Linear orbits of smooth plane curves

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§1. INTRODUCTION

Consider a general codimension-8 linear subspace of the $\mathbb{P}^{14}$ parametrizing plane quartic curves. There is a generically finite dominant rational map from this $\mathbb{P}^6$ to the moduli space of curves of genus 3; what is the degree of this map?

To approach this kind of questions, we embark in this paper on the study of the natural action of PGL(3), the group of automorphisms of $\mathbb{P}^2$, on the projective space $\mathbb{P}^N$ parametrizing plane curves of degree d (thus $N = d(d+3)/2$). We are concerned here with the orbits of (points corresponding to) smooth plane curves C of degree $d \geq 3$. These orbits O_C are 8-dimensional quasi-projective (in fact, affine) varieties. Their closures $\overline{O_C}$ (in $\mathbb{P}^N$) are 8-dimensional projective varieties, and one easily understands that the answer to the above question is nothing but the *degree* of $\overline{O_C}$ for the general plane quartic curve C. In §2 we explicitly construct a resolution of these varieties, which we use in §3 to compute their degree (for every smooth plane curve C of degree $d \geq 3$). In fact the construction gives more naturally the so-called *predegree* of the orbit closure $\overline{O_C}$: that is, the degree of $\overline{O_C}$ multiplied by the order of the PGL(3)-stabilizer of C. It turns out that, for a smooth curve C of degree d, the predegree depends only on d and the nature of the flexes of C. As is illustrated in §3.6, this has nice consequences related to the automorphism groups of smooth plane curves.

We now describe the contents of this paper more precisely. Associated with each plane curve C is a natural map $\mathrm{PGL}(3) \to \mathbb{P}^N$ with image O_C; we view this as a rational map ϕ_C from the $\mathbb{P}^8$ of 3×3 matrices to $\mathbb{P}^N$. In §2 our object is to resolve this map by a sequence of blow-ups over $\mathbb{P}^8$, in fact constructing a non-singular compactification of PGL(3) that dominates $\overline{O_C}$. For a smooth C, the base locus of ϕ_C is a subvariety of $\mathbb{P}^8$ isomorphic to $\mathbb{P}^2 \times C$, thus smooth; after blowing up this support, we find that the support of the base locus of the induced rational map from the blow-up to $\mathbb{P}^N$ is again smooth: and we choose it as the center of a second blow-up. Just continuing this process (which requires a fair amount of bookkeeping) gives a good resolution of the map. We find that the number of blow-ups needed equals the maximum order of contact of C with a line: so, for example, three blow-ups suffice for the general curve.

Having resolved the map ϕ_C, we compute in §3 the predegree of the closure of the orbit of C as the 8-fold self-intersection of the pull-back of the class of a 'point-condition', i.e., a hyperplane in $\mathbb{P}^N$ parametrizing the curves passing through a given point of $\mathbb{P}^2$. The main tool is an intersection formula for blow-ups from [**Aluffi1**]; to apply this formula, we extract from the geometry of the blow-ups detailed in §2 the relevant intersection-theoretic information, and particularly the normal bundles and intersection rings of the centers of the various blow-ups. This leads to explicit formulas for the predegree of $\overline{O_C}$ in terms of the degree of C and of four numbers encoding the number and type of the flexes of C. For example, the answer to the question posed in the beginning (that is, the degree for a general quartic) is 14,280 ($d = 4$ in the Corollary in §3.5).

Besides the applications to automorphism groups of plane curves already alluded to, and more examples given in §3.6 (e.g., we compute the degree of the trisecant variety to the d-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^N$), the computation of the degree of $\overline{O_C}$ also has some enumerative significance: it gives the number of translates of C that pass through 8 points in general position. On a more global level, the degree of the orbit closure of a general plane curve of degree d equals the degree of the natural map from a general codimension-8 linear subspace of $\mathbb{P}^N$ to the moduli space of smooth plane curves of degree d. A study of the boundaries of orbits and of 1-dimensional families of orbits reveals where this map is proper: these matters will be treated in a sequel to this paper. Also, we hope to be able to unify some scattered results we have concerning orbits of *singular* curves; and we plan to study the singular locus of orbit closures.

Excellent practice to become familiar with the techniques of the paper is to apply them to the easier case of the action of the group PGL(2) on the spaces $\mathbb{P}^d$ parametrizing d-tuples of points on a line. Only one blow-up of the $\mathbb{P}^3$ of 2×2 matrices is needed in this case, and one finds the following: if the d-tuple consists of points $p_1, \dots, p_s$, with multiplicities m_i (so that $\sum_{i=1}^s m_i = d$), and one puts $m^{(2)} = \sum m_i^2$, $m^{(3)} = \sum m_i^3$, then the predegree of its orbit-closure equals

$$d^3 - 3dm^{(2)} + 2m^{(3)},$$

so it depends only on d and on $m^{(2)}$, $m^{(3)}$ (one should compare this result with the degree of the orbit-closure of a smooth plane curve as computed in Theorem III(B)). Details of this computation, together with a discussion of the boundary and of 1-dimensional families of orbits, and multiplicity results, can be found in [**Aluffi-Faber**]. We should point out that in this case the degree can also be computed by using simple combinatorics.

Finally, to attract the attention of people working in representation theory, we remark that we deal here with the orbits of *general* vectors in *one* of the standard representations of *one* of the classical algebraic groups. Can these questions be approached in a more general context? From this point of view, a whole lot of work remains to be done.

ACKNOWLEDGEMENTS. We thank the University of Chicago, Oklahoma State University, the Universität Erlangen-Nürnberg and the Max-Planck-Institut für Mathematik for hospitality at various stages of this work. Discussions with many of our colleagues at these and other institutions were helpful. We also acknowledge partial support by the N.S.F. Finally, we are grateful to F. Schubert for writing his wonderful Fantasia in F minor for piano duet.

§2. A BLOW-UP CONSTRUCTION

In this section we construct a smooth projective variety surjecting onto the orbit closure $\overline{O_C}$ of a *smooth* plane curve $C \in \mathbb{P}^N = \mathbb{P}^{\frac{d(d+3)}{2}}$, where $d \geq 3$. As we will see, the construction depends essentially on the number and type of *flexes* of C.

