}}} + e^{-\frac{t}{\epsilon}}r(t,z) + \frac{1}{1-\epsilon}w\left(\frac{t}{\epsilon}, \frac{z}{\sqrt{\epsilon}}\right),$$

which clearly reveals the existence of a boundary layer, an initial layer, and an initial-boundary layer.

3. Asymptotic Analysis

In this section, we will derive the equations satisfied by the initial layer, boundary layer and initial- boundary layer, respectively. The approach we take is of Prandtl type. We will focus on deriving the Prandtl type equations near the boundary $z = 0$; the equations near $z = 1$ are entirely analogous. Motivated by the example in section 2, we formally assume the solutions of system (1.1) have an asymptotic expansion of the form

$$\begin{aligned} &\mathbf{v}^\epsilon = \mathbf{v}^0(t,x,z) + \mathbf{v}^I\big(t/\epsilon, x, z\big) + \mathbf{v}^B\big(t, x, z/\sqrt{\epsilon}\big) + \mathbf{v}^C\big(t/\epsilon, x, z/\sqrt{\epsilon}\big) + \cdots, \\ &p^\epsilon = p^0(t,x,z) + \cdots, \end{aligned} \tag{3.1}$$

with the superscripts I, B, C denoting the initial layer, boundary layer, and initial-boundary layer, respectively. Here $\mathbf{v}^I$ and $\mathbf{v}^B$ (more precisely v_1^B) take care of the difference in initial and boundary conditions between system (1.1) and (1.6), and $\mathbf{v}^C$ takes care of the extra boundary conditon introduced by $\mathbf{v}^I$ and the extra initial condition introduced by $\mathbf{v}^B$. Introducing the stretched variables $\tau = \frac{t}{\epsilon}$ and $Z = \frac{z}{\sqrt{\epsilon}}$, we impose the matching conditions as follows

$$\begin{aligned} &\mathbf{v}^I \to 0, \quad \text{as } \tau \to \infty, \\ &v_1^B \to 0, \quad \text{as } Z \to \infty, \\ &\mathbf{v}^C \to 0, \quad \text{as } \tau \to \infty, \\ &v_1^C \to 0, \quad \text{as } Z \to \infty. \end{aligned} \tag{3.2}$$

Note we did not impose decaying condition for v_2^B and v_2^C when $Z \to \infty$, since the boundary conditions in normal direction are the same in systems (1.1) and (1.6). We will see later that v_2^B is not of boundary layer type.

Plug the expansion (3.1) into equation (1.1) and keep all the $O(1)$ terms (in terms of $\sqrt{\epsilon}$). Outside the initial layer and boundary layer region ($\tau, Z \to \infty$, respectively), one rederives the Darcy's equation (1.6) by the matching conditions (3.2) and using the incompressibility condition. Within the initial layer region but outside the boundary layer region ($Z \to \infty$), one deduces the initial layer equation (3.4). Likewise, one has the boundary layer equation (3.6) within the boundary layer region and outside the initial layer region ($\tau \to \infty$). After subtracting the Darcy's equation, initial layer equation, and boundary layer equation, one is left with initial-boundary layer equation (3.16). The corresponding initial and boundary conditions are derived in the same way.

Note that the boundary layer and the initial-boundary layer exist both at $z = 0$ and $z = 1$. For constructing an approximate solution which satisfies the same boundary conditions as in (1.1), we have to modify the boundary layer and initial boundary layer profile. The rest of this section is devoted to the study of initial layer, and construction of modified boundary layer and initial-boundary layer. Throughout the rest of the paper, the following convention will be assumed

$$\begin{array}{l} a(t,x) = v_1^0(t,x,0), \quad b(t,x) = v_1^0(t,x,1), \quad c(x) = v_1^0(0,x,0), \\ d(x) = v_1^0(0,x,1), \quad \Omega_\infty = \{(x,Z)|x \in [0,1], Z \in (0,\infty)\}. \end{array} \tag{3.3}$$

3.1. Initial Layer

The initial layer $\mathbf{v}^I$ satisfies an ODE

$$\begin{cases} \epsilon \dfrac{\partial \mathbf{v}^I}{\partial t} + \mathbf{v}^I = 0, \\ \mathbf{v}^I|_{t=0} = \mathbf{v}_0(x,z) - \mathbf{v}^0(0,x,z), \end{cases} \tag{3.4}$$

where $\mathbf{v}^0$ is the solution to the Darcy equation (1.6). Its solution is given by $\big(\mathbf{v}_0(x,z) - \mathbf{v}^0(0,x,z)\big)e^{-\frac{t}{\epsilon}}$. It follows that

$$\begin{array}{l} ||\mathbf{v}^I, \quad \nabla \mathbf{v}^I, \quad \Delta \mathbf{v}^I||_{L^\infty(0,T;L^2(\Omega))} \leq C, \\ ||\mathbf{v}^I, \quad \nabla \mathbf{v}^I, \quad \Delta \mathbf{v}^I||_{L^2(0,T;L^2(\Omega))} \leq C\sqrt{\epsilon}, \end{array} \tag{3.5}$$

provided that $\mathbf{v}_0(x,z) - \mathbf{v}^0(0,x,z) \in H^2(\Omega)$. Note that $\nabla \cdot \mathbf{v}^I = 0$ since $\mathbf{v}_0$ is divergence free by assumption.

3.2. Boundary Layer

Recalling the stretched variable $Z = \frac{z}{\sqrt{\epsilon}}$, one finds that the leading order boundary layer $\mathbf{v}^{B,0}$ defined near $z = 0$ satisfies the following Prandtl type equation

$$\begin{cases} v_1^{B,0} - \partial_{ZZ} v_1^{B,0} = 0, \quad Z \in (0,\infty), \\ \dfrac{\partial v_1^{B,0}}{\partial x} + \dfrac{1}{\sqrt{\epsilon}} \dfrac{\partial v_2^{B,0}}{\partial Z} = 0, \quad Z \in (0,\infty), \quad x \in (0,1), \\ v_1^{B,0}\big|_{Z=0} = -a(t,x), v_2^{B,0}\big|_{Z=0} = 0, \\ v_1^{B,0} \to 0, \text{as } Z \to \infty. \end{cases} \tag{3.6}$$

The boundary layer near $z = 1$ satisfies a similar equation with the stretched variable $Z = \frac{1-z}{\sqrt{\epsilon}}$ and $a(t,x)$ replaced by $b(t,x)$. The solution to equation (3.6) is given as

$$v_1^{B,0} = -a(t,x)e^{-\frac{z}{\sqrt{\epsilon}}}, \quad v_2^{B,0} = \sqrt{\epsilon}\frac{\partial a}{\partial x}(1 - e^{-\frac{z}{\sqrt{\epsilon}}}). \tag{3.7}$$

Clearly $v_1^{B,0}$ and $v_2^{B,0}$ do not vanish at $z = 1$. Likewise, the boundary layer functions at $z = 1$ are not zero at the boundary $z = 0$. A truncation is needed to ensure the overall boundary layer profile satisfies the respective boundary conditions exactly. To maintain the divergence free condition, we take the truncation at the stream function level. Such an approach of truncation has been extensively used in the study of boundary layer problems for incompressible flow, cf. [54, 59, 60, 26] for instance.

Introduce a cut-off function $\rho \in C^\infty[0, 1]$ such that

$$\begin{cases} \rho = 1, \quad z \in [0, \dfrac{1}{4}], \\ \rho = 0, \quad z \in [\dfrac{1}{2}, 1]. \end{cases} \tag{3.8}$$

The truncated stream function is thus defined as, in view of the solution formula (3.7),

$$\psi^{B,0} = \sqrt{\epsilon} a(t, x)\big(1 - e^{-z/\sqrt{\epsilon}}\big)\rho(z).$$

Then the modified boundary layer profile at $z = 0$ can be defined as follows

$$\tilde{\mathbf{v}}^{B,0} = \nabla^\perp \psi^{B,0} = (-\frac{\partial \psi^{B,0}}{\partial z}, \frac{\partial \psi^{B,0}}{\partial x}).$$

Thus

$$\begin{aligned} \tilde{v}_1^{B,0} &= -a e^{-Z}\rho - \sqrt{\epsilon} a(1 - e^{-Z})\rho', \\ \tilde{v}_2^{B,0} &= \sqrt{\epsilon}\tfrac{\partial a}{\partial x}(1 - e^{-Z})\rho. \end{aligned} \tag{3.9}$$

The modified boundary layer profile $\tilde{\mathbf{v}}^{B,1}$ at $z = 1$ is constructed in a similar fashion. Letting

$$\tilde{\mathbf{v}}^B = \tilde{\mathbf{v}}^{B,0} + \tilde{\mathbf{v}}^{B,1}, \tag{3.10}$$

we see $\tilde{\mathbf{v}}^B$ is equal to the exact boundary layer profile $\tilde{\mathbf{v}}^{B,0}$ and $\tilde{\mathbf{v}}^{B,1}$ within a fixed width $\frac{1}{4}$ of the respective boundaries. Moreover, $\nabla \cdot \tilde{\mathbf{v}}^B = 0$ by construction, and $\mathrm{supp}\, \tilde{\mathbf{v}}^{B,0} \bigcap \mathrm{supp}\, \tilde{\mathbf{v}}^{B,1} = \emptyset$.

