The enumerative geometry of plane cubics. II: Nodal and Cuspidal Cubics

Paolo Aluffi

0. Introduction

In [**A1**] we study the basic enumerative question about the family of all *smooth* cubics: we compute its 'characteristic numbers', i.e. the number of smooth plane cubics tangent to n_ℓ general lines and containing $9-n_\ell$ general points of the plane. In this paper we study the analogous question for several families of *nodal* and *cuspidal* cubics, recovering as in [**A1**] classic results of Maillard and Zeuthen's.

Specifically, we will consider the families

- D of nodal cubics;
- $D\ell$ of cubics with node on a given line;
- Dp of cubics with node at a given point;
- C of cuspidal cubics;
- $C\ell$ of cubics with cusp on a given line;
- Cp of cubics with cusp at a given point.

(we will refer to a family by the subset of the $\mathbb{P}^9$ of plane cubics parametrizing it), and compute the list of characteristic numbers for each of them: i.e., for each family F we will compute the numbers $F(k)$ of elements of F that are tangent (at smooth points) to k lines and contain $(\dim F - k)$ points in general position in the plane. Also, we will compute for these families the numbers defined by considering conditions of tangency to lines *at specified points*. These results are listed in Theorem III, §3, and Theorem III′, §4.

The computation of the characteristic numbers for various families of plane cubics has been attacked successfully from a number of viewpoints, both in the XIX century ([**M**], [**Sc**], [**Z**]) and very recently ([**Sa**], [**KS**], [**MX**]): the problem stands out as a test-ground for techniques in enumerative geometry; and has a certain charm in itself, as do most problems so deceptively easy to state.

In both the classic and the modern approaches quoted above (for example, Kleiman-Speiser's excellent papers on the subject) the computation is carried out depending on successive degenerations, by relating the characteristic numbers for a family to the numbers for a more 'special' family. For example, the numbers for *cuspidal* cubics are used in obtaining the numbers for *nodal* ones, and these in turn are an ingredient of the computation for the family of *smooth* cubics. In fact, the numbers for cuspidal cubics are obtained by first studying families of reducible cubics, for which the enumerative problem is essentially combinatorial (modulo the enumerative geometry of conics).

In [**A1**] we have tried a different approach. In a sense, we have aimed to solving the enumerative question about any given family of reduced plane cubics independently from other families, at least for what concerns the contribution of degenerate

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elements. We produce a smooth variety of 'complete cubics', i.e. we resolve *all* indeterminacies of the map associating to each cubic its dual sextic at once: this is accomplished by a sequence of 5 blow-ups at smooth centers over $\mathbb{P}^9$ (the same sequence was considered independently by Sterz, [**St**]). Unfortunately, the construction doesn't provide an effective visualization of what a 'complete cubic' looks like, so the picture isn't nearly as informative and insightful as e.g. the one associated with the space of 'complete conics'. However, in this paper we would like to support the usefulness of that construction by employing it to recover Zeuthen's enumerative results on *singular* cubics (we address the reader to [**KS**] in particular for an alternative modern verification of most of these results, from a viewpoint close to Zeuthen's). .

Solving an enumerative problem about cubics amounts to computing the number of 'non-degenerate' points of intersection of suitable loci in $\mathbb{P}^9$. Modulo Bézout's theorem, this is equivalent to evaluating the contribution due to the set of *degenerate* points: in our case, this is the set S of *non-reduced* cubics (which are 'tangent' to all lines of the plane!). This brings naturally to trying to compute a certain Segre class of a scheme supported on S—for applications of this approach to enumerative problems on conics, see [**F**], Examples 9.1.8, 9.1.9. Now, computing Segre classes is in general very hard. In [**A1**] we essentially break the problem in five easier ones: let $B_0, \dots, B_4$ be the centers of the blow-ups, and let V_i be the i-th blow-up; if $F \subset \mathbb{P}^9$ parametrizes a family of reduced cubics, and F_i denotes the proper transform in V_i of the closure F_0 of F in $\mathbb{P}^9$, then the problem is reduced to the computation of the five classes $s(B_i \cap F_i, F_i)$, $i = 0, \dots, 4$. This is easier, because the B_i's are regularly embedded in the V_i's, and products $B_i \circ F_i = c(N_{B_i} V_i) s(B_i \cap F_i, F_i)$ (the 'full intersection classes' of [**A1**]) are relatively easy to handle. For example, in the case of the family of all *smooth* cubics, $B_i \circ F_i = [B_i]$, so the computation of the characteristic numbers for the family of all smooth cubics becomes particularly simple. For more general F, the enumerative problem is reduced explicitly in [**A1**] to the computation of the five classes $B_i \circ F_i$, $i = 0, \dots, 4$ (Theorems IV in [**A1**], which we recall as Theorem I in §1); as an example illustrating the more general case, we computed in [**A1**] the characteristic numbers for families of smooth cubics tangent to a line *at a given point*.

In this note we take the next step in the program: we compute the classes $B_i \circ F_i$ for some families of *singular* cubics. As an immediate application, we will recover classic enumerative results about these families, providing again a counterpoint to the degeneration method; however, perhaps the main motivation of this paper is to produce examples of computations of Segre classes in an interesting and natural geometric setting. We feel that more tools are needed for the computation of these important invariants of a closed embedding, and we hope that providing these examples might be of some help in this development.

In order to compute the classes corresponding to the families $D, D\ell$, etc. listed above, we realize the discriminant hypersurface D_0 of $\mathbb{P}^9$ (the closure of D) as the birational projection from $\mathbb{P}^2 \times \mathbb{P}^9$ of the codimension-3 subvariety $\widehat{D}_0$ of pairs (p, f) where $p \in \mathbb{P}^2$ and f is a cubic singular at p. If $\widehat{V}_i = \mathbb{P}^2 \times V_i$, $\widehat{B}_i = \mathbb{P}^2 \times B_i$, and $\widehat{D}_i$ denotes the proper transform of $\widehat{D}_0$ in $\widehat{V}_i$, then the birational invariance of Segre classes allows one to relate the classes $B_i \circ D_i$, $B_i \circ D\ell_i$ etc. to the classes $\widehat{B}_i \circ \widehat{D}_i$

(Propositions 2.1, 2.11 and 2.12). These latter are not too hard to compute, as the structure of $\widehat{D}_0$ is rather transparent; the results are listed in Theorem II, §2. The more technical tools used in the computation are presented in an appendix.

Once the classes for the loci D, $D\ell$, etc. are obtained, applying Theorem I furnishes us with the characteristic numbers for the families, 'counted with multiplicities'. A last step needs to be performed here, because of the singularity of the curves: for each configuration, a contribution to the 'weighted' characteristic numbers of one family might be due to another family. For example, among the nodal cubics tangent to 8 lines we find cubics tangent to 7 of the lines and with a node on the 8th, and cubics tangent to 6 of the lines and having the node at the intersection of the remaining 2. If we want to count only curves 'properly tangent' to the lines, then we'll have to evaluate the contribution due to the different possibilities. We dealt with this issue already in [**A3**] (for nodal and cuspidal curves of arbitrary degree), so here we will simply apply the tool obtained there (which we recall as Proposition 1.1).

Similarly, Theorem IV$'$ in [**A1**] will yield the characteristic numbers involving the additional condition of tangency to a given line *at a given point:* as seen in [**A1**], §5, no additional information is required for these results.

The last two of the blow-ups constructing the variety of complete cubics have been studied for arbitrary degree in [**A2**], and applied to derive some enumerative results for nodal and cuspidal curves in [**A3**]. In this paper we basically complete for degree 3 the partial computations worked out in [**A3**] for all degrees, and our methods here are similar to the ones employed there. Doing the same for e.g. degree 4 curves requires accomplishing first the construction of a variety of 'complete quartics', and is therefore beyond our reach at present.

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Computations in this paper were performed with Macsyma and Maple.

1. Preliminaries

We work over an algebraically closed field of characteristic 0. Consider the space $\mathbb{P}^9 = \mathbb{P}(H^0\mathcal{O}_{\mathbb{P}^2}(3))$ parametrizing cubic curves in the projective plane $\mathbb{P}^2$. In [**A1**] we give a sequence of five blow-ups

$$\widetilde{V} = V_5 \xrightarrow{\pi_5} V_4 \xrightarrow{\pi_4} V_3 \xrightarrow{\pi_3} V_2 \xrightarrow{\pi_2} V_1 \xrightarrow{\pi_1} V_0 = \mathbb{P}^9$$

at smooth centers producing a smooth projective variety $\widetilde{V}$ of 'complete cubics': i.e. a variety (birational to $\mathbb{P}^9$) on which the map associating to each smooth cubic its dual sextic extends to a regular map. In other terms, call 'line-condition' the hypersurface of $\mathbb{P}^9$ formed by all cubics tangent to a given line, and its proper transforms in the V_i's; then the intersection of all line-conditions in $\widetilde{V}$ is *empty* (Proposition 5.3 in [**A1**], §3.5). We will recall briefly a description of the centers of the blow-ups in §2, in the course of the main computation; the sequence of blow-ups accomplishes 'separating' the line-conditions over their intersection in $\mathbb{P}^9$, i.e. the set S of non-reduced cubics (the four-dimensional set of cubics $\lambda\mu^2$ consisting of a line λ and a double line μ^2). Now call $\widetilde{L}$ the class of the general line-condition in $\widetilde{V}$, and $\widetilde{P}$ the class of the general 'point-condition' (the proper transform of the hyperplane in $\mathbb{P}^9$ formed by cubics containing a given point); if F is (the parameter space of) a family of reduced cubics, call F_0 its closure in $\mathbb{P}^9$, F_i the proper transform of F_0 in V_i, and set $\widetilde{F} = F_5$. We observed in [**A1**], Theorem I, that the number of elements of F (thus, automatically non-degenerate) tangent to n_ℓ lines and containing n_p points is counted with multiplicity by the intersection product

$$(*) \qquad \widetilde{L}^{n_\ell} \cdot \widetilde{P}^{n_p} \cdot \widetilde{F} \quad ,$$

and furthermore elements 'properly' tangent to the lines (i.e., simply tangent at smooth points) count with multiplicity 1.

Our main task will be to compute the intersections (*) for the families $D, D\ell$, etc. listed in the introduction. After accomplishing this, taking account of elements contributing to (*) but not properly tangent will not be hard: denote the number of curves in F *properly* tangent to k lines and containing $\dim F - k$ points (in general position)—i.e., the k-th characteristic number of F—by $F(k)$; while denote (as in [**A1**]) by $N_F(n_pP, n_\ell L)$ the intersection product (*) above. Then:

PROPOSITION 1.1.

$$\begin{aligned}
D(k) &= N_D((8-k)P, kL) - 2kD\ell(k-1) - 4\binom{k}{2}Dp(k-2)\\
D\ell(k) &= N_{D\ell}((7-k)P, kL) - 2kDp(k-1)\\
Dp(k) &= N_{Dp}((6-k)P, kL)\\
C(k) &= N_C((7-k)P, kL) - 3kC\ell(k-1) - 9\binom{k}{2}Cp(k-2)\\
C\ell(k) &= N_{C\ell}((6-k)P, kL) - 3kCp(k-1)\\
Cp(k) &= N_{Cp}((5-k)P, kL)
\end{aligned}$$

PROOF: This is Theorem I in [**A3**], for degree 3, and with the above notations. ∎

Proposition 1.1 tells us that all we need to compute are the 'weighted' characteristic numbers $N_F(n_pP, n_\ell L)$, for $F = D, D\ell$, etc. This will be done by using Theorem IV from [**A1**]:

THEOREM I. *(Notations as above)*

$$N_F(n_pP, n_\ell L) = 4^{n_\ell} \cdot \deg(F_0) - \sum_{i=0}^{4} \int_{B_i} \frac{(B_i \circ P_i)^{n_p}(B_i \circ L_i)^{n_\ell}(B_i \circ F_i)}{c(N_{B_i}V_i)}$$

where $B_i \circ P_i$, $B_i \circ L_i$, $c(N_{B_i}V_i)$ are given explicitly in [**A1**], Theorem III, together with a description of the relevant intersection calculus of the B_i's. We see then that the only missing ingredients are the degrees of the closure F_0 and the classes $B_i \circ F_i$, for each family $F = D, D\ell$, etc. Of course there is nothing to the first item:

PROPOSITION 1.2.

$$\begin{array}{ll} \deg D_0 = 12 & \deg C_0 = 24 \\ \deg D\ell_0 = 6 & \deg C\ell_0 = 12 \\ \deg Dp_0 = 1 & \deg Cp_0 = 2 \end{array}$$

PROOF: These are well known (cf. Proposition 1.2 and 1.5 in [**A3**]). ∎

By contrast, the computation of the 'full intersection classes'

$$B_i \circ F_i = c(N_{B_i} V_i) s(B_i \cap F_i, F_i)$$

(where $c(\cdot)$ and $s(\cdot)$ denote resp. total Chern and Segre class) is non-trivial: this will be our task in §2.