Fix coordinates $(x_0 : x_1 : x_2)$ of $\mathbb{P}^2$, and assume the degree-d curve C has equation

$$F(x_0, x_1, x_2) = 0 \quad .$$

Consider the projective space $\mathbb{P}^8 = \mathbb{P}\mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3)$ of (homogeneous) 3×3 matrices $\alpha = (\alpha_{ij})_{i,j=0,1,2}$. So $\mathbb{P}^8$ is a compactification of $\mathrm{PGL}(3) = \{\alpha \in \mathbb{P}^8 : \deg\alpha \neq 0\}$. To ease notations, in this section we will refer to a point in $\mathbb{P}^8$ and to any 3×3 matrix representing it by the same term; in the same vein, for $\alpha \in \mathbb{P}^8$ we will call '$\ker\alpha$' the linear subspace of $\mathbb{P}^2$ on which the map determined by α is not defined, '$\mathrm{im}\,\alpha$' will be the image of this map, and the rank '$\mathrm{rk}\,\alpha$' of α will be $1+\dim(\mathrm{im}\,\alpha)$. So:

$$\alpha \in \mathrm{PGL}(3) \iff \ker\alpha = \emptyset \iff \mathrm{im}\,\alpha = \mathbb{P}^2 \iff \mathrm{rk}\,\alpha = 3.$$

The curve C determines a rational map

$$c : \mathbb{P}^8 \dashrightarrow \mathbb{P}^N$$

as follows: for $\alpha \in \mathbb{P}^8$, let $c(\alpha)$ be the curve defined by the degree-d polynomial equation $F(\alpha(x_0, x_1, x_2)) = 0$. So $c(\alpha)$ is defined as long as $F(\alpha(x_0, x_1, x_2))$ doesn't vanish identically; i.e., precisely if $\mathrm{im}\,\alpha \not\subset C$.

If $\alpha \in \mathrm{PGL}(3)$, then $c(\alpha)$ is the translate of C by α; therefore, $c(\mathrm{PGL}(3))$ is just the orbit O_C of C in $\mathbb{P}^N$ for the natural action of $\mathrm{PGL}(3)$.

As an alternative description for the map c, consider for any point $p \in \mathbb{P}^2$ the equation

$$F(\alpha(p)) = 0 \quad .$$

As an equation 'in p', this defines the translate $c(\alpha)$; as an equation 'in α' this defines the hypersurface of $\mathbb{P}^8$ consisting of all α that map p to a point of C. We will call these hypersurfaces, that will play an important role in our discussion, 'point-conditions'. The rational map defined above is clearly the map defined by the linear system generated by the point-conditions on $\mathbb{P}^8$.

Our task here is to resolve the indeterminacies of the map $c : \mathbb{P}^8 \dashrightarrow \mathbb{P}^N$, by a sequence of blow-ups at smooth centers: we will get a smooth projective variety $\widetilde{V}$ filling a commutative diagram

$$\begin{tikzcd}
\mathrm{PGL}(3) \arrow[r, phantom, "\subset"] \arrow[d, equal] & \widetilde{V} \arrow[r, "\widetilde{c}"] \arrow[d, "\pi"'] & \mathbb{P}^N \arrow[d, equal] \\
\mathrm{PGL}(3) \arrow[r, phantom, "\subset"] & \mathbb{P}^8 \arrow[r, dashed, "c"] & \mathbb{P}^N
\end{tikzcd}$$

The image of $\widetilde{V}$ in $\mathbb{P}^N$ by $\widetilde{c}$ will then be the orbit closure $\overline{O_C}$. In §3 we will use $\widetilde{c}$ to pull-back questions about $\overline{O_C}$ to $\widetilde{V}$; the explicit description of $\widetilde{V}$ obtained in this section will enable us to answer these questions.

The plan is to blow-up the support of the base locus of c; we will get a variety V_1 and a rational map $c_1 : V_1 \dashrightarrow \mathbb{P}^N$. We will then blow-up the support of the base locus of c_1, getting a variety V_2 and a rational map $c_2 : V_2 \dashrightarrow \mathbb{P}^N$; in the case we are considering here (i.e., the curve C is smooth to start with), repeating this process yields eventually a variety $\widetilde{V}$ as above. The support of the first base locus is in fact a copy of $\mathbb{P}^2 \times C$ in $\mathbb{P}^8$ (see §2.1); if $(k, q) \in \mathbb{P}^2 \times C$, and c_i denotes the map obtained at the i-th stage, we will find that c_i *still has indeterminacies over* (k, q) *if and only if the tangent line to* C *at* q *intersects* C *at* q *with multiplicity* $> i$.

So, for example, if C has only simple flexes then the map c_3 is regular (Proposition 2.9); and in general the number of blow-ups needed equals the highest possible multiplicity of intersection of a line with C.

We should point out that (even for smooth C) this is not the only way to construct a variety $\widetilde{V}$ as above: in fact, a different sequence of blow-ups is the one that seems to generalize naturally to approach the same problem for singular C.

§**2.1. The first blow-up.** The set of rank-1 matrices in $\mathbb{P}^8$ is the image of the Segre embedding

$$\check{\mathbb{P}}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$$

given in coordinates by

$$((k_0 : k_1 : k_2), (q_0 : q_1 : q_2)) \mapsto \begin{pmatrix} k_0q_0 & k_1q_0 & k_2q_0 \\ k_0q_1 & k_1q_1 & k_2q_1 \\ k_0q_2 & k_1q_2 & k_2q_2 \end{pmatrix}$$

where $k_0x_0 + k_1x_1 + k_2x_2 = 0$ is the kernel of the matrix, and $(q_0 : q_1 : q_2)$ is its image. Intrinsically, this is just the map induced from the map

$$\begin{aligned} \check{\mathbb{C}}^3 \oplus \mathbb{C}^3 &\to \check{\mathbb{C}}^3 \otimes \mathbb{C}^3 = \mathrm{Hom}(\mathbb{C}^3, \mathbb{C}^3) \\ (f, u) &\mapsto f \otimes u \end{aligned}$$

We have already observed that the map $c : \mathbb{P}^8 \dashrightarrow \mathbb{P}^N$ is *not* defined at $\alpha \in \mathbb{P}^8$ precisely when $\operatorname{im}\alpha \subset C$; if C is smooth (therefore irreducible), this means that the image of α is a point of C. Therefore:
the support of the base locus of c is the image of $\check{\mathbb{P}}^2 \times C$ in $\mathbb{P}^8$ via the Segre embedding identifying $\check{\mathbb{P}}^2 \times \mathbb{P}^2$ with the set of rank-1 matrices.