By using (3.9), we infer that $\tilde{\mathbf{v}}^B$ satisfies the following modified Prandtl type equation (in original variables)

$$\begin{cases} \tilde{\mathbf{v}}^B - \epsilon\Delta\tilde{\mathbf{v}}^B = \mathbf{f}^B, \\ \nabla \cdot \tilde{\mathbf{v}}^B = 0, \\ \tilde{\mathbf{v}}^B\big|_{z=0,1} = -\mathbf{v}^0\big|_{z=0,1}. \end{cases} \tag{3.11}$$

where $\mathbf{f}^B = \mathbf{f}^{B,0} + \mathbf{f}^{B,1}$ with $\mathbf{f}^{B,0} = (f_1^{B,0}, f_2^{B,0})$ defined as follows

$$
\begin{aligned}
f_1^{B,0} &= \epsilon^{\frac{3}{2}}\Delta(a\rho')(1 - e^{-\frac{z}{\sqrt{\epsilon}}}) + \epsilon\big(\tfrac{\partial^2 a}{\partial x^2}\rho + 3a\rho''\big)e^{-\frac{z}{\sqrt{\epsilon}}} \\
&\qquad\qquad -2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho' - \sqrt{\epsilon}a\rho', \qquad (3.12a) \\
f_2^{B,0} &= -\epsilon^{\frac{3}{2}}\Delta(\tfrac{\partial a}{\partial x}\rho)(1 - e^{-\frac{z}{\sqrt{\epsilon}}}) - 2\epsilon\tfrac{\partial a}{\partial x}e^{-\frac{z}{\sqrt{\epsilon}}}\rho' + \sqrt{\epsilon}\tfrac{\partial a}{\partial x}\rho. \qquad (3.12b)
\end{aligned}
$$

$\mathbf{f}^{B,1}$ has similar terms.

Remark 1. *It is clear from* (3.12a) *that the truncation introduces extra error terms of order $\sqrt{\epsilon}$ which are not of boundary layer type (the functions do not decay when $Z \to \infty$).*

A direct calculation based on (3.9) gives

$$
\begin{aligned}
&||\tilde{v}_1^{B,0}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{4}}, \\
&||\partial_z\tilde{v}_1^{B,0}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{-\frac{1}{4}}, \\
&||\tilde{v}_2^{B,0}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \\
&||\partial_z\tilde{v}_2^{B,0}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{4}}.
\end{aligned}
\tag{3.13}
$$

with C a generic constant independent of ϵ. Moreover, one has

$$
||\partial_x^j\mathbf{f}^B||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \quad j = 0, 1. \tag{3.14}
$$

which follows easily from (3.12), provided that

$$
\partial_x^3 a = \partial_x^3 v_1^0(t, x, 0) \in L^\infty(0, T; L^2(\partial\Omega)).
$$

For the spatial uniform estimate, we recall here the following version of anisotropic Sobolev embedding (corollary 7.3 from [26], see also [54, 70])

Lemma 3.1. *There exists a constant C such that for any $u \in H^1_{0,per}(\Omega)$*

$$
||u||_{L^\infty(\Omega)} \leq C(||u||_{L^2}^{\frac{1}{2}}||\partial_z u||_{L^2}^{\frac{1}{2}} + ||\partial_x u||_{L^2}^{\frac{1}{2}}||\partial_z u||_{L^2}^{\frac{1}{2}} + ||u||_{L^2}^{\frac{1}{2}}||\partial_x\partial_z u||_{L^2}^{\frac{1}{2}}).
$$

where one or both sides of the inequality could be infinite.

Note taking derivative with respect to x or t does not change the estimate (3.13) as long as a is smooth in x and t. Combining this observation, (3.13) and lemma 3.1 we deduce

Proposition 3.2. *Assume* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ *with* $m \geq 4$. *The following estimates hold*

$$
\begin{aligned}
&||\partial_t^j \tilde{v}_1^B||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{4}}, \quad j = 0, 1 \\
&||\partial_t^j \tilde{v}_2^B||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \quad j = 0, 1 \\
&||\nabla \tilde{v}_1^B||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{-\frac{1}{4}}, \\
&||\nabla \tilde{v}_2^B||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{4}}, \\
&||\tilde{v}_1^B||_{L^\infty(0,T;L^\infty(\Omega))} \leq C, \\
&||\tilde{v}_2^B||_{L^\infty(0,T;L^\infty(\Omega))} \leq C\epsilon^{\frac{1}{4}}.
\end{aligned}
\tag{3.15}
$$

Remark 2. *It is clear from the explicit solution formula* (3.7) *that the estimates above are all optimal.*

3.3. Initial-Boundary Layer

Recall the definition of the stretched variables $\tau = \frac{t}{\epsilon}, Z = \frac{z}{\sqrt{\epsilon}}$. The initial-boundary layer $\mathbf{v}^{C,0}$ at the corner $t = 0$, $z = 0$ satisfies the following Prandtl type equation

$$
\begin{cases}
\dfrac{\partial v_1^{C,0}}{\partial \tau} + v_1^{C,0} - \partial_{ZZ} v_1^{C,0} = 0, \quad \tau > 0, \quad Z > 0, \\
\dfrac{\partial v_1^{C,0}}{\partial x} + \dfrac{1}{\sqrt{\epsilon}} \dfrac{\partial v_2^{C,0}}{\partial Z} = 0, \\
v_1^{C,0}\big|_{\tau=0} = -v_1^{B,0}\big|_{t=0} = c(x)e^{-Z}, \\
v_1^{C,0}\big|_{Z=0} = -v_1^{I}\big|_{z=0} = c(x)e^{-\tau}, \quad v_2^{C,0}\big|_{Z=0} = 0, \\
v_1^{C,0} \to 0, \quad \text{as } Z \to \infty.
\end{cases}
\tag{3.16}
$$

with $c(x) = v_1^0(0, x, 0)$ as defined in (3.3). In the derivation one has applied the compatibility condition $\mathbf{v}_0\big|_{z=0} = 0$.

Remark 3. *We claim that the initial-boundary layer system* (3.16) *is not trivial for generic initial condition* $\mathbf{v}_0$ *and source term* $\mathbf{F}$. *To see this, one observes that* $\mathbf{v}^0$ *is the Helmholtz projection of* $\mathbf{F}$ *into the divergence free space* H. *In general,* $v_1^0(0, x, z)\big|_{z=0,1} \neq 0$ *for generic* $\mathbf{F}$.

For system (3.16), one has the following a priori estimates

Lemma 3.3.

$$
\begin{aligned}
&|v_1^{C,0}| \le |c(x)|e^{-\frac{\tau}{2}-\frac{Z}{\sqrt{2}}},\\
&||v_1^{C,0}||_{L^\infty(0,\infty;L^2(\Omega_\infty))} \le C,\\
&||\partial_Z v_1^{C,0}||_{L^\infty(0,\infty;L^2(\Omega_\infty))} \le C,\\
&||v_1^{C,0}||_{L^2(0,\infty;L^2(\Omega_\infty))} \le C,\\
&||\partial_Z v_1^{C,0}||_{L^2(0,\infty;L^2(\Omega_\infty))} \le C.
\end{aligned}
$$

PROOF. We only need to show the pointwise estimate

$$|v_1^{C,0}| \le |c(x)|e^{-\frac{\tau}{2}-\frac{Z}{\sqrt{2}}}.$$

The rest of the inequalities follow directly from the standard energy estimate for the equation satisfied by $v_1^{C,0} - c(x)e^{-\tau-Z}$. For that, we introduce an axillary function $k(\tau, x, Z) = |c(x)|e^{-\frac{\tau}{2}-\frac{Z}{\sqrt{2}}}$. It is clear that k satisfies the equation

$$\frac{\partial k}{\partial \tau} + k - \partial_{ZZ} k = 0,$$

but with

$$k\big|_{\tau=0} \ge v_1^{C,0}\big|_{\tau=0}, \quad k\big|_{Z=0} \ge v_1^{C,0}\big|_{Z=0}.$$

A comparison principle of parabolic equation (pp. 219 in [30]) gives $v_1^{C,0} \le k$. The same argument applies to $-v_1^{C,0}$, which concludes the proof.

Remark 4. *It is clear that $\partial_x v_1^{C,0}$ satisfies the same type of inequalities as $v_1^{C,0}$.*

With lemma 3.3, the following propostion can be established by using the change of variable and the divergence free condition.