NOTE. The classes $B_i \circ F_i$ live naturally in the Chow groups of $B_i \cap F_i$; we will actually compute their *push-forward* in the Chow group of B_i; we will still denote the push-forward by $B_i \circ F_i$, for convenience of notation.

To prepare for the computation, we want to highlight here a basic fact that we will systematically apply in §2. For $B, F \subset V$ (with $B \overset{j}{\hookrightarrow} V$ a regular embedding of codimension d), denoting by $e_B F$ the multiplicity of F along B, and by $\{\cdot\}_m$ the m-th dimensional piece of the class within braces:

LEMMA 1.3.

(1) $\{B \circ F\}_{\dim B} = e_B F[B]$
(2) $\{B \circ F\}_{\dim F - d} = j^*[F] = B \cdot F$
(3) $\{B \circ F\}_i = 0$ *for* $i < \dim F - d$, $i > \dim F \cap B$

PROOF: (1) holds because $s(B \cap F, F) = e_B F[B] +$ lower dimensional terms, by [**F**], §4.3). (2), (3) are in [**A1**], Lemma in §2. ∎

So e.g. if F is a *divisor*, then simply

$$B \circ F = e_B F[B] + B \cdot F \quad .$$

In general, $B \circ F$ has non-zero terms in at most codim$F+1$ dimensions. In a sense, this is the reason why through this process it is easier to obtain results for the family of all smooth cubics rather than for more special families: as a general rule, the more special the family is, the higher the codimension, and the higher the number of terms to be computed.

Other (more technical) facts needed in the computations of §2 are listed in the appendix.

2. FULL INTERSECTION CLASSES

Our aim in this section is the computation of the classes

$$B_i \circ F_i \quad , \quad i = 0, \dots, 4$$

where $B_0, \dots, B_4$ are the centers of the blow-ups given in [**A1**], $F = D, D\ell, Dp, C, C\ell, Cp$ are the families listed in the introduction, and F_i denotes the proper transform in V_i of the closure F_0 of F in $\mathbb{P}^9$.

Each is to be expressed in terms of the generators given in [**A1**], Theorem III for the intersection rings of the B_i's, i.e. various subsets of the list $h, \epsilon, \varphi, \ell, m, e$. The result will be:

Theorem II. *With the above notations, the classes $B_i \circ F_i$ for the six families $F = D, D\ell, Dp, C, C\ell, Cp$, and $i = 0, \ldots, 4$, are resp.:*

$$\begin{cases} 8+36h \\ 5+(36h-8\epsilon) \\ 3+(36h-8\epsilon-5\varphi) \\ 6+(12\ell+24m-24e) \\ 6+(-6\ell+6m) \end{cases}$$

$$\begin{cases} 2+22h+54h^2 \\ 1+(13h-2\epsilon)+(54h^2-22\epsilon h+2\epsilon^2) \\ (9h-2\epsilon-\varphi)+(54h^2-22\epsilon h-13\varphi h+2\epsilon^2+2\epsilon\varphi+\varphi^2) \\ 1+(4\ell+11m-8e)+(6\ell^2+24\ell m+24m^2-66e\ell+18e^2) \\ 1+(-2\ell+5m)+(3\ell^2-3\ell m-6e\ell+2e^2) \end{cases}$$

$$\begin{cases} 2h+14h^2 \\ h+(8h^2-2\epsilon h)+(-14\epsilon h^2+2\epsilon^2 h) \\ (6h^2-2\epsilon h-\varphi h)+(-14\epsilon h^2-8\varphi h^2+2\epsilon^2 h+2\epsilon\varphi h+\varphi^2 h) \\ m+(4\ell m+5m^2-8e\ell)+(6\ell^2 m+12\ell m^2-42e\ell^2+18e^2\ell) \\ m+(-2\ell m-m^2)+(3\ell^2 m+3\ell m^2-6e\ell^2+2e^2\ell) \end{cases}$$

$$\begin{cases} 8+84h+216h^2 \\ 10+(102h-21\epsilon)+(216h^2-84\epsilon h+8\epsilon^2) \\ (18h-3\epsilon-6\varphi)+(216h^2-84\epsilon h-102\varphi h+8\epsilon^2+21\epsilon\varphi+10\varphi^2) \\ 6+(24\ell+48m-48e)+(24\ell^2+96\ell m+96m^2-288e\ell+96e^2) \\ 6+(-12\ell+12m)+(6\ell^2-12\ell m+6m^2) \end{cases}$$

$$\begin{cases} 2+34h+186h^2 \\ 2+(32h-5\epsilon)+(186h^2-69\epsilon h+6\epsilon^2)+(-186\epsilon h^2+34\epsilon^2 h-2\epsilon^3) \\ (54h^2-21\epsilon h-24\varphi h+2\epsilon^2+5\epsilon\varphi+2\varphi^2)+(-186\epsilon h^2-186\varphi h^2+34\epsilon^2 h \\ \quad +69\epsilon\varphi h+32\varphi^2 h-2\epsilon^3-6\epsilon^2\varphi-5\epsilon\varphi^2-2\varphi^3) \\ 1+(6\ell+15m-12e)+(14\ell^2+62\ell m+68m^2-174e\ell+50e^2)+(72\ell^2 m \\ \quad +144\ell m^2-612e\ell^2+372e^2\ell-72e^3) \\ 1+(-3\ell+6m)+(5\ell^2-10\ell m+5m^2-6e\ell+2e^2)+(6\ell^2 m-3\ell m^2) \end{cases}$$

$$\begin{cases} 2h+26h^2 \\ 2h+(22h^2-5\epsilon h)+(-48\epsilon h^2+6\epsilon^2 h)+(26\epsilon^2 h^2-2\epsilon^3 h) \\ (-18\epsilon h^2-18\varphi h^2+2\epsilon^2 h+5\epsilon\varphi h+2\varphi^2 h)+(26\epsilon^2 h^2+48\epsilon\varphi h+22\varphi^2 h^2-2\epsilon^3 h \\ \quad -6\epsilon^2\varphi h-5\epsilon\varphi^2 h-2\varphi^3 h) \\ m+(6\ell m+9m^2-12e\ell)+(14\ell^2 m+38\ell m^2-126e\ell^2+50e^2\ell)+(48\ell^2 m^2 \\ \quad +276e^2\ell^2-72e^3\ell) \\ m-3\ell m+(5\ell^2 m+2\ell m^2-6e\ell^2+2e^2\ell) \end{cases}$$

These expressions carry (admittedly, rather cryptically) concrete geometric information about the objects we are considering. Of course the enumerative results of §§3,4 will best illustrate this point; however, one instance in which this is very explicit is the first brace, corresponding to the family of nodal cubics D: the information carried by the expressions consists of the degree of the discriminant (the hyperplane in $\mathbb{P}^9$ pulls-back to $3h$ on B_0, so the class of the discriminant pulls-back to $36h$), and of the multiplicity of the discriminant and its proper transforms along the centers of the blow-ups (the constant terms in the expressions: 8, 5, 3, 6, 6). This is all the information needed to compute the 'weighted' characteristic numbers $N_D(n_pP, n_\ell L)$ (in fact, even less is needed: cf. [**A4**], Theorem I).

Proving Theorem II will take us the rest of this section; our approach is along the same lines as the computation in §2 of [**A3**]. Give coordinates $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$, and consider the codimension-3 subvariety $\widehat{D}_0$ of the product $\mathbb{P}^2 \times \mathbb{P}^9$ defined by

$$(p, f) \in \widehat{D}_0 \iff \begin{cases} \dfrac{\partial f}{\partial x_0}(p) = 0 \\ \dfrac{\partial f}{\partial x_1}(p) = 0 \\ \dfrac{\partial f}{\partial x_2}(p) = 0 \end{cases} .$$

So $(p, f) \in \widehat{D}_0$ if and only if f is a cubic singular at p. The projection $p_1 : \mathbb{P}^2 \times \mathbb{P}^9 \longrightarrow \mathbb{P}^2$ restricts to a map $\widehat{D}_0 \longrightarrow \mathbb{P}^2$ realizing $\widehat{D}_0$ as a $\mathbb{P}^6$-bundle over $\mathbb{P}^2$: the fiber over p being the $\mathbb{P}^6$ of cubics singular at p. The projection $p_2 : \mathbb{P}^2 \times \mathbb{P}^9 \longrightarrow \mathbb{P}^9$ restricts to a birational morphism from $\widehat{D}_0$ to the discriminant hypersurface $D_0 \subset \mathbb{P}^9$: the fiber over $f \in D_0$ consists of the singular locus of f. Observe that p_2 restricts to an isomorphism over the set D of *nodal* cubics. Now for each $V_0 = \mathbb{P}^9, V_1, \dots,$ define $\widehat{V}_i = \mathbb{P}^2 \times V_i$, and for each center B_i define $\widehat{B}_i = \mathbb{P}^2 \times B_i$. It is clear then that each $\widehat{V}_i$ $(i > 0)$ is the blow-up of $\widehat{V}_{i-1}$ along $\widehat{B}_{i-1}$, and we can consider the proper transform $\widehat{D}_i$ of $\widehat{D}_0$ in $\widehat{V}_i$. The projection on the second factor will then restrict to birational morphisms

$$\widehat{D}_i \longrightarrow D_i \quad ,$$

that will be our main tool: we will argue now that the classes $\widehat{B}_i \circ \widehat{D}_i$ contain all the information we need concerning families of nodal curves (cf. Lemma 2.2 etc. in [**A3**]).

Let k denote the hyperplane class in $\mathbb{P}^2$. So classes in $\widehat{B}_i$ will be polynomials of degree ≤ 2 in k, with coefficients in the intersection rings of the B_i.

PROPOSITION 2.1. *For* $i = 0, \dots, 4$

$B_i \circ D_i$= *coefficient of* k^2 *in* $\widehat{B}_i \circ \widehat{D}_i$
$B_i \circ D\ell_i$= *coefficient of* k^1 *in* $\widehat{B}_i \circ \widehat{D}_i$
$B_i \circ Dp_i$= *coefficient of* k^0 *in* $\widehat{B}_i \circ \widehat{D}_i$

PROOF: These follow easily from the birational invariance of Segre classes: write $\widehat{B}_i \circ \widehat{D}_i = A_2 + A_1 k + A_0 k^2$, with A_0, A_1, A_2 classes on B_i; if $p^{(i)}$ is the projection $\mathbb{P}^2 \times V_i \longrightarrow V_i$, then by the projection formula

$$A_0 = p^{(i)}_*(A_2 + A_1 k + A_0 k^2) \quad ,$$

since $p_*^{(i)}(k^0) = p_*^{(i)}(k^1) = 0$, $p_*^{(i)}(k^2) = 1$; so

$$\begin{aligned} A_0 &= p_*^{(i)}(\widehat{B}_i \circ \widehat{D}_i) \\ &= p_*^{(i)} c(N_{\widehat{B}_i} \widehat{V}_i) s(\widehat{B}_i \cap \widehat{D}_i, \widehat{D}_i) \\ &= c(N_{B_i} V_i) p_*^{(i)} s(\widehat{B}_i \cap \widehat{D}_i, \widehat{D}_i) \quad \text{since } N_{\widehat{B}_i} \widehat{V}_i = {p^{(i)}}^* N_{B_i} V_i \\ &= c(N_{B_i} V_i) s(B_i \cap D_i, D_i) \qquad \text{by the bir. inv. of Segre classes} \\ &= B_i \circ D_i \end{aligned}$$

which is the first claim.