In particular, the support of the base locus of c is *smooth*, since C is. We let then $B = \check{\mathbb{P}}^2 \times C$, and we let $V_1 \xrightarrow{\pi_1} \mathbb{P}^8$ be the blow-up of $\mathbb{P}^8$ along B. Since $B \cap \mathrm{PGL}(3) = \emptyset$, V_1 contains a dense open set which we can identify with $\mathrm{PGL}(3)$. Also, the linear system generated by the proper transforms in V_1 of the point-conditions (which we will call 'point-conditions in V_1'), defines a rational map $c_1 : V_1 \dashrightarrow \mathbb{P}^N$ making the diagram

$$\begin{array}{ccccc} \mathrm{PGL}(3) & \subset & V_1 & \overset{c_1}{\dashrightarrow} & \mathbb{P}^N \\ \| & & {\scriptstyle\pi_1}\downarrow & & \| \\ \mathrm{PGL}(3) & \subset & \mathbb{P}^8 & \overset{c}{\dashrightarrow} & \mathbb{P}^N \end{array}$$

commutative. The exceptional divisor E_1 in V_1 is the projectivized normal bundle of B in $\mathbb{P}^8$: $E_1 = \mathbb{P}(N_B\mathbb{P}^8)$. We will show now that *the base locus of c_1 is supported on a $\mathbb{P}^1$-subbundle of E_1 over B.*

Let (k, q) be a point of $B = \check{\mathbb{P}}^2 \times C$: i.e., a rank-1 $\alpha \in \mathbb{P}^8$ with $\ker\alpha = k$, $\operatorname{im}\alpha = q \in C$. Also, let ℓ be the line tangent to C at q, let p be a point of $\mathbb{P}^2$, and denote by P the point-condition in $\mathbb{P}^8$ corresponding to p.

Lemma 2.1. *(i) The tangent space to B at (k, q) consists of all $\varphi \in \mathbb{P}^8$ such that $\operatorname{im}\varphi \subset \ell$ and $\varphi(k) \subset q$.*

(ii) P is non-singular at (k,q), and the tangent space to P at (k,q) consists of all $\varphi \in \mathbb{P}^8$ such that $\varphi(p) \subset \ell$.

We are using our notations rather freely here. For example, in (i) $\alpha = (k,q)$ is in the tangent space since $\alpha(k) = \emptyset$ (as α is not defined along k).

Proof: (i) The tangent space to B at (k,q) is spanned by the plane $\{(k',q) \in B : k' \in \check{\mathbb{P}}^2\} = \{\varphi \in \mathbb{P}^8 : \operatorname{im}\varphi = q\}$ and by the line $\{(k,q') \in B : q' \in \ell\} = \{\varphi \in \mathbb{P}^8 : \ker\varphi = k, \operatorname{im}\varphi \in \ell\}$. Both these subspaces of $\mathbb{P}^8$ are contained in $\{\varphi \in \mathbb{P}^8 : \operatorname{im}\varphi \subset \ell, \varphi(k) \subset q\}$; since this latter has clearly dimension 3, we are done.

(ii) For $\alpha = (k,q)$ and $\varphi \in \mathbb{P}^8$ consider the line $\alpha + \varphi\, t$. Restricting the equation for P to this line gives the polynomial equation in t

$$F((\alpha + \varphi\, t)(p)) = 0 \quad \text{, i.e.}$$
$$F(\alpha(p)) + \sum_i \left(\frac{\partial F}{\partial x_i}\right)_{\alpha(p)} \varphi_i(p)\, t + \cdots = 0$$

(where $\varphi_i(p)$ denotes the i-th coordinate of $\varphi(p)$).

$F(\alpha(p)) = 0$ since $\operatorname{im}\alpha = q \in C$; the line is tangent to P at α when the linear term also vanishes, i.e. if $\sum_i (\partial F/\partial x_i)_q \varphi_i(p) = 0$. This says precisely $\varphi(p) \subset \ell$, as claimed.

P is non-singular at α because any φ *not* satisfying the condition $\varphi(p) \subset \ell$ gives a line $\alpha + \varphi\, t$ intersecting P with multiplicity 1 at α, by the above computation. ∎

With the same notations, the tangent space to $\check{\mathbb{P}}^2 \times \mathbb{P}^2$ at α consists of all φ with $\varphi(\ker\alpha) \subset \operatorname{im}\alpha$ (intrinsically, all transformations φ inducing a map $\operatorname{coim}\alpha \longrightarrow \operatorname{coker}\alpha$).

The set of all φ such that $\operatorname{im}\varphi \subset \ell$ forms (for any α) a 5-dimensional space containing the tangent space to B at α, and therefore determines a 2-dimensional subspace of the fiber of $N_B\mathbb{P}^8$ over α. As α moves in B we get a rank-2 subbundle of $N_B\mathbb{P}^8$, and hence a $\mathbb{P}^1$-subbundle of $E_1 = \mathbb{P}(N_B\mathbb{P}^8)$, which we denote B_1. Notice that B_1 is non-singular, as a $\mathbb{P}^1$-bundle over the non-singular B.

Proposition 2.2. *The base locus of the map $c_1 : V_1 \dashrightarrow \mathbb{P}^N$ is supported on B_1.*

Proof: Since c_1 is defined by the linear system generated by all point-conditions in V_1, we simply need to show that the intersection of all point-conditions in V_1 is set-theoretically B_1. This assertion can be checked fiberwise over $\alpha = (k,q) \in B$; so all we need to observe is that the intersection of the tangent spaces to all point-conditions at α consists (by Lemma 2.1 (ii)) of the $\varphi \in \mathbb{P}^8$ such that $\varphi(p) \subset \ell$ for *all* p; i.e., the 5-dimensional space used above to define B_1. ∎

If $P_1^{(p)}$ denotes the point-condition in V_1 corresponding to $p \in \mathbb{P}^2$, we have just shown $\bigcap_{p\in\mathbb{P}^2} P_1^{(p)}$ is supported on B_1. The proof says a little more:

Remark 2.3. $\bigcap_{p\in\mathbb{P}^2} P_1^{(p)} \cap E_1 = B_1$ *(scheme-theoretically).*

Indeed on each fiber of E_1 (say over $\alpha \in B$) the fiber of B_1, a linear subspace, is cut out by the fibers of the $P_1^{(p)} \cap E_1$, linear subspaces themselves; and the situation clearly globalizes as α moves in B.