Proposition 3.4. *Assume $\mathbf{v}_0 \in V \cap H^m(\Omega)$ and $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ with $m \ge 4$. One has the following estimate*

$$
\begin{aligned}
&||v_1^{C,0}||_{L^\infty(0,T;L^2(\Omega)} \le C\epsilon^{\frac{1}{4}},\\
&||v_2^{C,0}||_{L^\infty(0,T;L^2(\Omega)} \le C\epsilon^{\frac{1}{2}},\\
&||v_1^{C,0}||_{L^2(0,T;L^2(\Omega)} \le C\epsilon^{\frac{3}{4}},\\
&||v_2^{C,0}||_{L^2(0,T;L^2(\Omega)} \le C\epsilon,
\end{aligned}
$$

$$
\begin{aligned}
&||\partial_z v_1^{C,0}||_{L^2(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{4}},\\
&||\partial_z v_1^{C,0}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{-\frac{1}{4}},\\
&||\partial_z v_2^{C,0}||_{L^2(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{3}{4}},\\
&||\partial_z v_2^{C,0}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{4}}.
\end{aligned}
$$

We proceed to construct a modified initial-boundary layer profile by truncating the stream function of system (3.16). Finding the stream function for $(v_1^{C,0}, v_2^{C,0})$ is a reverse of engineering. Since by definition

$$
v_1^{C,0} = -\frac{\partial \psi^{C,0}}{\partial z} = -\frac{1}{\sqrt{\epsilon}}\frac{\partial \psi^{C,0}}{\partial Z},
$$

we infer from equation (3.16) that $\psi^{C,0}$ should satisfy, assuming $\psi^{C,0}\big|_{Z=0} = 0$ which is consistent with $\frac{\partial \psi^{C,0}}{\partial x}\big|_{Z=0} = 0$

$$
\begin{cases}
\dfrac{\partial \psi^{C,0}}{\partial \tau} + \psi^{C,0} - \partial_{ZZ}\psi^{C,0} = f(\tau, x),\\
\psi^{C,0}\big|_{\tau=0} = -\sqrt{\epsilon}c(x)(1-e^{-Z}),\\
\dfrac{\partial \psi^{C,0}}{\partial Z}\big|_{Z=0} = -\sqrt{\epsilon}c(x)e^{-\tau},\\
\dfrac{\partial \psi^{C,0}}{\partial Z} \to 0, \text{as } Z \to \infty,\\
\psi^{C,0}\big|_{Z=0} = 0,
\end{cases}
\tag{3.17}
$$

with an integral constant (function) $f(\tau, x)$ to be determined so that the overdetermined system is solvable. To find $\psi^{C,0}$, we first solve the following equation

$$
\begin{cases}
\dfrac{\partial \psi_1}{\partial \tau} + \psi_1 - \partial_{ZZ}\psi_1 = 0,\\
\psi_1\big|_{\tau=0} = -\sqrt{\epsilon}c(x)(1-e^{-Z}),\\
\dfrac{\partial \psi_1}{\partial Z}\big|_{Z=0} = -\sqrt{\epsilon}c(x)e^{-\tau},\\
\dfrac{\partial \psi_1}{\partial Z} \to 0, \text{ as } Z \to +\infty.
\end{cases}
\tag{3.18}
$$

Its solution can be found by using Green's function approach, cf. [42, 11]

$$
\psi_1 = -\sqrt{\epsilon}c(x)e^{-\tau}\left\{\frac{1}{\sqrt{4\pi\tau}}\int_0^{+\infty}(1-e^{-Z_0})\left(e^{-\frac{(Z-Z_0)^2}{4\tau}} + e^{-\frac{(Z+Z_0)^2}{4\tau}}\right) dZ_0\right.
$$

$$-2\int_0^\tau \frac{1}{\sqrt{4\pi s}} e^{-\frac{Z^2}{4s}}\, ds\Big\}. \tag{3.19}$$

Next, we take $\psi_2 = -\psi_1\big|_{Z=0}$ so that, by simple calculation

$$\psi_2 = 2\sqrt{\epsilon}c(x)e^{-\tau}\big(-\sqrt{\frac{\tau}{\pi}} + \frac{1}{2} - \frac{1}{\sqrt{\pi}}\int_{\sqrt{\tau}}^{+\infty} e^{-Z_0^2}\, dZ_0\big).$$

Now defining $\psi^{C,0} = \psi_1 + \psi_2$, we see that $\psi^{C,0}$ satisfies the system (3.17) with

$$f(\tau, x) = -\frac{2\sqrt{\epsilon}c(x)e^{-\tau}}{\sqrt{4\pi\tau}}\int_0^{+\infty} e^{-\frac{z^2}{4\tau}-z}\, dz = -\frac{2\sqrt{\epsilon}c(x)}{\sqrt{\pi}}\int_{\sqrt{\tau}}^{+\infty} e^{-z^2}\, dz. \tag{3.20}$$

Pursuing the same line of thought as in the truncation of boundary layer, we define the truncated stream function as

$$\tilde{\psi}^{C,0} = \psi^{C,0}\rho(z),$$

so that the modified initial-boundary layer is given by

$$\tilde{\mathbf{v}}^{C,0} = (-\frac{\partial\tilde{\psi}^{C,0}}{\partial z}, \frac{\partial\tilde{\psi}^{C,0}}{\partial x}).$$

Thus

$$\begin{aligned} \tilde{v}_1^{C,0} &= v_1^{C,0}\rho(z) - \psi^{C,0}\rho'(z), \\ \tilde{v}_2^{C,0} &= v_2^{C,0}\rho(z). \end{aligned} \tag{3.21}$$

The corresponding $\tilde{\mathbf{v}}^{C,1}$ at the boundary $z = 1$ is defined similarly. Note that $\tilde{\mathbf{v}}^{C,0}(0, x, Z) = -\tilde{\mathbf{v}}^{B,0}(0, x, Z)$.

Taking

$$\tilde{\mathbf{v}}^C = \tilde{\mathbf{v}}^{C,0} + \tilde{\mathbf{v}}^{C,1}, \tag{3.22}$$

one finds that $\tilde{\mathbf{v}}^C$ satisfies the following system

$$\begin{cases} \epsilon\dfrac{\partial\tilde{\mathbf{v}}^C}{\partial t} + \tilde{\mathbf{v}}^C - \epsilon\Delta\tilde{\mathbf{v}}^C = \mathbf{f}^C, \quad t \in (0,T), (x,z) \in \Omega, \\ \nabla\cdot\tilde{\mathbf{v}}^C = 0, \\ \tilde{\mathbf{v}}^C\big|_{t=0} = -\tilde{\mathbf{v}}^B\big|_{t=0}, \\ \tilde{\mathbf{v}}^C\big|_{z=0,1} = -\mathbf{v}^I\big|_{z=0,1}. \end{cases} \tag{3.23}$$

Here $\mathbf{f}^C = \mathbf{f}_1^{C,0} + \mathbf{f}_2^{C,0}$, and $\mathbf{f}_1^{C,0} = (f_1^{C,0}, f_2^{C,0})$ with

$$\begin{aligned} f_1^{C,0} = & -f\rho' - 2\epsilon\frac{\partial v_1^{C,0}}{\partial z}\rho' - 3\epsilon v_1^{C,0}\rho'' \\ & + \epsilon\psi^{C,0}\rho''' - \epsilon\frac{\partial^2 v_1^{C,0}}{\partial x^2}\rho + \epsilon\frac{\partial v_2^{C,0}}{\partial x}\rho', \end{aligned} \tag{3.24a}$$

$$f_2^{C,0} = \sqrt{\epsilon}\big(2\frac{\partial v_1^{C,0}}{\partial x}\rho' - v_2^{C,0}\rho'' - \frac{\partial^2 v_2^{C,0}}{\partial x^2}\rho\big). \tag{3.24b}$$

In deriving $f_1^{C,0}$ one has utilized the equation (3.17).

For $\tilde{\mathbf{v}}^C$ and $\mathbf{f}^C$, one has the following estimates

Lemma 3.5. *The assumption is the same as the one in proposition 3.4. Then the following inequalities hold*

$$\begin{aligned} &||\tilde{v}_1^C||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{4}}, \\ &||\tilde{v}_2^C||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{2}}, \\ &||\nabla\tilde{v}_1^C||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{-\frac{1}{4}}, \\ &||\nabla\tilde{v}_2^C||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{4}}, \\ &||\tilde{v}_1^C||_{L^\infty(0,T;L^\infty(\Omega))} \le C, \\ &||\tilde{v}_2^C||_{L^\infty(0,T;L^\infty(\Omega))} \le C\epsilon^{\frac{1}{4}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} &||\partial_x^j\mathbf{f}^C||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{2}} \quad j = 0, 1. \\ &||\partial_x^j\mathbf{f}^C||_{L^2(0,T;L^2(\Omega))} \le C\epsilon \quad j = 0, 1. \end{aligned}$$

PROOF. Since

$$\psi^{C,0} = -\sqrt{\epsilon}\int_0^Z v_1^{C,0}\,dZ_0,$$

lemma 3.3 implies that $\psi^{C,0}$ observes the same estimate as $v_2^{C,0}$. Thus $\tilde{\mathbf{v}}^{C,0}$ has the same estimate as $\mathbf{v}^{C,0}$ in proposition 3.4. The estimates of $\tilde{\mathbf{v}}^C$ then follow from proposition 3.4 and lemma 3.1.

For the estimate of $\mathbf{f}^C$, one only needs to control the term $f\rho'$ in (3.24). It follows from (3.20)

$$|f(\tau,x)| \le \frac{2\sqrt{\epsilon}|c(x)|e^{-\tau}}{\sqrt{4\pi\tau}}\int_0^{+\infty} e^{-\frac{z^2}{4\tau}}\,dz = \sqrt{\epsilon}|c(x)|e^{-\tau},$$

which yields

$$||f\rho'||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}},$$

and by the change of variable $\tau = \frac{t}{\epsilon}$

$$||f\rho'||_{L^2(0,T;L^2(\Omega))} \leq C\epsilon.$$

We thus proved the lemma.