For the other equalities in the statement, define $\widehat{D\ell}_i$ = proper transform of $\widehat{D\ell}_0 = \mathbb{P}^2 \times D\ell_0$, and similarly $\widehat{Dp}_i$ = proper transform of $\widehat{Dp}_0 = \mathbb{P}^2 \times Dp_0$. The classes of $\widehat{D\ell}_i, \widehat{Dp}$ in $\widehat{D}_i$ are clearly resp. (the pull-backs of) k, k^2; also, $\widehat{D\ell}_i, \widehat{Dp}_i$ cut transversally in $\widehat{D}_i$ the support of the cone of $\widehat{B}_i \cap \widehat{D}_i$ in $\widehat{D}_i$, so

$$\begin{aligned} s(\widehat{B}_i \cap \widehat{D\ell}_i, \widehat{D\ell}_i) &= k \cdot s(\widehat{B}_i \cap \widehat{D}_i, \widehat{D}_i) \qquad \text{and} \\ s(\widehat{B}_i \cap \widehat{Dp}_i, \widehat{Dp}_i) &= k^2 \cdot s(\widehat{B}_i \cap \widehat{D}_i, \widehat{D}_i) \qquad , \end{aligned}$$

by Lemma A.3. Then one argues as above, starting from $A_1 = p_*^{(i)}[k \cdot (A_0 + A_1 k + A_2 k^2)]$ and $A_2 = p_*^{(i)}[k^2 \cdot (A_2 + A_1 k + A_0 k^2)]$ ∎

By Proposition 2.1, the five classes $\widehat{B}_i \circ \widehat{D}_i$ are the objects we have to compute to prove the first part of Theorem II. We will analyze the five cases in some detail in §§2.0–4 below. The main proposition in each section will give the corresponding class $\widehat{B}_i \circ \widehat{D}_i$, from which (by Proposition 2.1) one reads the i^{th} row in the first three braces in the statement of Theorem II, by taking resp. the coefficient of k^2, k, and the constant term with respect to k. As we will see in §2.5, very little additional work is required to obtain the classes for families of cuspidal cubics (i.e. the last three braces in Theorem II).

Note. As a general convention, we omit the notation of pull-back whenever we feel that this choice doesn't create ambiguities.

§2.0. $\widehat{B}_0 \circ \widehat{D}_0$. Recall from [**A1**], §3.0 that the center of the first blow-up is the subvariety $B_0 \subset \mathbb{P}^9$ of cubics consisting of a 'triple line'; $B_0 \cong \mathbb{P}^2$ is in fact embedded in $\mathbb{P}^9$ by the third Veronese embedding. Points of $\widehat{B}_0 = \mathbb{P}^2 \times B_0$ will then be pairs (p, λ), where $p \in \mathbb{P}^2$ and λ is a line. We call h the hyperplane class in $B_0 \cong \mathbb{P}^2$, so the intersection ring of $\widehat{B}_0 \cong \mathbb{P}^2 \times B_0$ is generated by k, h, and the only non-zero monomial in dimension 0 is $h^2 k^2$. Also, the pull-back of the hyperplane class H of $\mathbb{P}^9$ via $B_0 \hookrightarrow \mathbb{P}^9$ is $3h$.

Lemma 2.2. $c(N_{\widehat{B}_0} \widehat{V}_0) = (1 + H + 2k)^3$.

Proof: This is clear from the equations for $\widehat{D}_0$ (linear in the coefficients of the cubic, and quadratic in $(x_0 : x_1 : x_2)$). ∎

The intersection $\widehat{B}_0 \cap \widehat{D}_0$ is supported on the incidence correspondence $\{(p, \lambda) \in \widehat{B}_0 \text{ s.t. } p \in \lambda\}$; in fact, restricting the equations for $\widehat{D}_0$ to $\widehat{B}_0$ we find that $\widehat{B}_0 \cap \widehat{D}_0$ is regularly embedded in $\widehat{B}_0$, as a divisor of class $2h + 2k$.

Proposition 2.3. $\widehat{B}_0 \circ \widehat{D}_0 = (2h+2k) + (14h^2 + 22hk + 8k^2) + (54h^2k + 36hk^2)$.

Proof: Both $\widehat{B}_0$ and $\widehat{D}_0$ are non-singular, so $\widehat{B}_0 \circ \widehat{D}_0 = \widehat{D}_0 \circ \widehat{B}_0$ by Lemma A.1. Now since $\widehat{B}_0 \cap \widehat{D}_0$ is a divisor in $\widehat{B}_0$, with class $2h+2k$, then (as a class in $\widehat{B}_0$)

$$s(\widehat{B}_0 \cap \widehat{D}_0, \widehat{B}_0) = (2h+2k) - (2h+2k)^2 + (2h+2k)^3 - (2h+2k)^4 \quad ;$$

while (Lemma 2.2) $c(N_{\widehat{D}_0}\widehat{V}_0)$ pulls-back on $\widehat{B}_0$ to $(1+3h+2k)^3$. So

$$\widehat{B}_0 \circ \widehat{D}_0 = (1+3h+2k)^3 \left\{(2h+2k) - (2h+2k)^2 + (2h+2k)^3 - (2h+2k)^4\right\} \quad ,$$

which gives the statement. ∎

§**2.1.** $\boldsymbol{\widehat{B}_1 \circ \widehat{D}_1}$**.** The center B_1 of the second blow-up is a $\mathbb{P}^2$-bundle over B_0 ([**A1**], §3.1); we interpret the fiber over a (triple) line $\lambda \in B_0$ as the plane of pairs of points on λ: so we will denote a point of $\widehat{B}_1 = \mathbb{P}^2 \times B_1$ by a triple $(p, \lambda, \{p_1, p_2\})$, where $p_1, p_2 \in \lambda$. The intersection ring of B_1 is generated by (the pull-back of) the class h from B_0 and by the class ϵ of the universal line bundle on B_1. In fact B_1 is a subbundle of the exceptional divisor $E_1 = \mathbb{P}(N_{B_0}V_0)$, so ϵ is the pull-back via $B_1 \hookrightarrow E_1 \hookrightarrow V_1$ of the class of E_1.

We can easily get equations for $\widehat{D}_1$ in an open set in $\widehat{V}$, by using the coordinates for V_1 given in [**A1**], §3.1: give homogeneous coordinates $(a_0 : a_1 : \cdots : a_9)$ in $\mathbb{P}^9$, so that the point $(a_0 : \cdots : a_9)$ corresponds to the cubic with equation

$$\begin{aligned} a_0x_0^3 + a_1x_0^2x_1 + a_2x_0^2x_2 + a_3x_0x_1^2 + a_4x_0x_1x_2 \\ + a_5x_0x_2^2 + a_6x_1^3 + a_7x_1^2x_2 + a_8x_1x_2^2 + a_9x_2^3 = 0 \quad . \end{aligned}$$

Then we can give coordinates $(b_1, \ldots, b_9)$ in an open in V_1, such that the blow-up map is given by

$$\begin{array}{lll} b_1 = a_1 & b_2 = a_2 & b_3 = 3a_3 - a_1^2 \\ b_4b_3 = 3a_4 - 2a_1a_2 & b_5b_3 = 3a_5 - a_2^2 & b_6b_3 = 9a_6 - a_1a_3 \\ b_7b_3 = 3a_7 - a_2a_3 & b_8b_3 = 3a_8 - a_1a_5 & b_9b_3 = 9a_9 - a_2a_5 \end{array}$$

In this description $b_3 = 0$ is the exceptional divisor, and the point of B_1 corresponding to a line $\lambda : x_0 + \lambda_1x_1 + \lambda_2x_2 = 0$ with pair of points (p_1, p_2) determined by $x_1^2 + \alpha x_1x_2 + \beta x_2^2$ has coordinates

$$(3\lambda_1, 3\lambda_2, 0, \alpha, \beta, 2\lambda_1, \lambda_1\alpha, \lambda_2\alpha, 2\lambda_2\beta) \quad .$$

On $\{a_0 \neq 0\}$, $\widehat{D}_0$ is cut out by the equations

$$\begin{cases} \dfrac{\partial f}{\partial x_0}(p) = 0 \\ \left\{\dfrac{\partial f}{\partial x_1} - \dfrac{a_1}{3}\dfrac{\partial f}{\partial x_0}\right\}(p) = 0 \\ \left\{\dfrac{\partial f}{\partial x_2} - \dfrac{a_2}{3}\dfrac{\partial f}{\partial x_0}\right\}(p) = 0 \end{cases} \quad ,$$

from which we get equations for $\widehat{D}_1$:

$$\begin{cases} (3x_0+b_1x_1+b_2x_2)^2+b_3(x_1^2+b_4x_1x_2+b_5x_2^2)=0 \\ 2x_0x_1+b_4x_0x_2+b_6x_1^2+\dfrac{2b_2+6b_7-b_1b_4}{3}x_1x_2+b_8x_2^2=0 \\ b_4x_0x_1+2b_5x_0x_2+b_7x_1^2+\dfrac{2b_1b_5+6b_8-b_2b_4}{3}x_1x_2+b_9x_2^2=0 \end{cases}$$

Restricting these equations to $\widehat{B}_1$, we find equations for $\widehat{B}_1\cap\widehat{D}_1$ in $\widehat{B}_1$: in terms of the above coordinates for $\widehat{B}_1$

$$\begin{cases} (x_0+\lambda_1x_1+\lambda_2x_2)^2=0 \\ (x_0+\lambda_1x_1+\lambda_2x_2)(2x_1+\alpha x_2)=0 \\ (x_0+\lambda_1x_1+\lambda_2x_2)(\alpha x_1+2\beta x_2)=0 \end{cases}$$

i.e., $\widehat{B}_1\cap\widehat{D}_1$ is the divisor of $\widehat{B}_1$

$$\{(p,\lambda,\{p_1,p_2\})\in\widehat{B}_1 \text{ s.t. } p\in\lambda\} \quad ,$$

with an embedded component on

$$\{(p,\lambda,\{p_1,p_2\})\in\widehat{B}_1 \text{ s.t. } p=p_1=p_2\} \quad .$$

Also, along $\widehat{B}_1\cap\widehat{D}_1$ one finds that $\widehat{D}_1$ is regularly embedded in $\widehat{V}_1$; and that $\widehat{D}_1$ is singular at points $(p,\lambda,\{p_1,p_2\})$ with $p\in\{p_1,p_2\}$.

Since $\widehat{B}_1\cap\widehat{D}_1$ is a divisor of class $h+k$ outside the embedded component (which has codimension 3 in $\widehat{B}_1$), we have

$$s(\widehat{B}_1\cap\widehat{D}_1,\widehat{B}_1)=(h+k)-(h+k)^2+\text{ higher codimensional terms.}$$

The omitted terms presumably are affected by the embedded component; however, we will not need to compute them. Similarly, we only list the relevant terms in the pull-back of $c(N_{\widehat{D}_1}\widehat{V}_1)$:

Lemma 2.4. *$c(N_{\widehat{B}_1}\widehat{V}_1)$ restricts to $1+9h+6k-2\epsilon+\dots$*

Proof: By Lemma A.5 in the appendix, this is

$$\begin{aligned} c(N_{\widehat{B}_0\cap\widehat{D}_0}\widehat{B}_0)c\left(\frac{N_{\widehat{D}_0}\widehat{V}_0}{N_{\widehat{B}_0\cap\widehat{D}_0}\widehat{B}_0}\otimes\mathcal{O}(1)\right) &= (1+2h+2k)\frac{(1+3h+2k-\epsilon)^3}{(1+2h+2k-\epsilon)} \\ &= 1+9h+6k-2\epsilon+\dots \end{aligned}$$

as claimed. ∎

The information we have collected is enough to obtain the first two terms of $\widehat{B}_1\circ\widehat{D}_1$. By Lemma 1.3, the third term is $\widehat{B}_1\cdot\widehat{D}_1$ and the remaining ones are 0:

Proposition 2.5.

$$\widehat{B}_1 \circ \widehat{D}_1 = (h+k) + (8h^2 + 13hk + 5k^2 - 2\epsilon h - 2\epsilon k) \\ + (54h^2k + 36hk^2 - 14\epsilon h^2 - 22\epsilon hk - 8\epsilon k^2 + 2\epsilon^2 h + 2\epsilon^2 k) \quad .$$

Proof: Since the embedded component of $\widehat{B}_1 \cap \widehat{D}_1$ has codimension 3 in $\widehat{B}_1$, we can discard it *in computing the codimension-1 and 2 terms in* $\widehat{B}_1 \circ \widehat{D}_1$, and assume $\widehat{B}_1 \cap \widehat{D}_1 \hookrightarrow \widehat{V}_1$, $\widehat{D}_1 \hookrightarrow \widehat{V}_1$ are both regular embeddings. Also, using the coordinate description above, one checks that (in codimension ≤ 2) the blow-up of $\widehat{D}_1$ along $\widehat{B}_1 \cap \widehat{D}_1$ is the residual scheme to the exceptional divisor in the blow-up of $\widehat{V}_1$ along $\widehat{B}_1 \cap \widehat{D}_1$, and is regularly embedded there. Thus, by Lemma A.2 in the appendix,

$$\widehat{B}_1 \circ \widehat{D}_1 = \widehat{D}_1 \circ \widehat{B}_1$$

in codimension ≤ 2 in $\widehat{B}_1$: so

$$\begin{aligned} \widehat{B}_1 \circ \widehat{D}_1 &= c(N_{\widehat{D}_1} \widehat{V}_1) s(\widehat{B}_1 \cap \widehat{D}_1, \widehat{B}_1) \qquad \text{(in cod. 2)} \\ &= (1 + 9h + 6k - 2\epsilon + \dots)((h+k) - (h+k)^2 + \dots), \end{aligned}$$

which gives the first two terms shown in the statement.

