Irregular Period Tripling Bifurcations in Axisymmetric Scalefree Potentials

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1. INTRODUCTION

We have examined a range of orbits in axisymmetric potentials with central cusps, focusing on SED potentials.[1] The latter arise from mass distributions with spheroidal equidensity surfaces.[2] They are characterized by two parameters - the power $-\beta$ of dependence on distance, and the flatness parameter (axis ratio) q. Our primary focus is on understanding period tripling bifurcations in these potentials, which are related to $4:3$ resonances between radial and vertical frequencies.

The period-3 orbits which bifurcate in stable/unstable pairs via a pitchfork bifurcation from the stable thin tube orbit at a 120° rotation angle at some critical L_z^2 are called *regular*.[3] However, of particular interest to us are *irregular* orbits which bifurcate away from the thin tube via a turning-point bifurcation.[3]

In our study of orbits in scale-free potentials, the most prevalent bifurcation we encountered has been the regular pitchfork bifurcation in which a stable/unstable period-3 orbit pair bifurcates near, or in some special cases, exactly from, the stable period-1 orbit (Figure 1). This paper focuses on investigating the dynamics of period-3 orbits that arise away from the stable period-1 orbit and away from the orbit in the central plane, the last invariant curve in the phase space (Figure 2). It is important to note that the difference between the number of stable and the number of unstable orbits is a conserved quantity in both regular and irregular bifurcations.[3]

We restrict our attention to the axisymmetric, scalefree SED potential when the axis ratio $q = 0.8$ and $\beta = 0.3$. This is an oblate case, as the axis ratio is less then unity.

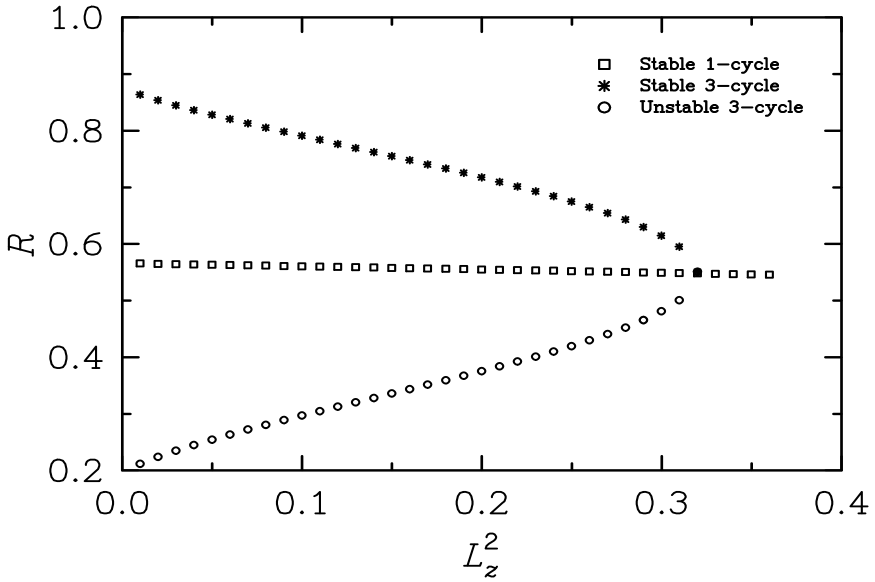

FIGURE 1. Regular pitchfork bifurcation. SED potential $q = 1.2$, $\beta = 0.35$.