Remark 5. *We point out that the estimates in lemma 3.5 are also optimal. In fact, since the initial-boundary layer equation* (3.16) *is a linear parabolic equation, the solution formula* (3.21) *can be given explicitly in terms of Green's function, cf. [42]. (Compare to* (3.18) *and* (3.19)*)*

4. Approximate Solution and Error Estimate

Based on the asymptotic analysis above, we define the approximate solution as follows

$$\begin{aligned} &\mathbf{v}^{app} = \mathbf{v}^0 + \mathbf{v}^I + \tilde{\mathbf{v}}^B + \tilde{\mathbf{v}}^C, \\ &p^{app} = p^0. \end{aligned} \tag{4.1}$$

Plugging the approximate solution $\mathbf{v}^{app}$ and p^{app} into the system (1.1), one finds they should satisfy

$$\begin{cases} \epsilon\left(\dfrac{\partial \mathbf{v}^{app}}{\partial t} + (\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{app}\right) + \mathbf{v}^{app} - \epsilon\Delta\mathbf{v}^{app} + \nabla p^{app} \\ \qquad = \mathbf{F} + \mathbf{F}^{err}, \\ \nabla\cdot\mathbf{v^{app}} = 0, \\ \mathbf{v^{app}}|_{z=0,1} = 0, \\ \mathbf{v^{app}}|_{t=0} = \mathbf{v}_0. \end{cases} \tag{4.2}$$

with the extra body forcing $\mathbf{F}^{err}$ defined as

$$\mathbf{F}^{err} = \mathbf{f}^B + \mathbf{f}^C + \epsilon\big(\frac{\partial \mathbf{v}^0}{\partial t} + \frac{\partial \tilde{\mathbf{v}}^B}{\partial t}\big) + \epsilon(\mathbf{v}^{app}\cdot\nabla)\mathbf{v}^{app} - \epsilon(\Delta\mathbf{v}^0 + \Delta\mathbf{v}^I). \tag{4.3}$$

In view of the estimates (3.5) and (3.14), proposition 3.2 and lemma 3.5, one can easily deduce the following estimate for the extra body forcing $\mathbf{F}^{err}$

Lemma 4.1. *Under the assumption of* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ *with* $m \geq 4$*, the following estimate holds*

$$||\partial_x^j \mathbf{F}^{err}||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \quad j = 0, 1. \tag{4.4}$$

Define the error functions

$$\mathbf{v}^{err} = \mathbf{v}^\epsilon - \mathbf{v}^{app}, \quad p^{err} = p^\epsilon - p^{app}.$$

Combining system (1.1) and (4.2), one can see $\mathbf{v}^{err}, p^{err}$ satisfy the following equations

$$\begin{aligned} &\epsilon\left(\frac{\partial \mathbf{v}^{err}}{\partial t} + (\mathbf{v^{err}} \cdot \nabla)\mathbf{v^{err}} + (\mathbf{v^{app}} \cdot \nabla)\mathbf{v^{err}} + (\mathbf{v^{err}} \cdot \nabla)\mathbf{v^{app}}\right) + \mathbf{v^{err}} \\ &\qquad - \epsilon\Delta\mathbf{v^{err}} + \nabla p^{err} = -\mathbf{F}^{err}, \\ &\nabla \cdot \mathbf{v^{err}} = 0, \\ &\mathbf{v^{err}}|_{z=0,1} = 0, \\ &\mathbf{v^{err}}|_{t=0} = 0. \end{aligned} \tag{4.5}$$

Now we are in a position to state our main theorem.

Theorem 4.2. *Assume* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ *with* $m \geq 4$*. The following convergence rates hold*

$$||\mathbf{v}^\epsilon - \mathbf{v}^{app}||_{L^\infty(0,T,L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{4.6a}$$

$$||\mathbf{v}^\epsilon - \mathbf{v}^{app}||_{L^\infty(0,T,H^1(\Omega))} \leq C, \tag{4.6b}$$

$$||\mathbf{v}^\epsilon - \mathbf{v}^{app}||_{L^\infty(0,T,L^\infty(\Omega))} \leq C\epsilon^{\frac{1}{4}}, \tag{4.6c}$$

$$||\nabla(p^\epsilon - p^0)||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon^{\frac{1}{2}}. \tag{4.6d}$$

Since $||\mathbf{v}^I + \tilde{\mathbf{v}}^B + \tilde{\mathbf{v}}^C||_{L^2(0,T,L^2(\Omega))}$ is of order $O(\epsilon^{\frac{1}{4}})$ (cf. Remark 2 and 5), estimate (4.6a) implies the following optimal vanishing viscosity limit result

Corollary 4.3. *The assumption is that in theorem* (4.2)*. One has*

$$C_1\epsilon^{\frac{1}{4}} \leq ||\mathbf{v}^\epsilon - \mathbf{v}^0||_{L^2(0,T,L^2(\Omega))} \leq C_2\epsilon^{\frac{1}{4}}, \tag{4.7}$$

with constants $C_1 < C_2$*.*

For the estimate of the nonlinear terms in equation (4.5), we need the following classical results, see [52, 16] for the detailed proof.

Lemma 4.4. *Let* $b(\mathbf{u}, \mathbf{v}, \mathbf{w})$ *be the trilinear form on* $V \times V \times V$ *defined by*

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{v} \cdot \mathbf{w} \, dx.$$

Then b *has the following properties*

$$b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0, \tag{4.8a}$$
$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}), \tag{4.8b}$$
$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C||\mathbf{u}||_{L^2}^{\frac{1}{2}} ||\nabla \mathbf{u}||_{L^2}^{\frac{1}{2}} ||\nabla \mathbf{v}||_{L^2} ||\mathbf{w}||_{L^2}^{\frac{1}{2}} ||\nabla \mathbf{w}||_{L^2}^{\frac{1}{2}}, \tag{4.8c}$$
$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \le C||\mathbf{u}||_{L^2}^{\frac{1}{2}} ||\nabla \mathbf{u}||_{L^2}^{\frac{1}{2}} ||\nabla \mathbf{v}||_{L^2}^{\frac{1}{2}} ||\Delta \mathbf{v}||_{L^2}^{\frac{1}{2}} ||\mathbf{w}||_{L^2}^{\frac{1}{2}}, \text{ provided } (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in V \times (V \cap H^2(\Omega)) \times H. \tag{4.8d}$$

Proof of theorem 4.2: Because of our construction, $\mathbf{v}^{err}$ satisfies homogeneous initial and boundary conditions in the z direction, and periodic in the x direction. Therefore, when performing integration by parts the boundary terms vanish, which helps to simplify our calculation. We divide the proof into four steps.

4.1. $L^\infty(L^2)$ Estimate.

Multiplying equation (4.5) by $\mathbf{v}^{err}$ on both sides and integrating over the domain gives

$$\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}||\mathbf{v}^{err}||_{L^2(\Omega)}^2 + ||\mathbf{v}^{err}||_{L^2(\Omega)}^2 + \epsilon||\nabla \mathbf{v}^{err}||_{L^2(\Omega)}^2 \\
&= -\epsilon\big((\mathbf{v}^{err} \cdot \nabla)\mathbf{v}^{app}, \mathbf{v}^{err}\big) - \big(F^\epsilon, \mathbf{v}^{err}\big) \\
&= \epsilon\big((\mathbf{v}^{err} \cdot \nabla)\mathbf{v}^{err}, \mathbf{v}^{app}\big) - \big(F^\epsilon, \mathbf{v}^{err}\big) \\
&\le \epsilon||\mathbf{v}^{app}||_{L^\infty}||\mathbf{v}^{err}||_{L^2(\Omega)}||\nabla \mathbf{v}^{err}||_{L^2(\Omega)} + ||\mathbf{v}^{err}||_{L^2(\Omega)}||\mathbf{F}^\epsilon||_{L^2(\Omega)} \\
&\le \frac{\epsilon}{2}||\mathbf{v}^{app}||_{L^\infty}^2||\mathbf{v}^{err}||_{L^2(\Omega)}^2 + \frac{\epsilon}{2}||\nabla \mathbf{v}^{err}||_{L^2(\Omega)}^2 + \frac{1}{2}||\mathbf{v}^{err}||_{L^2(\Omega)}^2 \\
&\quad + C||\mathbf{F}^\epsilon||_{L^2(\Omega)}^2,
\end{aligned}$$

which leads to, by lemma 4.1

$$\epsilon\frac{d}{dt}||\mathbf{v}^{err}||_{L^2(\Omega)}^2 + ||\mathbf{v}^{err}||_{L^2(\Omega)}^2 + \epsilon||\nabla \mathbf{v}^{err}||_{L^2(\Omega)}^2 \le C\epsilon||\mathbf{v}^{err}||_{L^2(\Omega)}^2 + C\epsilon, \tag{4.9}$$

where the uniform estimate $||\mathbf{v}^{app}||_{L^\infty}$ follows from Sobolev inequality, proposition 3.2, and lemma 3.5.