The codimension-3 term in $\widehat{B}_1 \circ \widehat{D}_1$ is the pull-back $\widehat{B}_1 \cdot \widehat{D}_1$ of the class of $\widehat{D}_1$ to B_1, by Lemma 1.3 (2): i.e., applying Lemma A.4,

$$54h^2k + 36hk^2 - \left\{ \frac{\widehat{B}_0 \circ \widehat{D}_0}{1+\epsilon} \right\}_{\text{codim3}} \quad ,$$

with the result listed in the statement.

By Lemma 1.3 (3) all other terms are 0, so we are done. ∎

§**2.2.** $\boldsymbol{\widehat{B}_2 \circ \widehat{D}_2}$. The center B_2 of the third blow-up is a $\mathbb{P}^3$ bundle over B_1 ([**A1**], §3.2); we interpret the fiber over a point of B_1 over a line λ as the $\mathbb{P}^3$ of triples of points on λ: so a point of $\widehat{B}_2 = \mathbb{P}^2 \times B_2$ will be a quadruple $(p, \lambda, \{p_1, p_2\}, \{q_1, q_2, q_3\})$ where p_1, p_2, q_1, q_2, q_3 are points of λ. The intersection ring of the exceptional divisor E_2 and of B_2 are generated by the classes h, ϵ from B_1, and by the class φ of the universal line-bundle; since B_2 is a subbundle of $E_2 = \mathbb{P}(N_{B_1} V_1)$, φ is the pull-back via $B_2 \hookrightarrow E_2 \hookrightarrow V_2$ of the class of E_2.

Concerning $\widehat{B}_2 \circ \widehat{D}_2$, Lemma 1.3 (2), (3) will give us the terms in codimension 3 and higher in $\widehat{B}_2$, i.e. in dimension 6 or lower. Since $\widehat{D}_2$ has dimension 8, and clearly does not contain $\widehat{B}_2$, the only term we must determine is the one in dimension 7, i.e. (by Lemma 1.3 (1)) the class of the components of $\widehat{B}_2 \cap \widehat{D}_2$ with coefficients depending on the multiplicity of $\widehat{D}_2$ along them.

To this purpose, we use coordinates again. From [**A3**], §3.2, we know we can give coordinates $(c_1, \dots, c_9)$ in an open in V_2 so that the blow-up map is given by

$$\begin{array}{lll} c_1 = b_1 & c_2 = b_2 & c_3 c_6 = b_3 \\ c_4 = b_4 & c_5 = b_5 & c_6 = 3b_6 - 2b_1 \\ c_7 c_6 = 3b_7 - b_1 b_4 & c_8 c_6 = 3b_8 - b_2 b_4 & c_9 c_6 = 9b_9 - b_2 b_5 \end{array}$$

With these coordinates, $c_6 = 0$ is the exceptional divisor; if λ is given by $x_0 + \lambda_1 x_1 + \lambda_2 x_2$, $\{p_1, p_2\}$ is determined by $Q = x_1^2 + \alpha x_1 x_2 + \beta x_2^2$, and $\{q_1, q_2, q_3\}$ by $K = x_1^3 + \rho x_1^2 x_2 + \sigma x_1 x_2^2 + \tau x_2^3$, then the point of B_2 specified by this data has coordinates

$$(3\lambda_1, 3\lambda_2, 0, \alpha, \beta, 0, \frac{\rho}{3}, \frac{\sigma}{3}, \tau).$$

Now, away from the embedded component $\{(p, \lambda, \{p, p\})\}$ of $\widehat{B}_1 \cap \widehat{D}_1$ (e.g. if $2x_1 + b_4 x_2 \neq 0$) one gets equations for $\widehat{D}_2$:

$$\begin{cases} (3x_0 + c_1x_1 + c_2x_2)(2x_1 + c_4x_2) + c_6(x_1^2 + 2c_7x_1x_2 + c_8x_2^2) = 0 \\ (3x_0 + c_1x_1 + c_2x_2)(x_1^2 + 2c_7x_1x_2 + c_8x_2^2) \\ \qquad -c_3(x_1^2 + c_4x_1x_2 + c_5x_2^2)(2x_1 + c_4x_2) = 0 \\ (c_4x_1 + 2c_5x_2)(x_1^2 + 2c_7x_1x_2 + c_8x_2^2) \\ \qquad -(2x_1 + c_4x_2)(c_7x_1^2 + 2c_8x_1x_2 + c_9x_2^2) = 0 \end{cases}$$

So (setting $c_6 = 0$ and observing that $2x_1 + c_4x_2 \neq 0$ since $2x_1 + b_4x_2 \neq 0$) $\widehat{E}_2 \cap \widehat{D}_2$ has equations

$$\begin{cases} c_6 = 0 \\ 3x_0 + c_1x_1 + c_2x_2 = 0 \\ c_3(x_1^2 + c_4x_1x_2 + c_5x_2^2) = 0 \\ (c_4x_1 + 2c_5x_2)(x_1^2 + 2c_7x_1x_2 + c_8x_2^2) \\ \qquad -(2x_1 + c_4x_2)(c_7x_1^2 + 2c_8x_1x_2 + c_9x_2^2) = 0 \end{cases}$$

in this open. We conclude that $\widehat{E}_2 \cap \widehat{D}_2$ consists of (at most) three 7-dimensional components:

–a component R_1 dominating the whole of $\widehat{B}_1 \cap \widehat{D}_1$, with dimension-2 fibers, and equations

$$\begin{cases} c_6 = 0 \\ 3x_0 + c_1x_1 + c_2x_2 = 0 \\ c_3 = 0 \\ (c_4x_1 + 2c_5x_2)(x_1^2 + 2c_7x_1x_2 + c_8x_2^2) \\ \qquad -(2x_1 + c_4x_2)(c_7x_1^2 + 2c_8x_1x_2 + c_9x_2^2) = 0 \end{cases}$$

–a component R_2 dominating the subset of $\widehat{B}_1 \cap \widehat{D}_1$

$$\{(p, \lambda, \{p_1, p_2\}) \quad \text{s.t. } p_1, p_2 \in \lambda,\ p = p_1 \text{ or } p = p_2\}$$

(which is the subset along which $\widehat{D}_1$ is singular) with dimension-3 fibers, and equations

$$\begin{cases} c_6 = 0 \\ 3x_0 + c_1x_1 + c_2x_2 = 0 \\ x_1^2 + c_4x_1x_2 + c_5x_2^2 = 0 \\ x_1^3 + 3c_7x_1^2x_2 + 3c_8x_1x_2^2 + c_9x_2^3 = 0 \end{cases} ;$$

–and a component R_3, dominating the embedded component of $\widehat{B}_1 \cap \widehat{D}_1$

$$\{(p, \lambda, \{p_1, p_2\}) \quad \text{s.t. } p = p_1 = p_2 \in \lambda\}$$

(as the above coordinates do not cover this locus, so there might be a component dominating it) with 4-dimensional fibers.

Now, the equations tell us that the only component of $\widehat{E}_2 \cap \widehat{D}_2$ contained in $\widehat{B}_2$ is R_1, with equations (in $\widehat{B}_2$)

$$\begin{cases} x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0 \\ (\alpha x_1 + 2\beta x_2)(3x_1^2 + 2\rho x_1 x_2 + \sigma x_2^2) - (2x_1 + \alpha x_2)(\rho x_1^2 + 2\sigma x_1 x_2 + 3\tau x_2^2) = 0 \end{cases}$$

and that $\widehat{D}_2$ is generically non-singular along it (in fact, D_2 is non-singular at $(p, \lambda, \{p_1, p_2\}, \{q_1, q_2, q_3\})$ if e.g. $p \notin \{p_1, p_2\}$). So $s(\widehat{B}_2 \cap \widehat{D}_2, \widehat{D}_2) = [R_1] + \ldots$, and using Lemma 1.3 we get:

$$\begin{aligned} \widehat{B}_2 \circ \widehat{D}_2 &= [R_1] + \text{ higher codimension terms} \\ &= [R_1] + \widehat{B}_2 \cdot \widehat{D}_2 \quad . \end{aligned}$$

To find the class of R_1 in $\widehat{B}_2$, observe that its first equation defines the divisor given by the pull-back of $\widehat{B}_1 \cap \widehat{D}_1$, i.e.

$$(h + k) \quad ;$$

the second is

$$\frac{\partial Q}{\partial x_2}\frac{\partial K}{\partial x_1} - \frac{\partial Q}{\partial x_1}\frac{\partial K}{\partial x_2} = 0 \quad ,$$

where $Q(x_1, x_2)$, $K(x_1, x_2)$ determine the pair $\{p_1, p_2\}$ and the triple $\{q_1, q_2, q_3\}$, as above; and $\partial Q/\partial x_i$, $\partial K/\partial x_i$ give global classes $3h - \epsilon + k$, $3h - \epsilon - \varphi + 2k$ resp., so the divisor defined by the above equation in $\widehat{B}_1$ has class

$$6h - 2\epsilon - \varphi + 3k \quad .$$

Now R_1 is the intersection of these two divisors: the above equations (and their mirror image obtained by assuming $b_4 x_1 + 2b_5 x_2 \neq 0$) show it away from the inverse image of the embedded component of $\widehat{B}_1 \cap \widehat{D}_1$, then globally since this has codimension 3 in $\widehat{B}_2$. So the class of R_1 is

$$(h + k)(6h + 3k - 2\epsilon - \varphi) = 6h^2 + 9hk + 3k^2 - 2\epsilon h - 2\epsilon k - \varphi h - \varphi k$$

Proposition 2.6.

$$\begin{aligned} \widehat{B}_2 \circ \widehat{D}_2 = (6h^2 + 9hk + 3k^2 - 2\epsilon h - 2\epsilon k - \varphi h - \varphi k) + (54h^2k + 36hk^2 - 14\epsilon h^2 \\ - 8\varphi h^2 - 22\epsilon hk - 13\varphi hk - 8\epsilon k^2 - 5\varphi k^2 + 2\epsilon^2 h + 2\epsilon\varphi h + \varphi^2 h + 2\epsilon^2 k + 2\epsilon\varphi k + \varphi^2 k) \end{aligned}$$

Proof: We have already observed $\widehat{B}_2 \circ \widehat{D}_2 = [R_1] + \widehat{B}_2 \cdot \widehat{D}_2$, and we have computed $[R_1]$ above. So all we need to get is $\widehat{B}_2 \cdot \widehat{D}_2$, for which one just applies Lemma A.4. ∎

§**2.3.** $\widehat{B}_3 \circ \widehat{D}_3$**.** The center B_3 of the fourth blow-up is a 4-dimensional non-singular variety, in fact isomorphic to the blow-up of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ along its diagonal. B_3 is the proper transform of the set of cubics consisting of a line and a 'double line' (each item parametrized by a factor of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$), cf. [**A1**], §3.3. The intersection ring of B_3 is generated by the pull-back of the classes ℓ, m of the hyperplane of the factors of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$, and by the exceptional divisor e. We choose the factors so that the pull-back of the hyperplane from $\mathbb{P}^9$ is $\ell + 2m$; and recall from [**A1**], §3 that the pull-backs of the first three exceptional divisors E_1, E_2, and E_3 are resp. $2e$, e, and e. Also, we have obvious relations $e\ell = em$, $\ell^3 = m^3 = 0$.