§**2.2. The second blow-up.** Let $V_2 \xrightarrow{\pi_2} V_1$ be the blow-up of V_1 along B_1. The new exceptional divisor is $E_2 = \mathbb{P}(N_{B_1}V_1)$; call 'point-conditions in V_2' the proper transforms of the point-conditions of V_1. The linear system generated by the point-conditions defines a rational map $c_2 : V_2 \dashrightarrow \mathbb{P}^N$; again, we obtain a diagram

$$\begin{array}{ccccc} \mathrm{PGL}(3) & \subset & V_2 & \overset{c_2}{\dashrightarrow} & \mathbb{P}^N \\ \Big\| & & \Big\downarrow{\scriptstyle \pi_2\circ\pi_1} & & \Big\| \\ \mathrm{PGL}(3) & \subset & \mathbb{P}^8 & \overset{c}{\dashrightarrow} & \mathbb{P}^N \end{array}$$

and we proceed to determine the support of the base locus of c_2.

Let $\widetilde{E}_1$ be the proper transform of E_1 in V_2. Then

Lemma 2.4. *The base locus of c_2 is disjoint from $\widetilde{E}_1$.*

Proof: This is basically a reformulation of Remark 2.3: $\widetilde{E}_1$ is the blow-up of E_1 along B_1, and B_1 is cut out scheme-theoretically by the intersections of E_1 with the point-conditions of V_1. So the intersection of the point-conditions in V_2 must be empty along $\widetilde{E}_1$, which is the claim. ∎

Lemma 2.4 reduces the determination of the support of the base locus of c_2 to a computation in $\mathbb{P}^8$. Denote by $\mathcal{B}$ the scheme-theoretic intersection of the point-conditions in $\mathbb{P}^8$, so the support of $\mathcal{B}$ is B. For $\alpha \in B$, let $th_\alpha(\mathcal{B})$ be the maximum length of the intersection with $\mathcal{B}$ of the germ of a smooth curve centered at α and transversal to B (the 'thickness' of $\mathcal{B}$ at α, in the terminology of [**Aluffi2**]).

Lemma 2.5. *The base locus of c_2 is disjoint from $(\pi_2 \circ \pi_1)^{-1}\alpha$ if $th_\alpha(\mathcal{B}) \le 2$.*

Proof: The base locus of c_2 is the intersection of all point-conditions in V_2, i.e. the set of all directions normal to B_1 and tangent to all point-conditions in V_1. Let then $\gamma(t)$ be a smooth curve germ centered at a point of B_1 above α, transversal to B_1, and tangent to all point-conditions in V_1. By Lemma 2.4, γ is transversal to E_1; therefore $\pi_1(\gamma(t))$ is a *smooth* curve germ centered at α and *transversal* to B. Since $\gamma(t)$ intersects all point-conditions in V_1 with multiplicity 2 or more, $\pi_1(\gamma(t))$ must intersect all point-conditions in $\mathbb{P}^8$ with multiplicity 3 or more; $\mathcal{B}$ is the intersection of all point-conditions in $\mathbb{P}^8$, so this forces $th_\alpha(\mathcal{B}) \ge 3$. ∎

Now the key computation is

Lemma 2.6. *If $\alpha = (k, q) \in B$, and ℓ is the line tangent to C at q, then $th_\alpha(\mathcal{B})$ equals the intersection multiplicity of ℓ and C at q.*

Proof: Let m be the intersection multiplicity of ℓ and C at q. To show $th_\alpha(\mathcal{B}) \ge m$, we just have to produce a curve normal to B and intersecting all point-conditions with multiplicity at least m at α; such is the line $\alpha + \varphi\, t$, with $\varphi \in \mathbb{P}^8$ such that $\operatorname{im}\varphi = \ell$ and $\varphi(k) \neq q$. Indeed, the last condition guarantees normality (Lemma 2.1 (i)); and, for general p, $q = \alpha(p)$ and $\varphi(p)$ span ℓ: so $F((\alpha + \varphi\, t)(p))$ is just the restriction of F to a parametrization of ℓ, and it must vanish exactly m times at $t = 0$. Notice that these directions are precisely those defining B_1.

To show $th_\alpha(\mathcal{B}) \le m$, let $\gamma(t)$ be any smooth curve germ normal to B and centered at α; we have to show that γ intersects some point-condition with multiplicity $\le m$ at α. In an affine open of $\mathbb{P}^8$ containing α, write

$$\gamma(t) = \alpha + \varphi\, t + \dots \quad .$$

The equation for the point-condition corresponding to p restricts on γ to

$$F((\alpha+\varphi\, t+\dots)(p)) = F(\alpha(p)) + \sum_i \left(\frac{\partial F}{\partial x_i}\right)_{\alpha(p)} \varphi_i(p)\, t + \cdots = 0 \quad ,$$

where $\varphi_i(p)$ denotes the i-th coordinate of $\varphi(p)$. The coefficient of t^m in this expansion is

$$(*) \quad \frac{1}{m!} \sum_{i_1,\dots,i_m} \left(\frac{\partial^m F}{\partial x_{i_1}\cdots\partial x_{i_m}}\right)_{\alpha(p)} \varphi_{i_1}(p)\cdots\varphi_{i_m}(p)$$
$$+ \text{ terms involving derivatives of lower order,}$$

and to conclude the proof we have to show that for some p this term doesn't vanish.