Applying first Gronwall's inequality to (4.9) yields

$$\epsilon||\mathbf{v}^{err}(t,\cdot)||^2_{L^2(\Omega)} + ||\mathbf{v}^{err}||^2_{L^2(0,t;L^2(\Omega))} + \epsilon||\nabla\mathbf{v}^{err}||^2_{L^2(0,t;L^2(\Omega))} \leq C\epsilon, \quad (4.10)$$

which gives

$$||\mathbf{v}^{err}||_{L^\infty(L^2)} \leq C, \quad (4.11a)$$
$$||\mathbf{v}^{err}||_{L^2(L^2)} \leq C\epsilon^{\frac{1}{2}}, \quad (4.11b)$$
$$||\mathbf{v}^{err}||_{L^2(H^1)} \leq C. \quad (4.11c)$$

By using the estimate (4.11a), inequality (4.9) can be simplified as

$$\epsilon\frac{d}{dt}||\mathbf{v}^{err}||^2_{L^2(\Omega)} + ||\mathbf{v}^{err}||^2_{L^2(\Omega)} \leq C\epsilon. \quad (4.12)$$

Multiplying the inequality above by an integration factor $e^{\frac{t}{\epsilon}}$ and then integrating the result, we obtain

$$||\mathbf{v}^{err}||_{L^\infty(L^2)} \leq C\epsilon^{\frac{1}{2}}, \quad (4.13)$$

where the constant C is independent of ϵ.

4.2. $L^\infty(H^1)$ Estimate

We follow the same line of proof as in [60, 37]. First, we try to control the tangential derivative (conormal derivative) $\partial_x\mathbf{v}^{err}$. Applying the operator ∂_x to both sides of equation (4.5), one obtains

$$\begin{aligned}&\epsilon\left(\frac{\partial(\partial_x\mathbf{v}^{err})}{\partial t} + (\mathbf{v^{err}}\cdot\nabla)\partial_x\mathbf{v^{err}} + (\mathbf{v^{app}}\cdot\nabla)\partial_x\mathbf{v^{err}} + (\mathbf{v^{err}}\cdot\nabla)\partial_x\mathbf{v^{app}}\right)\\ &+\partial_x\mathbf{v^{err}} - \epsilon\Delta\partial_x\mathbf{v^{err}} + \nabla\partial_x p^{err} = -\partial_x\mathbf{F}^{err}\\ &-\epsilon(\partial_x\mathbf{v}^{err}\cdot\nabla\mathbf{v}^{err} + \partial_x\mathbf{v}^{app}\cdot\mathbf{v}^{err} + \partial_x\mathbf{v}^{err}\cdot\nabla\mathbf{v}^{app}). \quad (4.14)\end{aligned}$$

Upon multiplying by $\partial_x\mathbf{v}^{err}$ on both sides of equation (4.14), we control the non-linear terms by using lemma 4.4 as follows

$$|b(\mathbf{v}^{err}, \partial_x\mathbf{v}^{app}, \partial_x\mathbf{v}^{err})| \leq C||\partial_x\mathbf{v}^{app}||^2_{L^\infty}||\mathbf{v}^{err}||^2_{L^2} + \frac{||\nabla\partial_x\mathbf{v}^{err}||^2_{L^2}}{5}, \quad (4.15)$$

$$|b(\partial_x\mathbf{v}^{err}, \mathbf{v}^{err}, \partial_x\mathbf{v}^{err})| \leq C||\partial_x\mathbf{v}^{err}||_{L^2}||\nabla\partial_x\mathbf{v}^{err}||_{L^2}||\nabla\mathbf{v}^{err}||_{L^2}$$

$$\leq C||\nabla \mathbf{v}^{err}||_{L^2}^2 ||\partial_x \mathbf{v}^{err}||_{L^2}^2 + \frac{1}{5}||\nabla \partial_x \mathbf{v}^{err}||_{L^2}^2, \tag{4.16}$$

$$\begin{aligned}
&|b(\partial_x \mathbf{v}^{app}, \mathbf{v}^{err}, \partial_x \mathbf{v}^{err})| = |b(\partial_x \mathbf{v}^{app}, \partial_x \mathbf{v}^{err}, \mathbf{v}^{err})| \\
&\leq ||\partial_x \mathbf{v}^{app}||_{L^\infty} ||\nabla \partial_x \mathbf{v}^{err}||_{L^2} ||\mathbf{v}^{err}||_{L^2} \\
&\leq C||\partial_x \mathbf{v}^{app}||_{L^\infty}^2 ||\mathbf{v}^{err}||_{L^2}^2 + \frac{1}{5}||\nabla \partial_x \mathbf{v}^{err}||_{L^2}^2,
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
&|b(\partial_x \mathbf{v}^{err}, \mathbf{v}^{app}, \partial_x \mathbf{v}^{err})| = |b(\partial_x \mathbf{v}^{err}, \partial_x \mathbf{v}^{err}, \mathbf{v}^{app})| \\
&\leq ||\mathbf{v}^{app}||_{L^\infty} ||\partial_x \mathbf{v}^{err}||_{L^2} ||\nabla \partial_x \mathbf{v}^{err}||_{L^2} \\
&\leq C||\partial_x \mathbf{v}^{app}||_{L^\infty}^2 ||\partial_x \mathbf{v}^{err}||_{L^2}^2 + \frac{1}{5}||\nabla \partial_x \mathbf{v}^{err}||_{L^2}^2.
\end{aligned} \tag{4.18}$$

Collecting inequalities (4.15)-(4.18) and using integration by parts, one obtains

$$\begin{aligned}
&\frac{\epsilon}{2}\frac{d}{dt}||\partial_x \mathbf{v}^{err}||_{L^2}^2 + ||\partial_x \mathbf{v}^{err}||_{L^2}^2 + \frac{\epsilon}{5}||\nabla \partial_x \mathbf{v}^{err}||_{L^2}^2 \leq C||\partial_x \mathbf{F}^\epsilon||_{L^2}^2 \\
&+ C\epsilon||\partial_x \mathbf{v}^{app}||_{L^\infty}^2 ||\mathbf{v}^{err}||_{L^2}^2 + C\epsilon(||\nabla \mathbf{v}^{err}||_{L^2}^2 + ||\partial_x \mathbf{v}^{app}||_{L^\infty}^2)||\partial_x \mathbf{v}^{err}||_{L^2}^2 \\
&\leq C\epsilon + C\epsilon||\partial_x \mathbf{v}^{err}||_{L^2}^2,
\end{aligned} \tag{4.19}$$

where one has used the estimate (4.11).

The same approach as deriving the $L^\infty(L^2)$ estimate applied to (4.19) leads to

$$||\partial_x \mathbf{v}^{err}||_{L^\infty(L^2)} \leq C\epsilon^{\frac{1}{2}}, \tag{4.20a}$$

$$||\partial_x \mathbf{v}^{err}||_{L^2(L^2)} \leq C\epsilon^{\frac{1}{2}}, \tag{4.20b}$$

$$||\partial_x \mathbf{v}^{err}||_{L^2(H^1)} \leq C. \tag{4.20c}$$

In a similar fashion, one can show

$$||\partial_{xx} \mathbf{v}^{err}||_{L^\infty(L^2)} \leq C\epsilon^{\frac{1}{2}}. \tag{4.21}$$

Indeed, by an induction argument one can show that $\partial_x^j \mathbf{v}^{err}$ observes the same estimate as $\mathbf{v}^{err}$ for any integer $j > 0$, provided enough regularity imposed on the data. This is within the expectation since the boundary layer effects only in the normal direction.

Now by using the divergence free condition

$$\frac{\partial v_1^{err}}{\partial x} + \frac{\partial v_2^{err}}{\partial z} = 0,$$

it follows from (4.20a), (4.21) and the remark above that

$$||\partial_z v_2^{err}||_{L^\infty(L^2)} \leq C\epsilon^{\frac{1}{2}}, \tag{4.22}$$

$$||\partial_{xz} v_2^{err}||_{L^\infty(L^2)} \leq C\epsilon^{\frac{1}{2}}. \tag{4.23}$$

In view of the anisotropic Sobolev embedding lemma 3.1, one concludes the uniform estimate for v_2^{err}

$$||\partial_x^j v_2^{err}||_{L^\infty(L^\infty)} \leq C\epsilon^{\frac{1}{2}}, \quad j = 0, 1. \tag{4.24}$$

With the help of the uniform estimate (4.24), we can now derive the $L^\infty(H^1)$ estimate. For that, we multiply equation (4.5) by $-\Delta \mathbf{v}^{err}$ and integrate over the domain Ω. We examine the trilinear term one by one. First of all, by using lemma 4.4 one gets

$$\begin{aligned}
&|b(\mathbf{v}^{err}, \mathbf{v}^{err}, \Delta \mathbf{v}^{err})| \\
&\leq C||\mathbf{v}^{err}||_{L^2}^{\frac{1}{2}} ||\nabla \mathbf{v}^{err}||_{L^2}^{\frac{1}{2}} ||\nabla \mathbf{v}^{err}||_{L^2}^{\frac{1}{2}} ||\Delta \mathbf{v}^{err}||_{L^2}^{\frac{1}{2}} ||\Delta \mathbf{v}^{err}||_{L^2} \\
&= C||\mathbf{v}^{err}||_{L^2}^{\frac{1}{2}} ||\nabla \mathbf{v}^{err}||_{L^2} ||\Delta \mathbf{v}^{err}||_{L^2}^{\frac{3}{2}} \\
&\leq C||\mathbf{v}^{err}||_{L^2}^2 ||\nabla \mathbf{v}^{err}||_{L^2}^4 + \frac{1}{6}||\Delta \mathbf{v}^{err}||_{L^2}^2 \\
&\leq C\epsilon||\nabla \mathbf{v}^{err}||_{L^2}^4 + \frac{1}{6}||\Delta \mathbf{v}^{err}||_{L^2}^2.
\end{aligned} \tag{4.25}$$