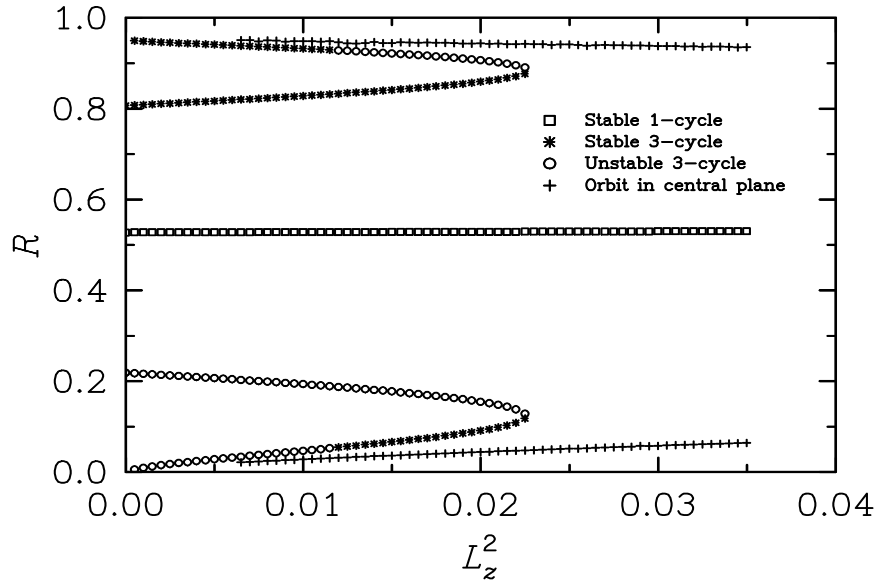

FIGURE 2. Irregular turning point bifurcation. SED model $q = 0.8$, $\beta = 0.3$.

2. IRREGULAR PERIOD TRIPLING BIFURCATIONS

We first observed these stable/unstable pairs of period-3 orbits in our surface of section (SOS) graphs that we generated by integration (FIGURE 3). These are graphs of the phase space or Poincaré space, the standard tool we use for analysis of the dynamics. This surface of section graph shows two stable/unstable pairs of period-3 orbits, shortly after they bifurcated. We see that they are originating in the region away from both the stable period-1 orbit at the center and the orbit in plane.

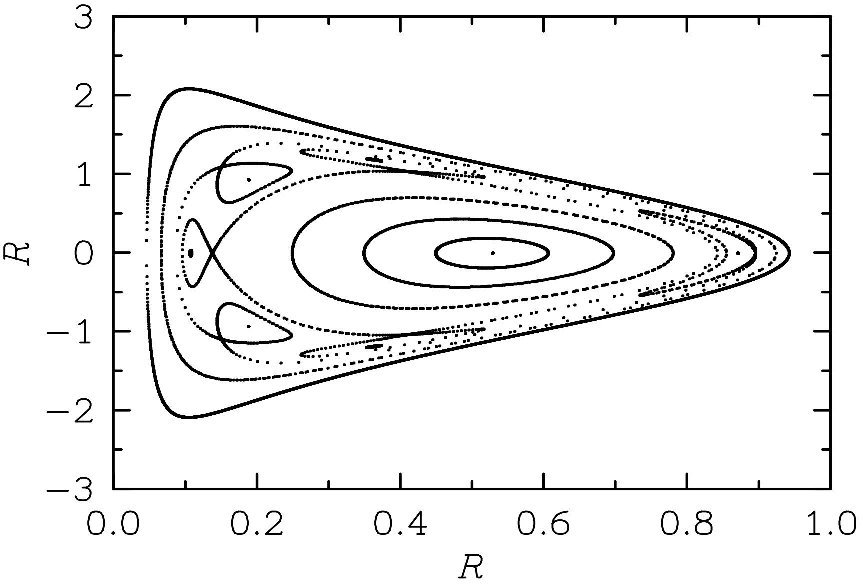

FIGURE 3. Surface of section (R,$\dot{R}$) for SED potential $q = 0.8$, $\beta = 0.3$, $L_z^2 = 0.022$ (oblate case). Two stable/unstable pairs of period-3 orbits have just bifurcated in the outer regions.