Our picture for $\widehat{B}_3 = \mathbb{P}^2 \times B_3$ is the following: a point in $\widehat{B}_3$ is a triple $(p, (\lambda, \mu), q)$, where $p \in \mathbb{P}^2$, $\lambda, \mu \in \check{\mathbb{P}}^2$ are lines (so that the corresponding cubic is the union of λ and the double line supported on μ; we denote this cubic $\lambda\mu^2$ in [**A1**]), and $q \in \lambda \cap \mu$. So the exceptional divisor is the set of such triples where $\lambda = \mu$, and q plays the role of 'the' point of intersection of λ and μ (cf. [**A1**], Remarks 1.4 in §3.1). Notice that $\widehat{B}_3$ maps injectively 'already' to $\widehat{V}_1$, $\widehat{V}_2$: in fact, Remark 2.4 in [**A1**], §3.2 says that points $(p, (\lambda, \mu), q)$ of the exceptional divisor (so $\lambda = \mu$) map to points $(p, \mu, \{q, q\}, \{q, q, q\})$ of $\widehat{B}_2$. In particular, it follows that $\widehat{D}_3$ is smooth along $\widehat{B}_3$ away from triples $(p, (\lambda, \mu), q)$ with $\lambda = \mu$ and $p = q$ (because $\widehat{D}_3$ is the blow-up of $\widehat{D}_2$ along $\widehat{B}_2 \cap \widehat{D}_2$, so it's smooth over points where both these are smooth): these form a set of codimension 3 in $\widehat{B}_3$, so Lemma A.1 tells us

$$\widehat{B}_3 \circ \widehat{D}_3 = \widehat{D}_3 \circ \widehat{B}_3 \qquad \text{in codimension} \leq 2 \text{ in } \widehat{B}_3.$$

Much as in §2.1, the computation is then reduced to finding the first terms of $s(\widehat{B}_3 \cap \widehat{D}_3, \widehat{B}_3)$ and of the restriction of $c(N_{\widehat{D}_3}\widehat{V}_3)$ to $\widehat{B}_3 \cap \widehat{D}_3$.

Lemma 2.7. *$c(N_{\widehat{D}_3}\widehat{V}_3)$ restricts to* $1 + 3\ell + 6m + 6k - 7e + \dots$.

Proof: Apply Lemma A.5 to the first three blow-ups, and restrict to $\widehat{B}_3$: c_1 of the normal bundle to $\widehat{D}_0$ in $\widehat{V}_0$ restricts to $3\ell + 6m + 6k$ (by Lemma 2.2), and via the blow-ups this gets modified by $-2\widehat{E}_1 - 2\widehat{E}_2 - \widehat{E}_3$, restricting on $\widehat{B}_3$ to $-7e$. ∎

Proposition 2.8.

$$\widehat{B}_3 \circ \widehat{D}_3 = (m+k) + (4\ell m + 5m^2 + 4k\ell + 11km + 6k^2 - 8e\ell - 8ek) + (6\ell^2 m + 12\ell m^2 + 6\ell^2 k + 24\ell mk + 24m^2 k + 12\ell k^2 + 24mk^2 - 42e\ell^2 - 66e\ell k - 24ek^2 + 18e^2\ell + 18e^2 k)$$

Proof: By Lemma 1.3 terms in codimension ≥ 4 in $\widehat{B}_3$ are 0, and the term in codimension 3 is $\widehat{B}_3 \cdot \widehat{D}_3$. For the codimension 1 and 2 terms, the only missing ingredient is (part of) $s(\widehat{B}_3 \cap \widehat{D}_3, \widehat{B}_3)$. To get equations for $\widehat{B}_3 \cap \widehat{D}_3$ in $\widehat{B}_3$, use the coordinates of §2.1: give coordinates $(\alpha_1, \alpha_2; u, t)$ to $\widehat{B}_3$, so that the blow-up map to $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ is given by

$$(\alpha_1, \alpha_2; u, t) \mapsto ((\alpha_1 + u, \alpha_2 + ut), (\alpha_1, \alpha_2))$$

(with obvious choices of coordinates for $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$); then in terms of $(b_1, \dots, b_9)$ one has ([**A1**], §3.1)

$$(\alpha_1, \alpha_2; u, t) \mapsto (3\alpha_1 + u, 3\alpha_2 + ut, -u^2, 2t, t^2, 2\alpha_1, 2\alpha_1 t, 2\alpha_2 t, 2\alpha_2 t^2)$$

Restricting the equations for $\widehat{D}_1$ gives equations

$$\begin{cases} (x_0+\alpha_1 x_1+\alpha_2 x_2)^2=0 \\ (x_0+\alpha_1 x_1+\alpha_2 x_2)(x_1+tx_2)=0 \end{cases} ;$$

these lift to equations for $\widehat{B}_3 \cap \widehat{D}_3$ in $\widehat{B}_3$. So $\widehat{B}_3 \cap \widehat{D}_3$ consists of the divisor of triples

$$\{(p,(\lambda,\mu),q) \in \widehat{B}_3 \text{ s.t. } p \in \mu\} \quad ,$$

with an embedded component along

$$\{(p,(\lambda,\mu),q) \in \widehat{B}_3 \text{ s.t. } p=q\} \quad .$$

The first has class $m+k$, the second is a divisor in the first, with class $\ell+m-e$. It follows easily that

$$\begin{aligned} s(\widehat{B}_3 \cap \widehat{D}_3, \widehat{B}_3) &= (m+k)-(m+k)^2+(m+k)(\ell+k-e)+ \text{ higher cod. terms} \\ &= (m+k)+(m+k)(\ell-m-e)+ \text{ higher cod. terms.} \end{aligned}$$

Therefore, by Lemma 2.7

$$\begin{aligned} \widehat{D}_3 \circ \widehat{B}_3 &= c(N_{\widehat{D}_3}\widehat{V}_3)s(\widehat{B}_3 \cap \widehat{D}_3, \widehat{B}_3) \\ &= (1+3\ell+6m+6k-7e+\dots)((m+k)+(m+k)(\ell-m-e)+\dots) \\ &= (m+k)+(4\ell m+5m^2+4k\ell+11km+6k^2-8e\ell-8ek)+\dots \end{aligned}$$

We are done, as we observed already that $\widehat{B}_3 \circ \widehat{D}_3 = \widehat{D}_3 \circ \widehat{B}_3$ in codimension ≤ 2, and the codimension-3 term, i.e. $\widehat{B}_3 \cdot \widehat{D}_3$, is given by a straightforward application of Lemma A.4. ∎

§**2.4.** $\boldsymbol{\widehat{B}_4 \circ \widehat{D}_4}$**.** The center B_4 of the fifth blow-up is isomorphic to B_3, therefore to the blow-up of $\check{\mathbb{P}}^2 \times \check{\mathbb{P}}^2$ along the diagonal ([**A1**], §3.4); the exceptional divisor E_4 in V_4 restricts to $3\ell+3m-4e$ on $\widehat{B}_4$ ([**A1**], Lemma 4.2). Lemmas 1.3 and A.4 will give easily the terms of $\widehat{B}_4 \circ \widehat{D}_4$ of codimension ≥ 3 in $\widehat{B}_4$; so, as in §2.3, we just have to determine the terms of $\widehat{B}_4 \circ \widehat{D}_4$ in codimension ≤ 2. The main problem here is analyzing the situation over the embedded component of $\widehat{B}_3 \cap \widehat{D}_3$ (which has codimension 2 in $\widehat{B}_3$, so affects the terms we have to compute). For this we introduce an 'intermediate' blow-up $\widehat{V}_3'$ of $\widehat{V}_3$ along the incidence correspondence

$$I = \{(p,(\lambda,\mu),q) \in \widehat{B}_3 \text{ s.t. } p=q\} \quad ,$$

on which the embedded component is supported (cf. §2.3). Next, let $\widehat{V}_4'$ be the blow-up of $\widehat{V}_3'$ along the proper transform $\widehat{B}_3'$ of $\widehat{B}_3$ in $\widehat{V}_3'$. By the universal property of blow-ups, $\widehat{V}_4'$ is also the blow-up of $\widehat{V}_4$ along the inverse image J of I, so one has the commutative diagram

$$\begin{tikzcd} \widehat{V}_4' \arrow[r, "\text{blow-up } J"] \arrow[d, "\text{blow-up } \widehat{B}_3'"'] & \widehat{V}_4 \arrow[d, "\text{blow-up } \widehat{B}_3"] \\ \widehat{V}_3' \arrow[r, "\text{blow-up I}"] & \widehat{V}_3 \end{tikzcd}$$

If $((\lambda, \mu), q) \in B_3$, look at the plane $\mathbb{P}^2 = \mathbb{P}^2 \times ((\lambda, \mu), q) \subset \widehat{B}_3$. Then $\widehat{D}_3$ intersects this $\mathbb{P}^2$ along μ, with embedded point at q. In $\widehat{V}_3'$, the proper transform of this plane is its blow-up $\widetilde{\mathbb{P}}^2$ at q, and the proper transform $\widehat{D}_3'$ of $\widehat{D}_3$ in $\widehat{V}_3'$ intersects $\widetilde{\mathbb{P}}^2$ along the inverse image of μ (use the equations for $\widehat{B}_3 \cap \widehat{D}_3$ in §2.3, proof of Proposition 2.8). As $((\lambda, \mu), q)$ moves in B_3, we find that $\widehat{D}_3'$ intersects $\widehat{B}_3'$ along the inverse image of the support of $\widehat{B}_3 \cap \widehat{D}_3$, which consists of two components; so the exceptional divisor of the blow-up of $\widehat{D}_3'$ along $\widehat{B}_3' \cap \widehat{D}_3'$ (i.e., the intersection of the exceptional divisor with the proper transform $\widehat{D}_4'$ of $\widehat{D}_3'$ in $\widehat{V}_4'$) will have two components $E^{(1)'}$, $E^{(2)'}$. Also, the top map doesn't contract either of these components; we conclude that, in $\widehat{V}_4$, $\widehat{E}_4 \cap \widehat{D}_4$ consists of two components $E^{(1)}$, $E^{(2)}$, the first dominating the support of of $\widehat{B}_3 \cap \widehat{D}_3$, and the second dominating the embedded component of $\widehat{B}_3 \cap \widehat{D}_3$ (supported on I). Also, tracing the inverse image of $\widehat{B}_3$ in the diagram gives that $\widehat{E}_4$ pulls-back on $\widehat{D}_4$ to the divisor $E^{(1)} + 2E^{(2)}$.

The information we have just collected is needed to compute the restriction of $c_1(N_{\widehat{D}_4}\widehat{V}_4)$ to $\widehat{B}_4 \cap \widehat{D}_4$:

Lemma 2.9. *$c(N_{\widehat{D}_4}\widehat{V}_4)$ restricts to $1 - 2\ell + 7k + \dots$.*

Proof: If $\widehat{E}_4$ is the exceptional divisor in $\widehat{V}_4$, then (omitting pull-backs as usual)

$$c_1(T\widehat{V}_4) = c_1(T\widehat{V}_3) - 4\widehat{E}_4$$

since the codimension of $\widehat{B}_3$ in $\widehat{V}_3$ is 5.