In deriving the inequality, we employed the following Young's inequality with parameter $\nu > 0$

$$\alpha\beta \leq \frac{\nu}{p}\alpha^p + \frac{1}{q\nu^{q/p}}\beta^q, \quad \alpha, \beta > 0, p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Next,

$$|b(\mathbf{v}^{app}, \mathbf{v}^{err}, \Delta \mathbf{v}^{err})| \leq C||\mathbf{v}^{app}||_{L^\infty}^2 ||\nabla \mathbf{v}^{err}||_{L^2}^2 + \frac{1}{6}||\Delta \mathbf{v}^{err}||_{L^2}^2. \tag{4.26}$$

Last,

$$b(\mathbf{v}^{err}, \mathbf{v}^{app}, \Delta \mathbf{v}^{err}) := I_1 + I_2, \tag{4.27}$$

where

$$
\begin{aligned}
I_1 &= \int_\Omega v_1^{err} \partial_x \mathbf{v}^{app} \cdot \Delta \mathbf{v}^{err} \, dx \\
&\leq C ||\partial_x \mathbf{v}^{app}||_{L^\infty}^2 ||v_1^{err}||_{L^2}^2 + \frac{1}{6} ||\Delta \mathbf{v}^{err}||_{L^2}^2 \\
&\leq C\epsilon + \frac{1}{6} ||\Delta \mathbf{v}^{err}||_{L^2}^2,
\end{aligned} \tag{4.28}
$$

and, by proposition 3.2, lemma 3.5, and the uniform estimate (4.24)

$$
\begin{aligned}
I_2 &= \int_\Omega v_2^{err} \partial_z \mathbf{v}^{app} \cdot \Delta \mathbf{v}^{err} \, dx \\
&\leq C ||v_2^{err}||_{L^\infty}^2 ||\partial_z \mathbf{v}^{app}||_{L^2}^2 + \frac{1}{6} ||\Delta \mathbf{v}^{err}||_{L^2}^2 \\
&\leq C\epsilon^{\frac{1}{2}} + \frac{1}{6} ||\Delta \mathbf{v}^{err}||_{L^2}^2.
\end{aligned} \tag{4.29}
$$

We therefore have

$$
\begin{aligned}
&\frac{\epsilon}{2} \frac{d}{dt} ||\nabla \mathbf{v}^{err}||_{L^2}^2 + ||\nabla \mathbf{v}^{err}||_{L^2}^2 + \frac{\epsilon}{6} ||\Delta \mathbf{v}^{err}||_{L^2}^2 \leq C \\
&+ \epsilon(\epsilon ||\nabla \mathbf{v}^{err}||_{L^2}^2 + 1) ||\nabla \mathbf{v}^{err}||_{L^2}^2 + C\epsilon^2 + C\epsilon^{\frac{3}{2}}.
\end{aligned} \tag{4.30}
$$

Since $||\nabla \mathbf{v}^{err}||_{L^2(L^2)} \leq C$ by (4.11c), the application of Gronwall's inequality and the method of integration factor implies

$$
||\nabla \mathbf{v}^{err}||_{L^\infty(L^2)} \leq C. \tag{4.31}
$$

4.3. $L^\infty(L^\infty)$ Estimate

We already derived the uniform estimate for v_2^{err} in (4.24). In view of the anisotropic Sobolev imbedding lemma 3.1, one only needs to control $||\partial_x \partial_z \mathbf{v}^{err}||_{L^\infty(L^2)}$ in order to get the uniform estimate for v_1^{err}. Therefore, multiplying equation (4.14) by $-\Delta \partial_x \mathbf{v}^{err}$, using the same technique as in proving the $L^\infty(H^1)$ estimate, one has

$$
\begin{aligned}
&\frac{\epsilon}{2} \frac{d}{dt} ||\nabla \partial_x \mathbf{v}^{err}||_{L^2}^2 + ||\nabla \partial_x \mathbf{v}^{err}||_{L^2}^2 + \frac{\epsilon}{6} ||\Delta \partial_x \mathbf{v}^{err}||_{L^2}^2 \leq C \\
&+ \epsilon(\epsilon ||\nabla \partial_x \mathbf{v}^{err}||_{L^2}^2 + 1) ||\nabla \partial_x \mathbf{v}^{err}||_{L^2}^2 + C\epsilon^2 + C\epsilon^2 ||\partial_x \partial_z \mathbf{v}^{app}||_{L^2}^2.
\end{aligned} \tag{4.32}
$$

By using (4.20c), proposition 3.2, lemma 3.5, applying Gronwall's inequality and integration factor, one concludes

$$||\nabla\partial_x \mathbf{v}^{err}||_{L^\infty(L^2)} \leq C. \tag{4.33}$$

The application of lemma 3.1 implies

$$||v_1^{err}||_{L^\infty(L^\infty)} \leq C\epsilon^{\frac{1}{4}}. \tag{4.34}$$

4.4. Estimate of Pressure

From the analysis above, one has

$$||\mathbf{v}^{err}||_{L^\infty(L^\infty)} \leq C\epsilon^{\frac{1}{4}}, \quad ||\nabla\mathbf{v}^{err}||_{L^\infty(L^2)} \leq C.$$

Thus by using proposition 3.2 and lemma 3.5, one concludes an estimate for the nonlinear terms

$$||\epsilon\Big((\mathbf{v}^{err}\cdot\nabla)\mathbf{v}^{err} + (\mathbf{v}^{\mathbf{app}}\cdot\nabla)\mathbf{v}^{\mathbf{err}} + (\mathbf{v}^{\mathbf{err}}\cdot\nabla)\mathbf{v}^{\mathbf{app}}\Big)||_{L^\infty(L^2)} \leq C\epsilon.$$

One can then write the error equation (4.5) as a time-dependent Stokes system

$$\epsilon\frac{\partial\mathbf{v}^{err}}{\partial t} - \epsilon\Delta\mathbf{v}^{\mathbf{err}} + \nabla p^{err} = \tilde{\mathbf{F}}^{err}, \tag{4.35}$$

where $||\tilde{\mathbf{F}}^{err}||_{L^\infty(L^2)} \leq C\epsilon^{\frac{1}{2}}$. The regularity theory of Stokes system [52] (applied to $\epsilon\mathbf{v}^{err}$ and p^{err}) implies

$$||\nabla p^{err}||_{L^\infty(L^2)} \leq C\epsilon^{\frac{1}{2}}.$$

This completes the proof of Theorem 4.2.

5. Higher Order Expansion

It is clear from (3.9) and (3.21) that the truncation introduces extra errors of order $\sqrt{\epsilon}$ into the approximate equation (4.2) (the last terms in (3.12a) and the first term in (3.24a)). To correct these errors, it is natural to look at the expansion up to the order $\sqrt{\epsilon}$. Thus one assumes an Ansatz

$$\mathbf{v}^\epsilon = \mathbf{v}^0 + \big(\mathbf{v}^I + \tilde{\mathbf{v}}^B + \tilde{\mathbf{v}}^C\big) + \sqrt{\epsilon}\big(\mathbf{u}^0(t,x,z) + \mathbf{u}^I(t/\epsilon,x,z)\big)$$

$$
\begin{aligned}
&+ \sqrt{\epsilon}\big(\mathbf{u}^{B,0}(t,x,z/\sqrt{\epsilon}) + \mathbf{u}^{C,0}(t/\epsilon,x,z/\sqrt{\epsilon})\big) \\
&+ \sqrt{\epsilon}\big(\mathbf{u}^{B,1}(t,x,(1-z)/\sqrt{\epsilon}) + \mathbf{u}^{C,1}(t/\epsilon,x,(1-z)/\sqrt{\epsilon})\big) + O(\epsilon), \quad (5.1)
\end{aligned}
$$

$$
p^\epsilon = p^0 + \sqrt{\epsilon}p^1(t,x,z) + O(\epsilon). \tag{5.2}
$$

with the matching condition defined similarly as (3.2). One recalls the definition of $\tilde{\mathbf{v}}^B$ and $\tilde{\mathbf{v}}^C$ in (3.10) and (3.22), respectively.