2.1. Average Rotation Number

In order to understand this phenomenon of irregular period tripling bifurcations, we have analyzed the rotation numbers of invariant curves in $(R, \dot{R})$ Poincaré space. The rotation number of an invariant curve is defined to be a fraction of a full revolution around the period-1 orbit that a point traverses in one iteration. Hence, the rotation number is the mapping angle divided by 2π. If we rotate any point around the stable period-1 orbit by an angle α, the resulting point, is the same one we would have obtained if the rotation angle was $(2\pi - \alpha)$ in the opposite direcion of rotation. Therefore, we have two different rotation numbers for the same mapping. To avoid this ambiguity, we use Floquet perturbation analysis[4] for the linearized mapping in the immediate vicinity of the period-1 orbit. The analysis shows that mapping angles in our case are around 120°, rather than 240°. Therefore, the rotation number of these sets of period-3 orbits is 1/3 rather than −2/3 or any other multiple of 1/3.

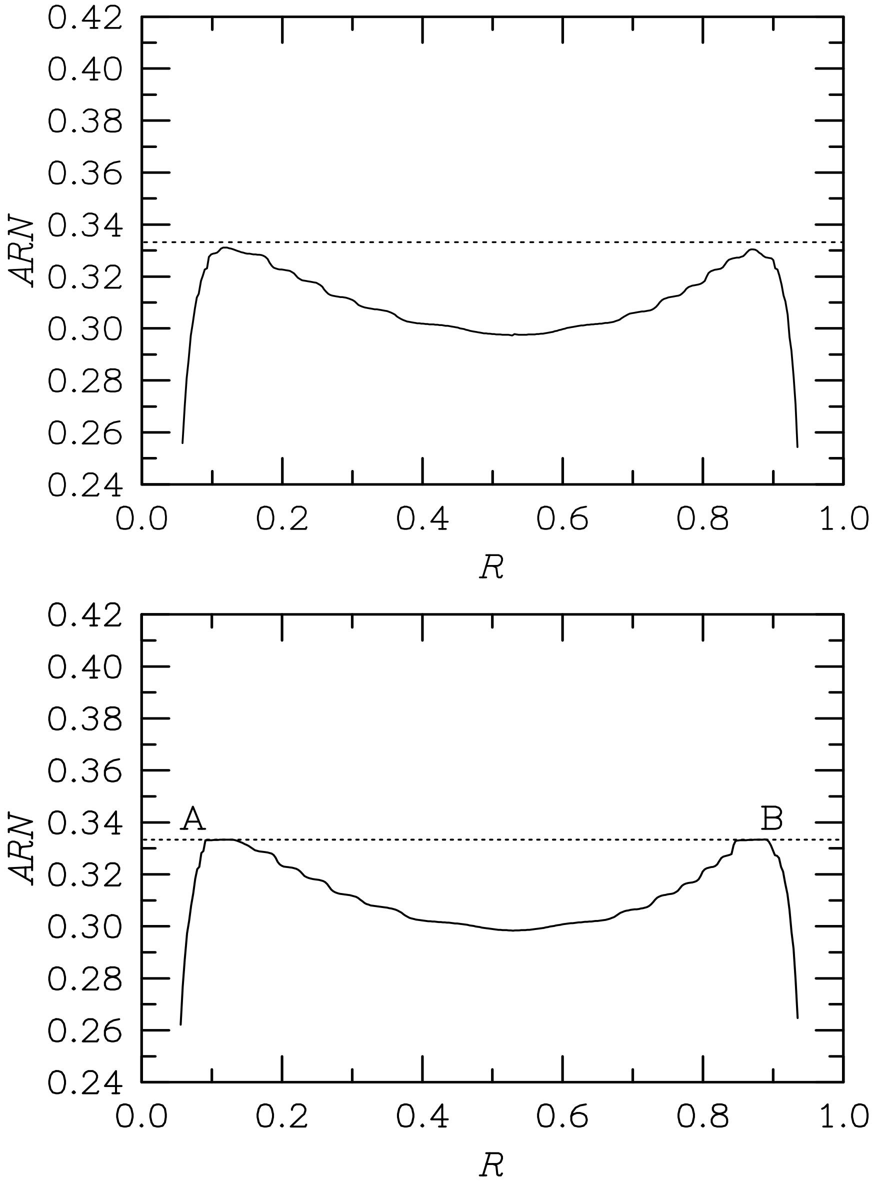

FIGURE 4. Average rotation number (ARN) as a function of R. SED potential $q = 0.8$, $\beta = 0.3$, and (**a**) $L_z^2 = 0.025$, (**b**) $L_z^2 = 0.0225$.