To see this, observe that since ℓ and C intersect with multiplicity exactly m at q, then the form

$$\sum_{i_1,\dots,i_m} \left(\frac{\partial^m F}{\partial x_{i_1}\cdots\partial x_{i_m}}\right)_{\alpha(p)} x_{i_1}\cdots x_{i_m}$$

doesn't vanish identically on ℓ; since $\varphi(\ker\alpha) \not\subset q$ (γ is normal to B), this implies that the summand

$$\frac{1}{m!} \sum_{i_1,\dots,i_m} \left(\frac{\partial^m F}{\partial x_{i_1}\cdots\partial x_{i_m}}\right)_{\alpha(p)} \varphi_{i_1}(p)\cdots\varphi_{i_m}(p)$$

vanishes exactly $d-m$ times along the line $k = \ker\alpha$ (as a function of p). But since all the other summands in (*) involve derivatives of order $< m$, they vanish with order $> d-m$ along k. Therefore the order of vanishing of (*) along k must be exactly $d-m$, and in particular (*) can't be identically 0, as we claimed. ∎

We adopt the following convention:

Definition. *A point q of C is a 'flex of order r' if the line tangent to C at q intersects C at q with multiplicity $r+2$. We will say that q is a 'flex' of C if $r \geq 1$, and that q is a 'simple flex' if $r = 1$.*

Now we observe that there is a section $s : B_1 \longrightarrow E_2$: for $\alpha_1 \in B_1$, let $\alpha = \pi_1(\alpha_1) \in B$, say $\alpha = (k, q)$, and let ℓ be the line tangent to C at q. By the construction of B_1, there is a matrix $\varphi \in \mathbb{P}^8$ with $\operatorname{im}\varphi \subset \ell$ such that α_1 is the intersection of E_1 and the proper transform of the line $\alpha + \varphi\, t$ in V_1; now let $s(\alpha_1)$ be the intersection of E_2 and the proper transform of the line $\alpha+\varphi\, t$ in V_2 (it is clear that $s(\alpha_1)$ does not depend on the specific φ chosen to represent α_1).

Let B_2 be the image via s of the set $\{\alpha_1 \in B_1 : q \text{ is a flex of } C\}$. Thus B_2 consists of a number of smooth three-dimensional components, one for each flex of C: each component maps isomorphically to a $\mathbb{P}^1$-bundle over one of the planes $\{(k,q) \in B : q \text{ is a flex of } C\}$.

Proposition 2.7. *The base locus of the map $c_2 : V_2 \dashrightarrow \mathbb{P}^N$ is supported on B_2.*

Proof: Let $\alpha_1 \in B_1$, and $\alpha = (k, q)$ the image of α_1 in B, as above. Consider the intersection of the base locus of c_2 with the fiber $\pi_2^{-1}(\alpha_1) \cong \mathbb{P}^3$. By Lemma 2.5

and 2.6 this is *empty* if q is not a flex of C; even if q is a flex of C, the intersection is a linear subspace of $\mathbb{P}^3$ missing a $\mathbb{P}^2$ (by Lemma 2.4), thus it consists of at most *one point*. Thus all we have to show is that $s(\alpha_1)$ *is* in the base locus of c_2 if q is a flex of C (of order $r \geq 1$). But, as observed in the proof of Lemma 2.6, the line $\alpha+\varphi\, t$ determining α_1 intersects each point-condition in $\mathbb{P}^8$ with multiplicity at least $r+2 \geq 3$; therefore the proper transform of $\alpha+\varphi\, t$ is *tangent* to all point-conditions in V_1, and it follows that $s(\alpha_1) \in$ all point-conditions in V_2, as needed. ∎

§**2.3. The third blow-up.** Let $V_3 \xrightarrow{\pi_3} V_2$ be the blow-up of V_2 along B_2. The new exceptional divisor is E_3; the 'point-conditions of V_3' are the proper transforms of the point-conditions of V_2. The linear system generated by the point-conditions defines a rational map $c_3 : V_3 \dashrightarrow \mathbb{P}^N$, making the diagram

$$\begin{array}{ccccc} \mathrm{PGL}(3) & \subset & V_3 & \overset{c_3}{\dashrightarrow} & \mathbb{P}^N \\ \Big\| & & {\scriptstyle \pi_3\circ\pi_2\circ\pi_1}\big\downarrow & & \Big\| \\ \mathrm{PGL}(3) & \subset & \mathbb{P}^8 & \overset{c}{\dashrightarrow} & \mathbb{P}^N \end{array}$$

commute. We will show now that c_3 is a *regular map* if all the flexes of C are simple, so that in this case V_3 is the variety we are looking for. For each flex of order > 1, we will find a four-dimensional component in the base locus of c_3, and more blow-ups will be needed.

Call $\mathcal{B}_2$ the scheme-theoretic intersection of the point-conditions in V_2, so $\mathcal{B}_2$ is supported on B_2. For $\alpha_2 \in B_2$, define the thickness $th_{\alpha_2}(\mathcal{B}_2)$ of $\mathcal{B}_2$ at α_2 as we did above for $th_\alpha(\mathcal{B})$. Also, let $\alpha = (k, q)$ be the image of α_2 in B. With these notations:

Lemma 2.8. *If q is an flex of order r of C, then $th_{\alpha_2}(\mathcal{B}_2) = r$.*

Proof: We have to show that if $\gamma(t)$ is a smooth curve germ in V_2, centered at α_2 and transversal to B_2, then the maximum length of the intersection of $\mathcal{B}_2$ and γ at $t = 0$ is precisely r.

Suppose first that γ is transversal to E_2: then, as argued in the proof of Lemma 2.5, the image of γ in $\mathbb{P}^8$ is a smooth curve germ centered at α and transversal to B: by Lemma 2.6, the length of the intersection of $\mathcal{B}$ and this curve is at most $r+2$; it follows that the maximum length of the intersection of $\mathcal{B}_2$ and such γ's is indeed r (attained for example by the proper transform of $\alpha + \varphi\, t$, with φ as in the proof of Lemma 2.6).

Thus we may assume that γ is tangent to E_2, and we have to show that

Claim. *$\mathcal{B}_2 \cap \gamma(t)$ vanishes at most r times at $t = 0$.*

This is a lengthy but straightforward coordinate computation, which we leave to the reader. The outcome is that the maximum length is r, and it is attained in the direction normal to B_2 in the section $s(B_1) \subset E_2$ defined in §2.2. ∎

The next results are now easy consequences.

Proposition 2.9. *If all flexes of C are simple, then the map $c_3 : V_3 \dashrightarrow \mathbb{P}^N$ is regular.*

Proof: We have to show that c_3 has no base locus, i.e. that the intersection of all point-conditions in V_3 is *empty*. But a point in the intersection of all point-conditions in V_3 would determine a direction normal to B_2 and *tangent* to all point-conditions in V_2; the thickness of $\mathcal{B}_2$ would then be ≥ 2 at some point. By Lemma 2.8, if all flexes of C are simple (i.e., of order 1) the thickness of $\mathcal{B}_2$ is precisely 1 everywhere on B_2, so this can't happen. ∎

By Proposition 2.9, we are done in the case when C has only simple flexes: V_3 is the variety $\widetilde{V}$ we meant to construct. We will show now that for each flex of C of order $r > 1$, the base locus of c_3 has a smooth four-dimensional connected component.