Remark 6. *The ansatz* (5.2) *for pressure makes sense, since the estimate* (4.6d) *in Theorem 4.2 suggests there are no boundary layer or initial-boundary layer of leading order for pressure.*

Following the same approach as in section 3, we can derive the Prandtl type equations as follows

- $\mathbf{u}^0$ and p^1

$$
\begin{cases}
\mathbf{u}^O + \nabla p^1 = \mathbf{f}^O, \quad (x,z) \in \Omega \\
\nabla \cdot \mathbf{u}^O = 0, \\
u_2^o\big|_{z=0,1} = 0.
\end{cases} \tag{5.3}
$$

  where $\mathbf{f}^O = \bigg(\rho_0' a + \rho_1' b, -\frac{\partial a}{\partial x}\rho_1 - \frac{\partial b}{\partial x}\rho_2\bigg)$, ρ_0 and ρ_1 are the cut-off functions at $z=0$ and $z=1$, respectively. Since $\nabla \cdot \mathbf{f}^O = 0$, the pressure p^1 satisfies a Neumann problem

$$
\begin{cases}
\Delta p^1 = 0, \quad (x,z) \in \Omega \\
\dfrac{\partial p^1}{\partial z}\Big|_{z=0} = -\dfrac{\partial a}{\partial x}, \\
\dfrac{\partial p^1}{\partial z}\Big|_{z=1} = -\dfrac{\partial b}{\partial x}.
\end{cases} \tag{5.4}
$$

- $\mathbf{u}^I$

$$
\begin{cases}
\dfrac{\partial \mathbf{u}^I}{\partial \tau} + \mathbf{u}^I = \mathbf{f}^I, \\
\mathbf{u}^I\big|_{\tau=0} = -\mathbf{u}^0(0,x,z).
\end{cases} \tag{5.5}
$$

  where $\mathbf{f}^I = \bigg(-(c(x)\rho_0' + d(x)\rho_1')\frac{2}{\sqrt{\pi}}\int_{\sqrt{\tau}}^\infty e^{-z^2}\,dz, 0\bigg)$

- $\mathbf{u}^{B,0}$

$$
\begin{cases}
u_1^{B,0} - \partial_{ZZ} u_1^{B,0} = 0, \quad Z \in (0,\infty) \\
\dfrac{\partial u_1^{B,0}}{\partial x} + \dfrac{1}{\sqrt{\epsilon}} \dfrac{\partial u_2^{B,0}}{\partial Z} = 0, \quad Z \in (0,\infty) \\
u_1^{B,0}\big|_{Z=0} = -u_1^{O}\big|_{z=0}, \quad u_2^{B,0}\big|_{Z=0} = 0, \\
u_1^{B,0} \to 0, \quad \text{as } Z \to \infty.
\end{cases}
\tag{5.6}
$$

Remark 7. *Note that the boundary layer type function* $-2ae^{-\frac{z}{\sqrt{\epsilon}}}\rho'$ *from* (3.12) *is not included in the the equation* (5.6)*. In the sequel we will show that*

$$
|| - 2ae^{-\frac{z}{\sqrt{\epsilon}}}\rho'||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^N
$$

for any $N \ge 1$ *with* C *a constant depending on* N*.*

- $\mathbf{u}^{C,0}$

$$
\begin{cases}
\dfrac{\partial u_1^{C,0}}{\partial_\tau} + u_1^{C,0} - \partial_{ZZ} u_1^{C,0} = 0, \quad Z \in (0,\infty) \\
\dfrac{\partial u_1^{C,0}}{\partial x} + \dfrac{1}{\sqrt{\epsilon}} \dfrac{\partial u_2^{C,0}}{\partial Z} = 0, \quad Z \in (0,\infty) \\
u_1^{C,0}\big|_{\tau=0} = -u_1^{B,0}\big|_{t=0}, \\
u_1^{C,0}\big|_{Z=0} = -u_1^{I}\big|_{z=0}, \quad u_2^{B,0}\big|_{Z=0} = 0, \\
u_1^{C,0} \to 0, \quad \text{as } Z \to \infty.
\end{cases}
\tag{5.7}
$$

One observes the Prandtl type equations (5.5)-(5.7) are entirely analogous to the ones studied in section 3. Therefore the truncated boundary layer profile $\tilde{\mathbf{u}}^B$ and initial-boundary layer profile $\tilde{\mathbf{u}}^C$ can be constructed in the same way as $\tilde{\mathbf{v}}^B$ and $\tilde{\mathbf{v}}^C$. Moreover $\tilde{\mathbf{u}}^B$ and $\tilde{\mathbf{u}}^C$ follow the same estimates as in proposition 3.2 and lemma 3.5.

We define the approximate solution as follows

$$
\tilde{\mathbf{v}}^{app} = \mathbf{v}^0 + \mathbf{v}^I + \tilde{\mathbf{v}}^B + \tilde{\mathbf{v}}^C + \sqrt{\epsilon}\big(\mathbf{u}^0 + \mathbf{u}^I + \tilde{\mathbf{u}}^B + \tilde{\mathbf{u}}^C\big), \tag{5.8a}
$$

$$
\tilde{p}^{app} = p^0 + \sqrt{\epsilon} p^1. \tag{5.8b}
$$

We see $\tilde{\mathbf{v}}^{app}$ and $\tilde{p}^{app}$ satisfy a similar equation as (4.2) with a different forcing term

$$\begin{cases} \epsilon\left(\dfrac{\partial \tilde{\mathbf{v}}^{app}}{\partial t} + (\tilde{\mathbf{v}}^{app}\cdot\nabla)\tilde{\mathbf{v}}^{app}\right) + \tilde{\mathbf{v}}^{app} - \epsilon\Delta\tilde{\mathbf{v}}^{app} + \nabla\tilde{p}^{app} \\ \qquad = \mathbf{F} + \tilde{\mathbf{F}}^{err}, \\ \nabla\cdot\tilde{\mathbf{v}}^{app} = 0, \\ \tilde{\mathbf{v}}^{app}|_{z=0,1} = 0, \\ \tilde{\mathbf{v}}^{app}|_{t=0} = \mathbf{v}_0, \end{cases} \tag{5.9}$$

where the forcing term $\tilde{\mathbf{F}}^{err}$ takes the form of

$$\begin{aligned} \tilde{\mathbf{F}}^{err} = &\epsilon\big(\frac{\partial \mathbf{v}^0}{\partial t} + \frac{\partial \tilde{\mathbf{v}}^B}{\partial t}\big) + \epsilon(\tilde{\mathbf{v}}^{app}\cdot\nabla)\tilde{\mathbf{v}}^{app} - \epsilon\big(\Delta\mathbf{v}^0 + \Delta\mathbf{v}^I\big) \\ &+\epsilon^{\frac{3}{2}}\big(\frac{\partial \mathbf{u}^0}{\partial t} + \frac{\partial \tilde{\mathbf{u}}^B}{\partial t}\big) - \epsilon^{\frac{3}{2}}\big(\Delta\mathbf{u}^0 + \Delta\mathbf{u}^I\big) + \tilde{\mathbf{f}}^B + \tilde{\mathbf{f}}^C. \end{aligned} \tag{5.10}$$

Here $\tilde{\mathbf{f}}^B$ and $\tilde{\mathbf{f}}^C$ have similar terms as $\mathbf{f}^B$ and $\mathbf{f}^C$ except those $O(\sqrt{\epsilon})$ terms. As an illustration, we give the explicit formulation of $\tilde{\mathbf{f}}^B = \tilde{\mathbf{f}}^{B,0} + \tilde{\mathbf{f}}^{B,1}$:

$$\begin{aligned} \tilde{f}_1^{B,0} = &\epsilon^{\frac{3}{2}}\Delta(a\rho')(1 - e^{-\frac{z}{\sqrt{\epsilon}}}) + \epsilon\big(\frac{\partial^2 a}{\partial x^2}\rho + 3a\rho''\big)e^{-\frac{z}{\sqrt{\epsilon}}} \\ &- 2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho' - \epsilon^{\frac{3}{2}}\Delta u_1^O + \epsilon^2\Delta(\bar{a}\rho')(1 - e^{-\frac{z}{\sqrt{\epsilon}}}) \\ &+ \epsilon^{\frac{3}{2}}\big(\frac{\partial^2\bar{a}}{\partial x^2}\rho + 3\bar{a}\rho''\big)e^{-\frac{z}{\sqrt{\epsilon}}} - 2\epsilon\bar{a}e^{-\frac{z}{\sqrt{\epsilon}}}\rho' - \epsilon\bar{a}\rho', \end{aligned} \tag{5.11a}$$

$$\begin{aligned} \tilde{f}_2^{B,0} = &-\epsilon^{\frac{3}{2}}\Delta(\frac{\partial a}{\partial x}\rho)(1 - e^{-\frac{z}{\sqrt{\epsilon}}}) - 2\epsilon\frac{\partial a}{\partial x}e^{-\frac{z}{\sqrt{\epsilon}}}\rho' - \epsilon^{\frac{3}{2}}\Delta u_2^O \\ &- \epsilon^2\Delta(\frac{\partial\bar{a}}{\partial x}\rho)(1 - e^{-\frac{z}{\sqrt{\epsilon}}}) - 2\epsilon^{\frac{3}{2}}\frac{\partial\bar{a}}{\partial x}e^{-\frac{z}{\sqrt{\epsilon}}}\rho' + \epsilon\frac{\partial\bar{a}}{\partial x}\rho, \end{aligned} \tag{5.11b}$$

with $\bar{a} = u_1^O(t, x, 0)$.