To get $c_1(T\widehat{D}_4)$, we restrict the above blow-up diagram to the $\widehat{D}$'s:

$$\begin{tikzcd}
\widehat{D}_4' \arrow[r, "\text{blow-up } J\cap\widehat{D}_4"] \arrow[d, "\text{blow-up } \widehat{B}_3'\cap\widehat{D}_3'"'] & \widehat{D}_4 \arrow[d, "\text{blow-up } \widehat{B}_3\cap\widehat{D}_3"] \\
\widehat{D}_3' \arrow[r, "\text{blow-up I}"] & \widehat{D}_3
\end{tikzcd}$$

Let F_3 be the exceptional divisor of the bottom blow-up. The exceptional divisor of the leftmost blow-up consists (as we have seen) of two components $E^{(1)'}$, $E^{(2)'}$; F_3 contains one of the two components blown up on the left, and the top map contracts the proper transform $F_4 = F_3 - E^{(2)'}$ of F_3 in $\widehat{V}_4'$. Away from F_4 and its image in $\widehat{D}_4$ (which has codimension > 1), $\widehat{D}_4'$ and $\widehat{D}_4$ are isomorphic, the former being the blow-up of the latter along the divisor $E^{(2)}$; so $c_1(T\widehat{D}_4')$ restricts to (the pull-back of) $c_1(T\widehat{D}_4)$ on the complement of F_4. Now

$$\begin{aligned}
c_1(T\widehat{D}_4') &= c_1(T\widehat{D}_3') - 2E^{(1)'} - 2E^{(2)'} \\
&= c_1(T\widehat{D}_3) - 3F_3 - 2E^{(1)'} - 2E^{(2)'} \\
&= c_1(T\widehat{D}_3) - 3F_4 - 2E^{(1)'} - 5E^{(2)'}
\end{aligned}$$

restricts to $c_1(T\widehat{D}_3) - 2E^{(1)} - 5E^{(2)}$ on the complement of F_4, so recalling that $\widehat{E}_4$ pulls-back to $E^{(1)} + 2E^{(2)}$ on $\widehat{D}_4$ we find

$$c_1(T\widehat{D}_4) = c_1(T\widehat{D}_3) - 2\widehat{E}_4 - E^{(2)} \quad .$$

Thus

$$\begin{aligned} c_1(N_{\widehat{D}_4}\widehat{V}_4) &= c_1(T\widehat{V}_4) - c_1(T\widehat{D}_4) \\ &= c_1(T\widehat{V}_3) - 4\widehat{E}_4 - c_1(T\widehat{D}_3) + 2\widehat{E}_4 + E^{(2)} \\ &= c_1(N_{\widehat{D}_3}\widehat{V}_3) - 2\widehat{E}_4 + E^{(2)} \quad . \end{aligned}$$

Finally, the class of $E^{(2)}$ restricts on $\widehat{B}_4 \cap \widehat{D}_4$ to $\ell + k - e$: indeed, we'll see in a moment that $\widehat{B}_4 \cap \widehat{D}_4$ is supported on the pull-back of the support of $\widehat{B}_3 \cap \widehat{D}_3$; and $E^{(2)} \cap \widehat{B}_4$ is the pull-back in $\widehat{B}_4 \cap \widehat{D}_4$ of the divisor I of $\widehat{B}_3 \cap \widehat{D}_3$, which has class $\ell + k - e$.

Putting all together (and recalling that $\widehat{E}_4$ restricts to $3\ell + 3m - 4e$, beginning of this section)

$$\begin{aligned} c_1(N_{\widehat{D}_4}\widehat{V}_4) &= (3\ell + 6m + 6k - 7e) - 2(3\ell + 3m - 4e) + (\ell + k - e) \\ &= -2\ell + 7k \quad , \end{aligned}$$

which is the claim. ∎

Proposition 2.10.

$$\begin{aligned} \widehat{B}_4 \circ \widehat{D}_4 = (m+k) &+ (-2\ell m - m^2 - 2\ell k + 5mk + 6k^2) + (3\ell^2 m \\ &+ 3\ell m^2 + 3k\ell^2 - 3k\ell m - 6k^2\ell + 6k^2 m - 6e\ell^2 - 6ekl + 2e^2\ell + 2e^2 k) \end{aligned}$$

Proof: Once more we argue $\widehat{B}_4 \circ \widehat{D}_4 = \widehat{D}_4 \circ \widehat{B}_4$ (in codimension ≤ 2), and proceed to compute the first couple of terms in $s(\widehat{B}_4 \cap \widehat{D}_4, \widehat{B}_4)$. Now we claim that $\widehat{B}_4 \cap \widehat{D}_4$ is the divisor of $\widehat{B}_4$ dominating the support of $\widehat{B}_3 \cap \widehat{D}_3$, this time *without* embedded components. This is another coordinate computation: the key step is to show that the divisor is cut out scheme-theoretically (without embedded components); for this, it suffices to produce a divisor of $\widehat{V}_4$ containing $\widehat{D}_4$ and intersecting $\widehat{B}_4$ scheme-theoretically along the support of $\widehat{B}_4 \cap \widehat{D}_4$. For example, one sees that the proper transform of

$$2\left(a_0 \frac{\partial f}{\partial x_2} - \frac{a_2}{3}\frac{\partial f}{\partial x_0}\right)(3a_3 - a_1^2) - \left(a_0 \frac{\partial f}{\partial x_1} - \frac{a_1}{3}\frac{\partial f}{\partial x_0}\right)(3a_4 - 2a_1 a_2) = 0$$

satisfies this requirement over $\{a_0 \neq 0,\ x_2 \neq 0\}$.

So $\widehat{B}_4 \cap \widehat{D}_4$ is a divisor of $\widehat{B}_4$, with class $m+k$ (the class of the support of $\widehat{B}_3 \cap \widehat{D}_3$ in $\widehat{B}_3$, cf. §2.3), and therefore

$$(\widehat{B}_4 \cap \widehat{D}_4, \widehat{B}_4) = (m+k) - (m+k)^2 + \dots \quad .$$

Now using Lemma 2.9:

$$\widehat{D}_4 \circ \widehat{B}_4 = (1 - 2\ell + 7k + \dots)((m+k) - (m+k)^2 + \dots) \quad ,$$

so

$$\widehat{B}_4 \circ \widehat{D}_4 = (m+k) + (-2\ell m - m^2 - 2\ell k + 5mk + 6k^2) + \widehat{B}_4 \cdot \widehat{D}_4 \quad .$$

Finally, Lemma A.4 yields $\widehat{B}_4 \cdot \widehat{D}_4$, with the result given in the statement. ∎

§**2.5. Proof of Theorem II.** As observed already, the first part of Theorem II follows from the computations performed in §§2.0–4, by reading off each class $\widehat{B}_i \circ \widehat{D}_i$ the coefficient of k^2, k, and the constant term with respect to k. The results obtained give the classes for the three families of nodal cubics we are considering, and are enough to compute the characteristic numbers for such families. We will see now that the classes $\widehat{B}_i \circ \widehat{D}_i$ contain actually most of the information needed to compute the classes for families of cuspidal cubics as well.

As in [**A3**], §1.2, we describe the closure C_0 of the set of cuspidal curves as the projection from $\mathbb{P}^2 \times \mathbb{P}^9$ of the divisor $\widehat{C}_0$ of $\widehat{D}_0$ defined by

$$(p,f) \in \widehat{C}_0 \iff \begin{cases} \dfrac{\partial f}{\partial x_0}(p) = 0 \\ \dfrac{\partial f}{\partial x_1}(p) = 0 \\ \dfrac{\partial f}{\partial x_2}(p) = 0 \end{cases} \quad , \quad \begin{cases} \left[\left(\dfrac{\partial^2 f}{\partial x_0 \partial x_1}\right)^2 - \dfrac{\partial^2 f}{\partial x_0^2}\dfrac{\partial^2 f}{\partial x_1^2}\right](p) = 0 \\ \left[\left(\dfrac{\partial^2 f}{\partial x_0 \partial x_2}\right)^2 - \dfrac{\partial^2 f}{\partial x_0^2}\dfrac{\partial^2 f}{\partial x_2^2}\right](p) = 0 \\ \left[\left(\dfrac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 - \dfrac{\partial^2 f}{\partial x_1^2}\dfrac{\partial^2 f}{\partial x_2^2}\right](p) = 0 \end{cases} \quad .$$

(so $(p,f) \in \widehat{C}_0$ if and only if f is a cubic singular at p, whose tangent cone at p contains a double line). The projection $\mathbb{P}^2 \times \mathbb{P}^9 \longrightarrow \mathbb{P}^2$ restricts to a map $\widehat{C}_0 \longrightarrow \mathbb{P}^2$ whose fibers are quadrics in the fibers of $\widehat{D}_0$. The other projection, $\mathbb{P}^2 \times \mathbb{P}^9 \longrightarrow \mathbb{P}^9$, restricts to a birational morphism from $\widehat{C}_0$ to the closure of the set of cuspidal cubics in $\mathbb{P}^9$. We let $\widehat{C}_i$ be the proper transform of $\widehat{C}_0$ in $\widehat{V}_i$; then we obtain birational morphisms

$$\widehat{C}_i \longrightarrow C_i \quad .$$

Proposition 2.11. *For* $i = 0, \dots, 4$
$B_i \circ C_i =$ *coefficient of* k^2 *in* $\widehat{B}_i \circ \widehat{C}_i$
$B_i \circ C\ell_i =$ *coefficient of* k^1 *in* $\widehat{B}_i \circ \widehat{C}_i$
$B_i \circ Cp_i =$ *coefficient of* k^0 *in* $\widehat{B}_i \circ \widehat{C}_i$

Proof: The argument mirrors the proof of Proposition 2.1, and we leave it to the reader. ∎

So all we need to compute in order to complete the proof of Theorem II are the five classes $\widehat{B}_0 \circ \widehat{C}_0 \dots, \widehat{B}_4 \circ \widehat{C}_4$. Since each $\widehat{C}_i$ is a divisor in $\widehat{D}_i$, applying Lemma A.3 from the appendix reduces the computation to finding the 'multiplicity' of each $\widehat{C}_i$ along $\widehat{B}_i \cap \widehat{D}_i$.

Proposition 2.12.

$$\begin{aligned} \widehat{B}_0 \circ \widehat{C}_0 &= (1 + 6h)\widehat{B}_0 \circ \widehat{D}_0 \\ \widehat{B}_1 \circ \widehat{C}_1 &= (2 + 6h - \epsilon)\widehat{B}_1 \circ \widehat{D}_1 \\ \widehat{B}_2 \circ \widehat{C}_2 &= (6h - \epsilon - 2\varphi)\widehat{B}_2 \circ \widehat{D}_2 \\ \widehat{B}_3 \circ \widehat{C}_3 &= (1 + 2\ell + 4m - 4e)\widehat{B}_3 \circ \widehat{D}_3 \\ \widehat{B}_4 \circ \widehat{C}_4 &= (1 - \ell + m)\widehat{B}_4 \circ \widehat{D}_4 \end{aligned}$$

PROOF: We apply Lemma A.3 from the appendix. Obtaining the multiplicity of the $\widehat{C}_i$ along $\widehat{B}_i \circ \widehat{D}_i$ is done by computing the highest power of a local equation for the exceptional divisor that divides the pull-back to $\widehat{D}_{i+1}$ of a local equation for $\widehat{C}_i$ in $\widehat{D}_i$ (to start, observe that e.g. over $\{x_2 \neq 0\}$

$$\left[\left(\frac{\partial^2 f}{\partial x_0 \partial x_1}\right)^2 - \frac{\partial^2 f}{\partial x_0^2}\frac{\partial^2 f}{\partial x_1^2}\right](p) = 0$$

gives a local equation for $\widehat{C}_0$ in $\widehat{D}_0$).

This computation gives the constant terms $1, 2, 0, 1, 1$ of the linear factors in the statement.

For the other terms, the class of $\widehat{C}_0$ in $\widehat{D}_0$ is $2H$ by Lemma 1.4 in [**A3**] (H is the hyperplane class in $\mathbb{P}^9$, as in §2.0); therefore the multiplicity computation gives the classes of the $\widehat{C}_i$ in the $\widehat{D}_i$ as the pull-back of:

$$\begin{array}{ll}
2H & i=0 \\
2H - \widehat{E}_1 & i=1 \\
2H - \widehat{E}_1 - 2\widehat{E}_2 & i=2 \\
2H - \widehat{E}_1 - 2\widehat{E}_2 & i=3 \\
2H - \widehat{E}_1 - 2\widehat{E}_2 - \widehat{E}_4 & i=4
\end{array}$$

restricting on $\widehat{B}_i$ to

$$\begin{array}{ll}
6h & i=0 \\
6h - \epsilon & i=1 \\
6h - \epsilon - 2\varphi & i=2 \\
2\ell + 4m - 4e & i=3 \\
-\ell + m & i=4
\end{array}$$

($\widehat{E}_1$ restricts to $\epsilon, 2e$; $\widehat{E}_2$ to φ, e; $\widehat{E}_4$ to $3\ell + 3m - 4e$, see §§2.0–4) giving the other terms in the linear factors in the statement. ∎

Propositions 2.11 and 2.12 complete the proof of Theorem II. For example, by Proposition 2.12, $\widehat{B}_0 \circ \widehat{C}_0$ is

$$\begin{aligned}
(1+6h)(\widehat{B}_0 \circ \widehat{D}_0) &= (1+6h)(2h + 2k + 14h^2 + 22hk + 8k^2 + 54h^2k + 36hk^2) \\
&= 2h + 2k + 26h^2 + 34hk + 8k^2 + 186h^2k + 84hk^2 + 216h^2k^2 \\
&= (8 + 84h + 216h^2)k^2 + (2 + 34h + 186h^2)k + (2h + 26h^2) \quad ,
\end{aligned}$$

giving the first row of the last three braces in the statement of Theorem II, by Proposition 2.11.