We calculated the average rotation number of invariant curves for different values of L_z^2, the square of the z-component of the angular momentum, which is an integral of motion due to the axisymmetric nature of the potential. In FIGURE 4, we follow the transition of the rotation curve as L_z^2 is decreased through the bifurcation point. In the SOS graphs where the irregular period-3 orbits were initially observed, bifurcations occur at $L_z^2 = 0.0225$. At that point, as can be seen in FIGURE 4(b), the rotation curve touches the 1/3 line. It is important to note that it is the same invariant curve in the SOS graphs that touches 1/3 line at both A and B, as these are its two crossings of the $\dot{R} = 0$ in the phase space. Therefore, these two pairs of stable/unstable 3-cyles bifurcate simultaneously. This analysis shows that the irregular period tripling bifurcations occur in the phase space when the average rotation number of one of the invariant curves equals 1/3.

In the SOS graphs, the stable period-3 orbits form small islands. These are invariant curves which also have the rotation number equal to 1/3 around the stable 1-cycle. This is manifested by the flattenings of the rotation curve at 1/3 near the stable period-3 orbits.

As L_z^2 decreases, the maxima of the rotation curve increase, while the flattenings around the stable period-3 orbits on the left get smaller, due to shrinking of the islands in the phase space. The rotation curve will cross the 1/3 line at both unstable period-3 orbits, but no flattening will be present as no islands around the unstable point exist in the phase space. In the outer parts of the rotation curve smaller flattenings are observed. These flattenings correspond to higher resonances responsible for generation of orbits of higher periodicity.

2.2. Separatrix Reconnection and Exchange of Stability of Outer Period-3 Orbit

Another important feature of these irregular bifurcations is that of separatrix reconnection. The *separatrix* is the line separating the regions of different dynamics. We observe in (Figures 5(a) and 5(b)), that the separatrices delimiting the inner and the outer sets of period-3 orbits reconnect (at some intermediate value of L_z^2) so that their position reverses; that is, the inner set moves out, and the outer moves in. In this process of reconnection, only the stable period-3 orbits reverse their positions in reference to one another, while the unstable ones remain at their previous locations. As L_z^2 further decreases, the invariant curves around the outer stable period-3 orbit become more elongated until, finally, the stability of the outer pair of stable/unstable period-3 orbits reverses - the unstable period-3 orbit becomes stable and the stable one destabilizes (Figure 5(c)).