Let $\alpha_2 \in B_2$, mapping to $\alpha = (k, q)$ in B, and assume q is a flex of C of order $r > 1$. B_2 is 3-dimensional, so the fiber $\pi_3^{-1}(\alpha_2)$ of $E_3 = \mathbb{P}(N_{B_2}V_2)$ over α_2 is a $\mathbb{P}^4$. We have two special points in this $\mathbb{P}^4$, namely the point determined by the proper transform of the line $\alpha + \varphi t$ used in §2.2 to define s, and the direction normal to B_2 in the section $s(B_1)$. We have seen in the proof of Lemma 2.8 that the length of the intersection of these directions with $\mathcal{B}_2$ is exactly r; also, these points are *distinct* for all α_2 (since one of them corresponds to a direction contained in E_2, while the other comes from a direction transversal to E_2), so they determine a $\mathbb{P}^1$ in the fiber $\pi_3^{-1}(\alpha_2)$. As α_2 moves in the component of B_2 over q, this $\mathbb{P}^1$ traces a $\mathbb{P}^1$-bundle over that component, a smooth four-dimensional subvariety $B_3^{(q)}$ of E_3. Call B_3 the union of all these (disjoint) subvarieties of E_3, arising from non-simple flexes of C.

Proposition 2.10. *The base locus of the map $c_3 : V_3 \dashrightarrow \mathbb{P}^N$ is supported on B_3.*

Proof: The argument here is somewhat analogous to the argument in the proof of 2.7. We have to show that in each fiber $\pi_3^{-1}(\alpha_2) \cong \mathbb{P}^4$ as above, the intersection of all point-conditions is supported on the specified $\mathbb{P}^1$. Observe that each point-condition determines a hyperplane in this $\mathbb{P}^4$, so that the intersection of the base locus of c_3 with $\pi_3^{-1}(\alpha_2)$ must be a linear subspace of this $\mathbb{P}^4$. Secondly, for the same reason, no directions tangent to the fiber of E_2 containing α_2 can be tangent to all point-conditions in V_2. The fibers of E_2 are three-dimensional and transversal to B_2, thus this shows that the base locus of c_3 must miss a $\mathbb{P}^2$ in the fiber $\pi_3^{-1}(\alpha_2)$. Thus, the intersection of the base locus of c_3 with $\pi_3^{-1}(\alpha_2)$ can consist of at most a $\mathbb{P}^1$.

Therefore, we just have to show that the two points of $\pi_3^{-1}(\alpha_2)$ used in the construction of B_3 are contained in all point-conditions of V_3; or, equivalently, the two directions in V_2 used to define these points are *tangent* to all point-conditions in V_2. But this is precisely the result of the computation in the proof of Lemma 2.8: the length of the intersection of these curves with all point-conditions is $r \geq 2$. ∎

§**2.4 Further blow-ups.** As we have seen in §2.3, each non-simple flex q of C gives rise to a smooth four-dimensional component of the support B_3 of the base locus of c_3; and B_3 is the union of all such components. The plan is still to blow-up the support of the base-locus; since the components are disjoint, we can concentrate on a specific one: say $B_3^{(q)}$, corresponding to a flex q of C of order $r \geq 2$.

Let $V_3^{(q)}$ be the complement of all components of B_3 *other than* $B_3^{(q)}$ in V_3. Let $V_4^{(q)} \to V_3^{(q)}$ be the blow-up of $V_3^{(q)}$ along $B_3^{(q)}$; again, the proper transforms in

$V_4^{(q)}$ of the point-conditions define a map $c_4^{(q)} : V_4^{(q)} \dashrightarrow \mathbb{P}^N$. The base locus of $c_4^{(q)}$ might have components over $B_3^{(q)}$, whose union we denote $B_4^{(q)}$; in this case, we will let $V_5^{(q)}$ be the blow-up of $V_4^{(q)}$ along $B_4^{(q)}$. Iterating this process we get a tower of varieties and maps:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
B_{i+1}^{(q)} & \subset & V_{i+1}^{(q)} & \overset{c_{i+1}^{(q)}}{\dashrightarrow} & \mathbb{P}^N \\
\big\downarrow & & \big\downarrow & & \Vert \\
B_i^{(q)} & \subset & V_i^{(q)} & \overset{c_i^{(q)}}{\dashrightarrow} & \mathbb{P}^N \\
\big\downarrow & & \big\downarrow & & \Vert \\
\vdots & & \vdots & & \vdots \\
B_3^{(q)} & \subset & V_3^{(q)} & \overset{c_3}{\dashrightarrow} & \mathbb{P}^N
\end{array}$$

where, inductively for $i \geq 4$: $V_i^{(q)} \longrightarrow V_{i-1}^{(q)}$ is the blow-up of $V_{i-1}^{(q)}$ along $B_{i-1}^{(q)}$; $c_i^{(q)} : V_i^{(q)} \dashrightarrow \mathbb{P}^N$ is defined by the proper transforms in $V_i^{(q)}$ of the point-conditions (i.e., the 'point-conditions in $V_i^{(q)}$'); and (for $i \geq 3$) $B_i^{(q)}$ is the support of the intersection $\mathcal{B}_i^{(q)}$ of the point-conditions in $V_i^{(q)}$ (i.e., the base locus of $c_i^{(q)}$). Also, for $i \geq 3$ let $E_i^{(q)}$ be the exceptional divisor in $V_i^{(q)}$, and let $\widetilde{E}_i^{(q)}$ be the proper transform of $E_i^{(q)}$ in $V_{i+1}^{(q)}$.