3. Characteristic numbers

The computation of the characteristic numbers is now a straightforward application of Propositions 1.1 and Theorem I from §1 to the classes computed in Theorem II, §2: Theorem I gives the 'weighted' characteristic numbers $N_F(n_pP, n_\ell L)$ for each of the families $D, D\ell, Dp$, $C, C\ell, Cp$; these in turn give the characteristic numbers *proper*, via Proposition 1.1.

Proposition 3.1. *The weighted characteristic numbers $N_F(n_pP, n_\ell L)$ (where $n_p = \dim F - n_\ell$) are:*

n_ℓ	N_D	$N_{D\ell}$	N_{Dp}	N_C	$N_{C\ell}$	N_{Cp}
0	*12*	*6*	*1*	*24*	*12*	*2*
1	*48*	*24*	*4*	*96*	*48*	*8*
2	*192*	*96*	*16*	*384*	*144*	*20*
3	*768*	*336*	*52*	*1248*	*348*	*38*
4	*2784*	*1020*	*142*	*3264*	*642*	*44*
5	*8832*	*2466*	*256*	*6324*	*792*	*32*
6	*21828*	*4284*	*304*	*8376*	*648*	
7	*39072*	*5256*		*7584*		
8	*50448*					

Proof: For example, for the family of cuspidal cubics, and $n_\ell = 7$:

$$N_C(0P, 7L) = 4^7 \cdot 24 - \sum_{i=0}^{4} \int_{B_i} \frac{(B_i \circ L_i)^7 (B_i \circ C_i)}{c(N_{B_i} V_i)}$$

by Theorem I, i.e. (reading $B_i \circ C_i$ from Theorem II in §2, and $B_i \circ L_i$, $c(N_{B_i} V_i)$ from Theorem III in [**A1**])

$$\begin{aligned}
N_C(0P, 7L) = 16384 \cdot 24 &- \int_{B_0} \frac{(2+12h)^7(8+84h+216h^2)(1+h)^3}{(1+3h)^{10}} \\
&- \int_{B_1} \frac{(1+12h-2\epsilon)^7(10+102h-21\epsilon+216h^2+\dots)(1+2h-\epsilon)^6}{(1+\epsilon)(1+3h-\epsilon)^{10}} \\
&- \int_{B_2} \frac{(1+12h-2\epsilon-\varphi)^7(18h-3\epsilon-6\varphi+216h^2-84\epsilon h+\dots)}{(1+\varphi)(1+\epsilon-\varphi)} \\
&- \int_{B_3} \frac{(1+4\ell+8m-6e)^7(6+24\ell+48m-48e+24\ell^2+\dots)}{(1+7\ell+17m-16e+126m^2+\dots)} \\
&- \int_{B_4} \frac{(1+\ell+5m-2e)^7(6-12\ell+12m+6\ell^2-12\ell m+6m^2)}{(1-5\ell+5m+18m^2-27\ell m+3\ell^2+\dots)} \\
&= 393216 - 219648 - 127902 - 115554 + 67338 + 10134 \\
&= 7584
\end{aligned}$$

(each term is computed by expanding the fraction as a power series, selecting the term of degree $= \dim B_i$, and applying the relations given in [**A1**], Theorem III).

We list here the intermediate contributions for all families, obtained as above, for those n_ℓ giving non-zero terms.

D:

n_ℓ	$\int_{B_0}$	$\int_{B_1}$	$\int_{B_2}$	$\int_{B_3}$	$\int_{B_4}$
4	0	0	0	144	144
5	0	0	0	2052	1404
6	4608	2043	8901	6912	4860
7	59904	21807	73809	–3636	5652
8	439296	120966	289914	–97722	–16470

$D\ell$:

n_ℓ	$\int_{B_0}$	$\int_{B_1}$	$\int_{B_2}$	$\int_{B_3}$	$\int_{B_4}$
3	0	0	0	24	24
4	0	0	0	312	204
5	576	297	1071	1047	687
6	7680	3180	9228	–564	768
7	56832	17571	36405	–15402	–2358

Dp:

n_ℓ	$\int_{B_0}$	$\int_{B_1}$	$\int_{B_2}$	$\int_{B_3}$	$\int_{B_4}$
3	0	0	0	6	6
4	0	0	0	72	42
5	192	99	357	57	63
6	2048	833	2087	–1032	–144

C:

n_ℓ	$\int_{B_0}$	$\int_{B_1}$	$\int_{B_2}$	$\int_{B_3}$	$\int_{B_4}$
3	0	0	0	144	144
4	0	0	0	1764	1116
5	2304	2925	5139	4752	3132
6	29952	26739	36621	–5796	2412
7	219648	127902	115554	–67338	–10134

$C\ell$:

n_ℓ	$\int_{B_0}$	$\int_{B_1}$	$\int_{B_2}$	$\int_{B_3}$	$\int_{B_4}$
2	0	0	0	24	24
3	0	0	0	264	156
4	288	405	603	711	423
5	3840	3750	4506	–894	294
6	28416	17889	14175	–10578	–1398

Cp:

n_ℓ	$\int_{B_0}$	$\int_{B_1}$	$\int_{B_2}$	$\int_{B_3}$	$\int_{B_4}$
2	0	0	0	6	6
3	0	0	0	60	30
4	96	135	201	9	27
5	1024	925	931	–774	–90

The results in the statement of the Proposition are obtained by subtracting the sum of the numbers in each row from $4^{n_\ell} \cdot \deg F_0$ (the degree of $D_0, D\ell_0$, etc. are listed in Proposition 1.2), as prescribed by Theorem I. ∎

Is there any general pattern ruling the numbers listed in Proposition 3.1 (and its proof)? The alert reader has probably noticed that the numbers $N_F(n_pP, n_\ell L)$ of the statement are in each case congruent to $\deg F_0$ modulo 3: this is always true when F_0 is a *hypersurface* of $\mathbb{P}^9$ (see [**A4**], §1, Corollary 2).

Proposition 1.1 now concludes the computation:

Theorem III. *The characteristic numbers for the families $D, D\ell, Dp$, $C, C\ell, Cp$ are*

k	D	$D\ell$	Dp	C	$C\ell$	Cp
0	*12*	*6*	*1*	*24*	*12*	*2*
1	*36*	*22*	*4*	*60*	*42*	*8*
2	*100*	*80*	*16*	*114*	*96*	*20*
3	*240*	*240*	*52*	*168*	*168*	*38*
4	*480*	*604*	*142*	*168*	*186*	*44*
5	*712*	*1046*	*256*	*114*	*132*	*32*
6	*756*	*1212*	*304*	*60*	*72*	
7	*600*	*1000*		*24*		
8	*400*					

where $F(k)$ denotes the number of elements of F tangent at smooth points to k lines and containing $\dim F - k$ *points in general position in the plane.*

Proof: This is now straightforward. For example,

$$\begin{aligned} Cp(5) &= N_{Cp}(0P, 5L) = 32, \quad \text{so} \\ C\ell(6) &= N_{C\ell}(0P, 6L) - 18 \cdot 32 = 72, \quad \text{and} \\ C(7) &= N_C(0P, 7L) - 21 \cdot 72 - 9 \cdot 21 \cdot 32 = 24, \end{aligned}$$

by Propositions 1.1 and 3.1. ∎

4. Further characteristic numbers

In this last section we want to stress that the classes computed in §2 contain yet more enumerative information: no additional work is needed at this point to produce the characteristic numbers for the families obtained by further imposing conditions of tangency to a line *at a given point.*

Denote by $N_F(n_pP, n_\ell L, n_m M)$ the *weighted* number of elements of F containing n_p points, tangent to n_ℓ lines, and furthermore tangent to n_m lines at given points (where $n_p + n_\ell + 2n_m = \dim F$); then Theorem IV′ in [**A1**] gives

$$N_F(n_pP, n_\ell L, n_m M) = 4^{n_\ell} \cdot \deg F_0 - \sum_{i=0}^{4} \int_{B_i} \frac{(B_i \circ P_i)^{n_p}(B_i \circ L_i)^{n_\ell}(B_i \circ M_i)^{n_m}(B_i \circ F_i)}{c(N_{B_i}V_i)}$$

with notations as above, and $B_i \circ M_i$ given by Proposition 5.1 in [**A1**].

Proposition 4.1. *The 'weighted' numbers $N_F(n_pP, n_\ell L, n_m M)$ (where $n_p = \dim F - n_\ell - 2n_m$) are:*

—for $n_m = 1$

n_ℓ	N_D	$N_{D\ell}$	N_{Dp}	N_C	$N_{C\ell}$	N_{Cp}
0	12	6	1	24	12	2
1	48	24	4	96	36	6
2	192	84	14	312	90	12
3	696	258	40	816	168	14
4	2208	612	70	1536	210	
5	5232	1026		2004		
6	8868					

—for $n_m = 2$

n_ℓ	N_D	$N_{D\ell}$	N_{Dp}	N_C	$N_{C\ell}$	N_{Cp}
0	12	6	1	24	10	2
1	48	22	4	84	24	4
2	180	68	12	216	44	
3	576	156		384		
4	1296					

—for $n_m = 3$

n_ℓ	N_D	$N_{D\ell}$	N_{Dp}	N_C	$N_{C\ell}$	N_{Cp}
0	12	6	1	24	6	
1	48	18		60		
2	156					

PROOF: As for Proposition 3.1, we just list the relevant contributions one computes in applying the above formula:

—for $n_m = 1$:

	n_ℓ	$\int_{B_0}$	$\int_{B_1}$	$\int_{B_2}$	$\int_{B_3}$	$\int_{B_4}$
D :						
	3	0	0	0	36	36
	4	0	0	0	522	342
	5	1536	681	2967	972	900
	6	18432	6588	21636	–6282	–90
$D\ell$:						
	2	0	0	0	6	6
	3	0	0	0	78	48
	4	192	99	357	147	129
	5	2368	961	2719	–939	9
Dp :						
	2	0	0	0	1	1
	3	0	0	0	15	9
	4	64	33	119	–35	5
C :						
	2	0	0	0	36	36
	3	0	0	0	450	270
	4	768	975	1713	576	576
	5	9216	7938	10494	–4986	–90
$C\ell$:						
	1	0	0	0	6	6
	2	0	0	0	66	36
	3	96	135	201	87	81
	4	1184	1115	1301	–741	3
$Cp\ell$:						
	1	0	0	0	1	1
	2	0	0	0	13	7
	3	32	45	67	–33	3

—for $n_m = 2$:

	n_ℓ	$\int_{B_0}$	$\int_{B_1}$	$\int_{B_2}$	$\int_{B_3}$	$\int_{B_4}$
D :						
	2	0	0	0	6	6
	3	0	0	0	114	78
	4	512	227	989	–90	138
$D\ell$:						
	1	0	0	0	1	1
	2	0	0	0	17	11
	3	64	33	119	–11	23

Dp :					
2	0	0	0	2	2
C :					
1	0	0	0	6	6
2	0	0	0	102	66
3	256	325	571	–102	102
$C\ell$:					
0	0	0	0	1	1
1	0	0	0	15	9
2	32	45	67	–13	17
Cp :					
1	0	0	0	2	2

—for $n_m = 3$:

n_ℓ	$\int_{B_0}$	$\int_{B_1}$	$\int_{B_2}$	$\int_{B_3}$	$\int_{B_4}$
D :					
2	0	0	0	18	18
$D\ell$:					
1	0	0	0	3	3
C :					
1	0	0	0	18	18
$C\ell$:					
0	0	0	0	3	3

The statement of the proposition is obtained from these tables by subtracting the sum of the five numbers in each row from $4^{n_\ell} \cdot \deg F_0$. ∎

From Proposition 3.2 and the straightforward extension of Proposition 1.1 (which we leave to the reader) follow the characteristic numbers:

THEOREM III′. *Denote by $F^{(j)}(k)$ the number of elements of F tangent to k lines, containing $\dim F - k - 2j$ points, and tangent to j lines at given points (all choices being general). Then:*

k	$D^{(1)}$	$D\ell^{(1)}$	$Dp^{(1)}$	$C^{(1)}$	$C\ell^{(1)}$	$Cp^{(1)}$
0	*10*	*6*	*1*	*18*	*12*	*2*
1	*28*	*22*	*4*	*36*	*30*	*6*
2	*68*	*68*	*14*	*54*	*54*	*12*
3	*136*	*174*	*40*	*54*	*60*	*14*
4	*196*	*292*	*70*	*36*	*42*	
5	*200*	*326*		*18*		
6	*148*					

k	$D^{(2)}$	$D\ell^{(2)}$	$Dp^{(2)}$	$C^{(2)}$	$C\ell^{(2)}$	$Cp^{(2)}$
0	*8*	*6*	*1*	*12*	*10*	*2*
1	*20*	*20*	*4*	*18*	*18*	*4*
2	*40*	*52*	*12*	*18*	*20*	
3	*56*	*84*		*12*		
4	*56*					

k	$D^{(3)}$	$D\ell^{(3)}$	$Dp^{(3)}$	$C^{(3)}$	$C\ell^{(3)}$	$Cp^{(3)}$
0	*6*	*6*	*1*	*6*	*6*	
1	*12*	*16*		*6*		
2	*16*					

The enumerative results computed in Theorems III and III′ agree with Zeuthen's lists, with the exception of $D\ell(5)$ from Theorem III in §3 (the number of nodal cubics with node on a given line, containing three points and tangent to five lines in general position), a (very rare!) typo in [**Z**], p.607.