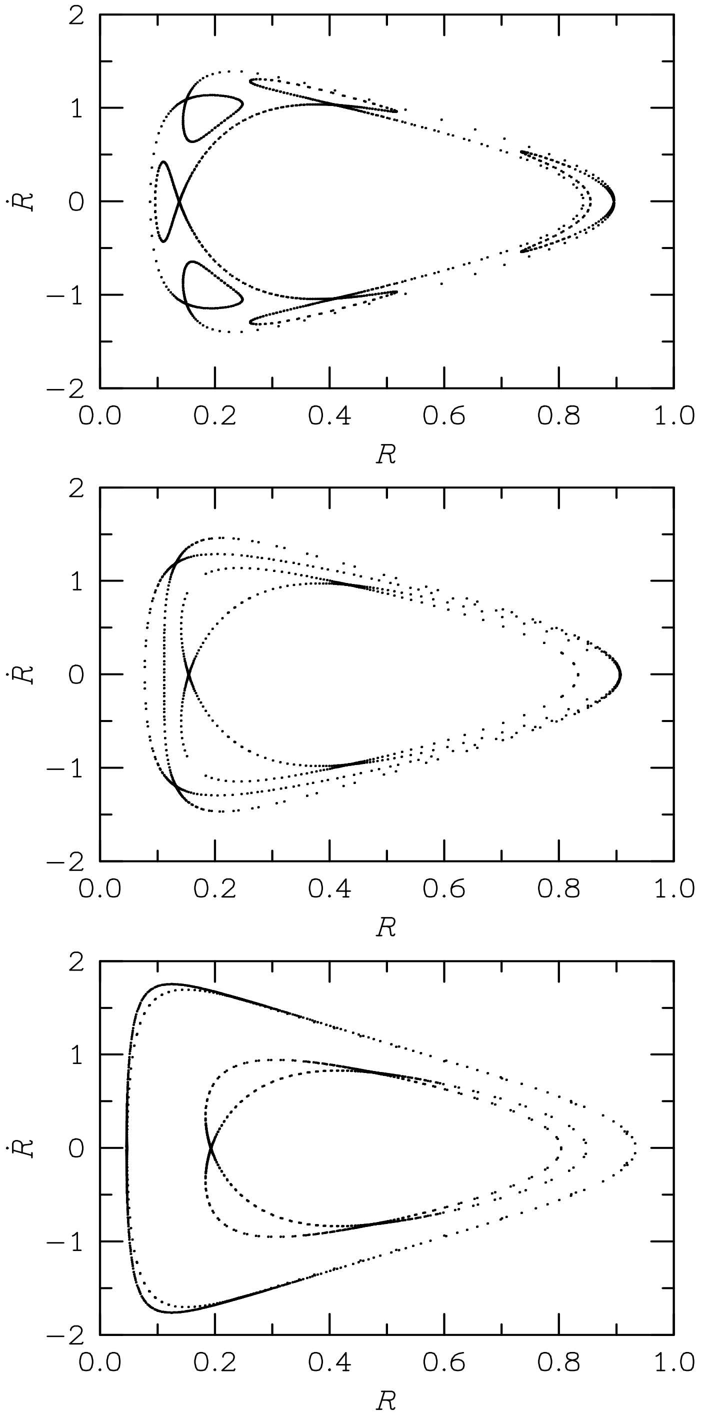

FIGURE 5. Separatrix reconnection occurs as L_z^2 decreases (between (**a**) and (**b**)). The stability of the outer pair of stable/unstable period-3 orbits reverses as L_z^2 decreases (between (**b**) and (**c**)). SED potential $q = 0.8$, $\beta = 0.3$, and (**a**) $L_z^2 = 0.022$, (**b**) $L_z^2 = 0.02$, (**c**) $L_z^2 = 0.01$.

2.3. Non-twist Maps

The phenomena that we encountered, periodic orbit formation and separatrix reconnection, are important features of the so-called *non-twist* maps that violate the twist condition. For detailed treatment of these non-twist maps and their dynamics, see del-Castillo-Negrete et al.[5] This clearly implies that the dynamics of this system cannot be described by the standard map. Our future efforts will be directed toward developing a non-twist map which will be able to describe qualitatively the dynamics of these kind of systems; this will give us a significant improvement in speed over traditional integration methods.

3. SUMMARY

In summary, it is important to realize that at all times the difference between the number of stable and unstable points is constant (in our case 1), that the irregular period tripling bifurcations occur when the average rotation number of an invariant curve in the phase space is 1/3 and that the creation as well as the transformation (change of stability) of these irregular period-3 orbit is simultaneous. The orbits in SED potentials seem to obey an effective third integral of motion in addition to energy and the z-component of the angular momentum, as we did not find any chaotic regions in SOS graphs. Thus, we conclude that this is at least a near-integrable, if not fully integrable, dynamical system. Further work is required for a more definitive resolution of this problem.

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