Lemma 2.11. *If q is a flex of order $r \geq 2$, then for $3 \leq i \leq r+1$:*

$(1)_i$: *$V_i^{(q)}$ is non-singular*

$(2)_i$: *the composition map $B_i^{(q)} \longrightarrow B_3^{(q)}$ is an isomorphism*

$(3)_i$: *the thickness of $\mathcal{B}_i^{(q)}$ is $r+2-i$ at each point of $B_i^{(q)}$*

$(4)_i$: *$B_{i+1}^{(q)} \cap \widetilde{E}_i^{(q)} = \emptyset$*

Proof: We have $(1)_3$, $(2)_3$ trivially, and $(3)_2$ by Lemma 2.8. Also, since B_3 is cut out by linear spaces in each fiber of E_3, we have $(4)_3$. Now we will show that:

Claim. *For $4 \leq i \leq r+1$, $(1)_{i-1}$, $(2)_{i-1}$, $(3)_{i-2}$ and $(4)_{i-1}$ imply $(1)_i$, $(2)_i$, $(3)_{i-1}$, and $(4)_i$.*

Also, we will show that $(3)_r$, $(4)_{r+1}$ imply $(3)_{r+1}$: this will prove the statement.

Proof of the Claim: In this proof we will drop the $^{(q)}$ notation, to ease the exposition. V_i is then the blow-up of V_{i-1} along B_{i-1}, and these are both non-singular by $(1)_{i-1}$, $(2)_{i-1}$: so V_i must also be non-singular, giving $(1)_i$.

Next, compute the thickness of $\mathcal{B}_{i-1}$: let $\gamma(t)$ be any smooth curve germ transversal to B_{i-1} and centered at any $\alpha_{i-1} \in B_{i-1}$. If γ is tangent to E_{i-1}, then by $(4)_{i-1}$ its proper transform will miss the general point-condition in V_i: i.e., the length of the intersection of $\gamma(t)$ with $\mathcal{B}_{i-1}$ at $t=0$ is 1. If γ is transversal to E_{i-1} (and B_{i-1}), then γ maps down to a *smooth* curve germ γ_* centered at a point of B_{i-2} and transversal to B_{i-2}. By $(3)_{i-2}$, the intersection of γ_* with the point-conditions in V_{i-2} has length at most $r-i+4$: it follows that the intersection of γ with the point-conditions in V_{i-1} has length at most $r-i+3 \geq 2$ (since $i \leq r+1$). Therefore the thickness of $\mathcal{B}_{i-1}$ at α_{i-1} is $r-i+3$, which gives $(3)_{i-1}$.

For $(2)_i$, look at the intersection of B_i with the fiber of E_i over an arbitrary $\alpha_{i-1} \in B_{i-1}$. First we argue this can't be empty: indeed, $th_{\alpha_{i-1}}(\mathcal{B}_{i-1}) = r-i+3 \geq 2$, so through every α_{i-1} in B_{i-1} there are directions tangent to all point-conditions in V_{i-1}. To get $(2)_i$, we need to show that the fiber of B_i over α_{i-1} consists (scheme-theoretically) of a simple point. But this is the intersection of B_i with the fiber of E_i ($\cong \mathbb{P}^3$) over α_{i-1}, thus a nonempty intersection of linear subspaces in $\mathbb{P}^3$ missing a hyperplane (by $(4)_{i-1}$): precisely a point, as needed for $(2)_i$.

Finally, we need $(4)_i$. Once more observe that $\mathcal{B}_i$ intersects each fiber of E_i in an intersection of linear spaces: thus there are no directions in the fibers of E_i and tangent to all point-conditions in V_i. This says that B_{i+1} must avoid the proper transforms in V_{i+1} of all fibers of E_i, and therefore $\widetilde{E}_i$, giving $(4)_i$.

This proves the Claim. The only case not covered yet is $(3)_{r+1}$: to obtain this and conclude the proof of 2.11, apply the same argument as above to $(3)_r$, $(4)_{r+1}$. ∎

Lemma 2.11 describes the sequence of blow-ups over V_3 that takes care of a specific flex q on C of order $r \geq 2$. The case $i = r+1$ of the statement says that the variety $V^{(q)}_{r+1}$ is non-singular, and the base locus of the map $c^{(q)}_{r+1} : V^{(q)}_{r+1} \dashrightarrow \mathbb{P}^N$ is supported on a variety $B^{(q)}_{r+1}$ isomorphic to $B^{(q)}_3$; moreover, for all $\alpha_{r+1} \in B^{(q)}_{r+1}$, we got $th_{\alpha_{r+1}}(\mathcal{B}_{r+1}) = 1$. Let then $V^{(q)}_{r+2} \longrightarrow V^{(q)}_{r+1}$ be the blow-up of $V^{(q)}_{r+1}$ along $B^{(q)}_{r+1}$, and denote by $c^{(q)}_{r+2}$ the rational map $V^{(q)}_{r+2} \dashrightarrow \mathbb{P}^N$ defined by the point-conditions in $V^{(q)}_{r+2}$. Then $V^{(q)}_{r+2}$ is clearly non-singular, and

Corollary 2.12. *$c^{(q)}_{r+2}$ is a regular map.*

Proof: Indeed, the point-conditions in $V^{(q)}_{r+2}$ cannot intersect anywhere along $E^{(q)}_{r+2}$: if they did, any intersection point would correspond to a direction normal to $B^{(q)}_{r+1}$ and tangent to all point-conditions in $V^{(q)}_{r+1}$, and the thickness of $\mathcal{B}^{(q)}_{r+1}$ would be ≥ 2, in contradiction with Lemma 2.11. ∎

By this last result, the sequence of $r-1$ blow-ups over V_3 just described resolves the indeterminacies of $c_3 : V_3 \dashrightarrow \mathbb{P}^N$ over the component $B^{(q)}_3$ of B_3. To resolve *all* indeterminacies of c_3, we just have to apply the construction simultaneously to all components of B_3: build the sequence

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
B_{i+1} & \subset & V_{i+1} & \overset{c_{i+1}}{\dashrightarrow} & \mathbb{P}^N \\
\downarrow & & \downarrow & & \| \\
B_i & \subset & V_i & \overset{c_i}{\dashrightarrow} & \mathbb{P}^N \\
\downarrow & & \downarrow & & \| \\
\vdots & & \vdots & & \vdots \\
B_3 & \subset & V_3 & \overset{c_3}{\dashrightarrow} & \mathbb{P}^N
\end{array}$$

where, for $i \geq 4$, $V_i \longrightarrow V_{i-1}$ is the blow-up of V_{i-1} along B_{i-1}, $c_i : V_i \dashrightarrow \mathbb{P}^N$ is defined by the proper transforms in V_i of the point-conditions, and B_i is the support of the base locus of c_i. By Lemma 2.11 and Corollary 2.12 all V_i's are non-singular, and, for each flex q of C of order r, B_i has either exactly one component mapping isomorphically to $B_3^{(q)}$ if $i \leq r+1$, or *no* component over $B_3^{(q)}$ if $i \geq r+2$.