For the estimate of $\tilde{\mathbf{F}}^{err}$, we need the following version of Hardy's inequality (lemma 13.4 of [50])

Lemma 5.1. *For $p > 1$, if $f \in L^p(R^+)$ and $g(t) = \frac{1}{t}\int_0^t f(s)\,ds$, then $g \in L^p(R^+)$ and $||g||_{L^p} \le \frac{p}{p-1}||f||_{L^p}$. For $p = \infty$, one replaces $\frac{p}{p-1}$ by 1.*

Now we are ready to prove

Lemma 5.2. *Assume $\mathbf{v}_0 \in V \cap H^m(\Omega)$ and $\mathbf{F} \in C^1([0, T]; H^m(\Omega))$ with $m \ge 5$. The following estimate holds*

$$||\partial_x^j\tilde{\mathbf{F}}^{err}||_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon, \quad j = 0, 1. \tag{5.12}$$

PROOF. In view of inequalities (3.5) and (3.14), proposition 3.2 and lemma 3.5, we only need to take care of three troublesome terms: $-2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho'$ from $\tilde{\mathbf{f}}^B$, $2\sqrt{\epsilon}\frac{\partial v_1^{C,0}}{\partial x}\rho'$ from $\tilde{\mathbf{f}}^C$, and $\epsilon(\tilde{\mathbf{v}}^{app}\cdot\nabla)\tilde{\mathbf{v}}^{app}$.

First of all, by the definition

$$
\begin{aligned}
||-2\sqrt{\epsilon}ae^{-\frac{z}{\sqrt{\epsilon}}}\rho'||_{L^2}^2 &\leq C\epsilon\int_{\frac{1}{4}}^{\frac{1}{2}} e^{-\frac{2z}{\sqrt{\epsilon}}}\rho'^2 dz \\
&\leq C\epsilon^2\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{z^2}{\epsilon}e^{-\frac{2z}{\sqrt{\epsilon}}}\rho'^2 dz \\
&\leq C\epsilon^{\frac{5}{2}}||Ze^{-Z}||_{L^2(0,\infty)}^2 \\
&\leq C\epsilon^{\frac{5}{2}}. \qquad (5.13)
\end{aligned}
$$

Next, note lemma 3.3 also implies

$$
|\frac{\partial v_1^{C,0}}{\partial x}| \leq |c'(x)|e^{-\frac{\tau}{2}-\frac{Z}{\sqrt{2}}}.
$$

Thus the above argument yields

$$
||2\sqrt{\epsilon}\frac{\partial v_1^{C,0}}{\partial x}\rho'||_{L^2} \leq C\epsilon^{\frac{5}{4}}. \qquad (5.14)
$$

For the estimate of $\epsilon(\tilde{\mathbf{v}}^{app}\cdot\nabla)\tilde{\mathbf{v}}^{app}$, we only need to control terms like $v_2^0\frac{\partial\tilde{v}_1^B}{\partial z}$ or $v_2^I\frac{\partial\tilde{v}_1^C}{\partial z}$. Since $v_2^0\big|_{z=0,1} = 0$, a direct application of Hardy's inequality lemma 5.1 yields

$$
\begin{aligned}
||v_2^0\frac{\partial\tilde{v}_1^B}{\partial z}||_{L^2} &\leq ||v_2^0\frac{\partial\tilde{v}_1^{B,0}}{\partial z}||_{L^2} + ||v_2^0\frac{\partial\tilde{v}_1^{B,1}}{\partial z}||_{L^2} \\
&\leq ||\partial_z v_2^0||_{L^\infty}\big(||z\frac{\partial\tilde{v}_1^{B,0}}{\partial z}||_{L^2} + ||(1-z)\frac{\partial\tilde{v}_1^{B,1}}{\partial z}||_{L^2}\big) \\
&\leq C\epsilon^{\frac{1}{4}}||Ze^{-Z}||_{L^2(0,\infty)}. \qquad (5.15)
\end{aligned}
$$

We thus proved the lemma.

The approach in proving theorem 4.2 then leads to

Theorem 5.3. *Assume* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ *with* $m \geq 5$. *The following convergence rates hold*

$$||\mathbf{v}^\epsilon - \tilde{\mathbf{v}}^{app}||_{L^\infty(0,T,L^2(\Omega))} \leq C\epsilon, \tag{5.16a}$$

$$||\mathbf{v}^\epsilon - \tilde{\mathbf{v}}^{app}||_{L^\infty(0,T,H^1(\Omega))} \leq C\epsilon^{\frac{1}{2}}, \tag{5.16b}$$

$$||\mathbf{v}^\epsilon - \tilde{\mathbf{v}}^{app}||_{L^\infty(0,T,L^\infty(\Omega))} \leq C\epsilon^{\frac{3}{4}}, \tag{5.16c}$$

$$||\nabla(p^\epsilon - \tilde{p}^{app})||_{L^\infty(0,T;L^2(\Omega))} \leq C\epsilon. \tag{5.16d}$$

Since (compare to Remark 2 and 5)

$$||\sqrt{\epsilon}(\mathbf{u}^0 + \mathbf{u}^I + \tilde{\mathbf{u}}^B + \tilde{\mathbf{u}}^C)||_{L^\infty(L^2)} \approx O(\epsilon^{\frac{1}{2}}), \tag{5.17a}$$

$$||\sqrt{\epsilon}(\mathbf{u}^0 + \mathbf{u}^I + \tilde{\mathbf{u}}^B + \tilde{\mathbf{u}}^C)||_{L^\infty(H^1)} \approx O(\epsilon^{\frac{1}{4}}), \tag{5.17b}$$

$$||\sqrt{\epsilon}(\mathbf{u}^0 + \mathbf{u}^I + \tilde{\mathbf{u}}^B + \tilde{\mathbf{u}}^C)||_{L^\infty(L^\infty)} \approx O(\epsilon^{\frac{1}{2}}), \tag{5.17c}$$

$$||\sqrt{\epsilon}p^1||_{L^\infty(L^2)} \approx O(\epsilon^{\frac{1}{2}}), \tag{5.17d}$$

theorem 5.3 immediately implies

Corollary 5.4. *Suppose that* $\mathbf{v}_0 \in V \cap H^m(\Omega)$ *and* $\mathbf{F} \in C^1([0,T]; H^m(\Omega))$ *with* $m \geq 5$. *Then the convergence rates* (4.6a) *and* (4.6d) *in theorem 4.2 are optimal. Moreover, one has the following improved optimal convergence rates*

$$||\mathbf{v}^\epsilon - \mathbf{v}^{app}||_{L^\infty(0,T,H^1(\Omega))} \leq C\epsilon^{\frac{1}{4}}, \tag{5.18a}$$

$$||\mathbf{v}^\epsilon - \mathbf{v}^{app}||_{L^\infty(0,T,L^\infty(\Omega))} \leq C\epsilon^{\frac{1}{2}}. \tag{5.18b}$$

6. Conclusion

In this paper, we have provided a detailed rigorous leading order asymptotic analysis of the nonlinear Darcy-Brinkman system in the vanishing Darcy number limit, which involves a boundary layer, an initial layer and their interaction-**initial-boundary layer**. The optimal convergence rates in Sobolev norms are proved rigorously by including the next order expansion. We remark that the analysis of the initial-boundary layer is novel, involving simultaneous two scale expansion in space and in time. The rigorous convergence result derived in this manuscript further validates the applicability of the Darcy model for flows in porous media if we view the nonlinear Darcy-Brinkman model as the ”true” model.

The convergence results are derived under the zeroth order compatibility assumption $\mathbf{v}_0\big|_{z=0,1} = 0$. Additional singular structures will emerge without this compatibility condition. In [39, 33], the authors used semiclassical techniques and layer potentials to study the boundary layer. This approach does not rely on the Prandtl theory and does not require any type of compatibility conditions between the initial and boundary data. However, it yields only convergence in $L^\infty(L^p)$ with $p \in [1, +\infty]$ and does not provide any estimate on normal gradients at the boundary.

A closely related model is the Bénard convection problem in a porous media region bounded by two parallel plates saturated with fluids. The bottom plate is kept at temperature T_2 and the top plate is kept at temperature T_1 with $T_2 > T_1$. Then the governing equations are the so called *Darcy-Brinkman-Oberbeck-Boussinesq system* in the non-dimensional form [41], see also [26]:

$$\begin{gathered}
\gamma_a \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) + \mathbf{v} - \tilde{Da}\Delta\mathbf{v} + \nabla p = Ra_D\,\mathbf{k}T,\\
\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \Delta T,\\
\operatorname{div}\mathbf{v} = 0,\\
\mathbf{v}\big|_{t=0} = \mathbf{v}_0, \quad T\big|_{t=0} = T_1,\\
\mathbf{v}\big|_{z=0,1} = 0, \quad T\big|_{z=0} = T_1, \quad T\big|_{z=1} = T_2.
\end{gathered}$$

where $\mathbf{k}$ is the unit normal vector directed upward (the positive z direction), Ra_D is the *Rayleigh-Darcy number*. One can also consider the vanishing Darcy number limit of the above system by taking $\gamma_a = \tilde{Da} = \epsilon$. We anticipate a similar result as Theorem 1.2 holds for this system. But the analysis would be more involved, and we leave it to a future work.

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