Appendix

In this appendix we list a few facts used in the computation of the full intersection classes in §2. Suppose $B, F \subset V$ are pure-dimensional schemes, with $B \hookrightarrow V$ a regular embedding. We set

$$B \circ F = c(N_B V) s(B \cap F, F) \quad ,$$

the 'full intersection class' of F by B in V (as usual, we omit pull-back notations). If $F \hookrightarrow V$ is also a regular embedding, then we can consider the class $F \circ B$ as well; unfortunately, $B \circ F \neq F \circ B$ in general: for example, let $B = p$ be a point in $V = \mathbb{P}^2$, and let F be any curve with a double point at p: then $B \circ F = 2[p]$, while $F \circ B = [p]$. However:

Lemma A.1. *If B, F, V are non-singular, then*

$$B \circ F = F \circ B \quad .$$

Proof: By [**F**], Example 4.2.6,

$$c(TF) s(B \cap F, F) = c(TB) s(B \cap F, B)$$

(this class is intrinsic of $B \cap F$). Multiplying by $\dfrac{c(TV)}{c(TF)c(TB)}$ gives then

$$\frac{c(TV)}{c(TB)} s(B \cap F, F) = \frac{c(TV)}{c(TF)} s(B \cap F, B) \quad , \quad \text{i.e.}$$
$$c(N_B V) s(B \cap F, F) = c(N_F V) s(B \cap F, B)$$

which is the claim. ∎

For example, in computing $B \circ F$, suppose that the hypotheses of A.1 hold in the complement of a subvariety W of B of codimension r. Then

$$\{B \circ F\}_i = \{F \circ B\}_i \quad \text{for } i > \dim B - r,$$

by Lemma A.1 (we say, a little improperly, '$B \circ F = F \circ B$ in codimension $< r$ in B'). Often the right-hand-side is easier to compute, and higher codimensional terms can be computed separately, e.g. by using Lemma 1.4 (2), (3). Notice that the right-hand-side above need not be defined in the whole of V, but just on $V - W$, because Segre classes are preserved via flat maps.

The commutativity of full intersection classes is strictly related to the following issue: suppose $W \subset X \subset V$ are closed embedding, and suppose $X \hookrightarrow V$ is regular.

Under what circumstances is

$$c(N_X V)^{-1} s(W, X)$$

independent of X?

The proof of A.1 works because this class is independent of X if X and V are non-singular. Other conditions can be considered; S. Keel has shown that this class is independent of X as long as the embedding $W \hookrightarrow X$ is 'linear' (see [**K**]): so $B \circ F = F \circ B$ if $B \hookrightarrow V$, $F \hookrightarrow V$ are regular embeddings and $B \cap F \hookrightarrow B$, $B \cap F \hookrightarrow F$ are *linear embeddings.* The following observation is also due to Keel:

Lemma (Keel). *Suppose $W \subset X \subset V$ are closed embeddings, with $W \hookrightarrow V$, $X \hookrightarrow V$ regular embeddings. Suppose the proper transform of X in the blow-up $B\ell_W V \xrightarrow{\pi} V$ of V along W is regularly embedded in $B\ell_W V$ as the residual scheme to the exceptional divisor in $\pi^{-1} X$. Then*

$$c(N_X V)^{-1} s(W, X) = s(W, V) \quad .$$

Proof: Let $\mathcal{I}$, $\mathcal{J}$ be the ideal sheaves of W, X resp. in $\mathcal{O}_V$. The exact sequence

$$\frac{\mathcal{J}}{\mathcal{J}^2} \to \frac{\mathcal{I}}{\mathcal{I}^2} \to \frac{\mathcal{I}}{\mathcal{I}^2 + \mathcal{J}} \to 0$$

induces an exact sequence of graded algebras

$$\frac{\mathcal{J}}{\mathcal{J}^2} \otimes \mathrm{Sym}\left(\frac{\mathcal{I}}{\mathcal{I}^2}\right)(-1) \to \mathrm{Sym}\left(\frac{\mathcal{I}}{\mathcal{I}^2}\right) \to \mathrm{Sym}\left(\frac{\mathcal{I}}{\mathcal{I}^2 + \mathcal{J}}\right) \to 0 \quad .$$

Since the embedding $W \hookrightarrow V$ is regular, the second term in this sequence is the homogeneous coordinate ring for $\mathbb{P}(N_W V)$; under the hypotheses, the embedding $W \hookrightarrow V$ is weakly linear (Theorem 1 in [**K**]), so the third term is the ring for $\mathbb{P}(C_W X)$. The image of the first is then the homogeneous ideal sheaf of $\mathbb{P}(C_W X)$ in $\mathbb{P}(N_W V)$, and we get the sequence of sheaves on $\mathbb{P}(N_W V)$

$$N_X V^* \otimes \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}(N_W V)} \to \mathcal{O}_{\mathbb{P}(C_W X)} \to 0 \quad .$$

Thus $\mathbb{P}(C_W X)$ is cut out by a section of $N_X V \otimes \mathcal{O}(1)$, which must be regular since the bundle has the right dimension and the embedding of $\mathbb{P}(C_W X)$ in $\mathbb{P}(N_W V)$ is regular. Therefore (using notation rather freely) if r is the codimension of X in V:

$$\begin{aligned} s(W, X) &= \sum_{i\geq 0} c_1(\mathcal{O}(1))^i \mathbb{P}(C_W X \oplus 1) \\ &= \sum_{i\geq 0} c_1(\mathcal{O}(1))^i c_r(N_X V \otimes \mathcal{O}(1)) \cap \mathbb{P}(N_W V \oplus 1) \\ &= \sum_{j\geq 0} c_{r-j}(N_X V) \sum_{i\geq j} c_i(\mathcal{O}(1)^i \mathbb{P}(N_W V \oplus 1) \\ &= c(N_X V) s(W, V). \ \blacksquare \end{aligned}$$

We use this fact in §2.1, in the form:

LEMMA A.2. *Suppose $B \subset V$ are non-singular irreducible varieties and $B \cap F \hookrightarrow V$, $F \hookrightarrow V$ are regular embeddings. Suppose the proper transform of F in the blow-up $B\ell_{B\cap F} V \xrightarrow{\pi} V$ of V along $B \cap F$ is regularly embedded in $B\ell_{B\cap F} V$ as the residual scheme to the exceptional divisor in $\pi^{-1} F$. Then*

$$B \circ F = F \circ B \quad .$$

PROOF: Since B, V are non-singular, $c(N_B V)^{-1} s(B \cap F, B) = s(B \cap F, V)$ (argue as in the proof of Lemma A.1); and $s(B \cap F, V) = c(N_F V)^{-1} s(B \cap F, F)$ by Keel's Lemma. The statement follows immediately. $\blacksquare$

The next Lemma focuses on the case of divisors:

LEMMA A.3. *Let $W \subset Y$ and $W \subset F$ be closed embeddings of pure-dimensional schemes, with W irreducible and Y a Cartier divisor of F, and suppose the proper transform $\widetilde{Y}$ in the blow-up of F along W is the residual scheme of the m^{th} multiple of the exceptional divisor. Then*

$$s(W \cap Y, Y) = (m + Y) \cdot s(W, F) \quad .$$

PROOF: We leave to the reader the case in which W is a component of Y (use Lemma 4.2 in [**F**]). If W is not a component of Y, then let $B\ell_W F \xrightarrow{\pi} F$ be the blow-up of F along W, and let E be the exceptional divisor. Then $E \cap \widetilde{Y}$ is the exceptional divisor of the blow-up of Y along $W \cap Y$, so that (by Corollary 4.2.2 in [**F**])

$$\begin{aligned} s(W \cap Y, Y) &= \pi_* \sum_{k\geq 1} (-1)^{k-1} (E \cap \widetilde{Y})^k \\ &= \pi_* \sum_{k\geq 1} (-1)^{k-1} E^k \cdot \widetilde{Y} \\ &= \pi_* \sum_{k\geq 1} (-1)^{k-1} E^k \cdot (\pi^* Y - mE) \\ &= m\pi_* \sum_{k\geq 1} (-1)^k E^{k+1} + \pi_*(\pi^* Y \cdot \sum_{k\geq 1} (-1)^{k-1} E^k) \\ &= (m + Y) \cdot \pi_* \sum_{k\geq 1} (-1)^{k-1} E^k \quad \text{by the projection formula} \\ &= (m + Y) s(W, F). \ \blacksquare \end{aligned}$$

Applying Lemma A.3 to the case in which $W = B \cap F$, with $B, F \subset V$ as in the beginning of this appendix, we get

$$B \circ Y = (m + Y)(B \circ F) \quad ,$$

which is the form we mainly need in §2.

Finally, we need two results about proper transforms. The first is Fulton's 'blow-up formula':

LEMMA A.4. *Let V be a variety, $B \hookrightarrow V$ a regular embedding, and let $F \subset V$ be a k-dimensional variety. Let $\widetilde{V}$ be the blow-up of V along B, and let $\widetilde{F}$ be the proper transform of F in $\widetilde{V}$; also, let $j : E \hookrightarrow \widetilde{V}$ be the exceptional divisor. Then (omitting pull-back notations)*

$$[\widetilde{F}] = [F] - j_* \left\{ \frac{B \circ F}{1 + E} \right\}_k \quad .$$

PROOF: This is Theorem 6.7 in [**F**]; or, set $r = 1$ in the Claim in [**A1**], Theorem II. ∎

The second computes the first Chern class of the normal bundle of a proper transform:

LEMMA A.5. *In the above situation, suppose the embeddings $B \cap F \hookrightarrow B$, $B \cap F \hookrightarrow F$ are regular. Then $\widetilde{F} \hookrightarrow \widetilde{V}$ is a regular embedding; and if $r = \mathrm{codim}_V B$, $s = \mathrm{codim}_F (B \cap F)$, then (omitting pull-backs)*

$$c_1(N_{\widetilde{F}} \widetilde{V}) = c_1(N_F V) - (r - s)E \quad .$$

PROOF: We leave the first claim to the reader. For the relation between Chern classes, clearly $c_1(N_{\widetilde{F}} \widetilde{V}) = c_1(N_F V) - kE$ for some k; to show $k = r - s$, we restrict to E. The class $c(N_{\widetilde{F}} \widetilde{V})$ restricts to $c(N_{E \cap \widetilde{F}} E)$, and $E = \mathbb{P}(N_B V)$, $E \cap \widetilde{F} = \mathbb{P}(N_{B \cap F} F)$; so chasing the Euler sequences for $\mathbb{P}(N_B V)$, $\mathbb{P}(N_{B \cap F} F)$ gives

$$c(N_{E \cap \widetilde{F}} E) = c(N_{B \cap F} B) c \left(\frac{N_F V}{N_{B \cap F} B} \otimes \mathcal{O}(1) \right) \quad ,$$

from which it follows that k equals the rank of $N_F V / N_{B \cap F} B$, i.e. $r - s$. ∎

Typically, to get into the hypotheses of this Lemma we have to restrict to open subsets of $V, \widetilde{V}$. However, since the statement only deals with the *first* Chern class, this will work since the open sets (implicitly) considered will always be the complement of subvarieties of codimension at least two.

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Mathematics Department, Oklahoma State University, Stillwater OK 74078