In particular, this construction will stop! If r is the maximum among the order of the flexes of C, let $\widetilde{V} = V_{r+2}$, $\widetilde{c} = c_{r+2}$, and let π be the composition of the $r+2$ blow-up maps; then we have shown

THEOREM II. $\widetilde{c} : \widetilde{V} \longrightarrow \mathbb{P}^N$ *is a regular map, and the diagram*

$$\begin{array}{ccccc} PGL(3) & \subset & \widetilde{V} & \xrightarrow{\widetilde{c}} & \mathbb{P}^N \\ \| & & {\scriptstyle\pi}\downarrow & & \| \\ PGL(3) & \subset & \mathbb{P}^8 & \overset{c}{\dashrightarrow} & \mathbb{P}^N \end{array}$$

commutes.

which was our objective.

§3. The degree of the orbit closure

In this section we employ the blow-up construction of §2 to compute the degree of the orbit closure $\overline{O_C}$ of a smooth plane curve $C \in \mathbb{P}^N = \mathbb{P}^{\frac{d(d+3)}{2}}$ with at most finitely many automorphisms (if $d = 3$, we should specify 'induced from PGL(3)'. This will be understood in the following). The degree will depend on just *six* natural numbers: the order of the group of automorphisms of C, the degree d of C, and four numbers encoding information about the number and order of the flexes of C. In fact, the blow-up construction of §2 yields most naturally the 'predegree' of O_C:

DEFINITION. *The 'predegree' of O_C is the 8-fold self-intersection $\widetilde{P}^8$ of the class $\widetilde{P}$ of a point-condition in $\widetilde{V}$.*

LEMMA 3.1. *The predegree of O_C equals the product of the degree of the orbit closure of C by the order o_C of the group of automorphisms of C induced from $PGL(3)$.*

PROOF: The map $\widetilde{c}$ is defined by the linear system generated by the point-conditions on $\widetilde{V}$, so $\widetilde{P}$ is the pull-back of the hyperplane class from $\mathbb{P}^N$. Therefore $\widetilde{P}^8$ computes the pull-back of the intersection of $\widetilde{c}(\widetilde{V}) = \overline{O_C}$ with 8 hyperplanes of $\mathbb{P}^N$: i.e., the product of $\deg(\overline{O_C})$ by the degree of the map $\widetilde{c}$. This latter equals o_C since, given a general $c(\alpha) \in O_C$ ($\alpha \in \mathbb{P}^8$), the fiber of $c(\alpha)$ consists of all products $\varphi\alpha$, where φ fixes C. ∎

Observe that for the general C of degree ≥ 4, the predegree of O_C equals the degree of the orbit closure. Our aim here is to compute the predegree of O_C, by using the construction of $\widetilde{V}$ described in §2: we will show that this number depends only on d and on the type of the flexes of C.

Our tool will be a formula relating intersection degrees under blow-ups:

Proposition 3.2. *Let $B \overset{i}{\hookrightarrow} V$ be non-singular projective varieties, and let $X \subset V$ be a codimension-1 subvariety, smooth along B. Let $\widetilde{V}$ be the blow-up of V along B, and let $\widetilde{X}$ be the proper transform of X. Then*

$$\int_{\widetilde{V}} [\widetilde{X}]^{\dim V} = \int_V [X]^{\dim V} - \int_B \frac{([B] + i^*[X])^{\dim V}}{c(N_B V)}$$

where $\int_{\widetilde{V}}$, etc. denote the degree of a class in $\widetilde{V}$, etc., cf. [**Fulton**], Def. 1.4. Note: we will omit the $\int$ sign and the class brackets $[\cdot]$ when this doesn't create ambiguities.

Proof: This follows from [**Aluffi1**], §2, Theorem II and Lemma (2), (3). ∎

We will compute the predegree of O_C (i.e. $\widetilde{P}^8$) by applying Proposition 3.2 to each blow-up in the sequence giving $\widetilde{V}$: the missing ingredients to be obtained at this point are the Chern classes of the normal bundles of the centers of the blow-ups, and calculations in their intersection rings.

In the following, $P, P_i, \widetilde{P}$ will denote resp. (the class of) point-conditions in $V, V_i, \widetilde{V}$. The embedding of B_j in V_j is denoted i_j, and p_{jk} will be used for the map $B_j \longrightarrow B_k$ (p_j will be p_{jj-1} for short). As a general convention, we will omit pull-back notations unless we fear ambiguity.

§**3.1. The first blow-up.** The center of the first blow-up is the variety $B = \check{\mathbb{P}}^2 \times C$; the embedding $i : B \hookrightarrow \mathbb{P}^8$ is given by composition with the Segre embedding:

$$B = \check{\mathbb{P}}^2 \times C \subset \check{\mathbb{P}}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^8 \quad .$$

Call h, k resp. the hyperplane class in $\mathbb{P}^2, \check{\mathbb{P}}^2$. Our convention on pull-backs allows us to write k, h for the pull-backs of k, h from the factors to $\check{\mathbb{P}}^2 \times \mathbb{P}^2$, and to $B \subset \check{\mathbb{P}}^2 \times \mathbb{P}^2$. Also, since the Segre embedding is linear on each factor, the hyperplane class of $\mathbb{P}^8$ pulls-back to $k + h$ on B.

Lemma 3.3. *If C has degree d:*

(i) In B: $k^3 = 0, k^2 h = d, kh^2 = 0, h^3 = 0$

(ii) $c(N_B \mathbb{P}^8) = \dfrac{(1 + k + h)^9 (1 + dh)}{(1 + k)^3 (1 + h)^3}$

(iii) $P^8 = d^8$; and P pulls-back to $dk + dh$.

Proof: (i) is immediate.

(ii) $c(N_B \mathbb{P}^8) = c(N_B \check{\mathbb{P}}^2 \times \mathbb{P}^2) c(N_{\check{\mathbb{P}}^2 \times \mathbb{P}^2} \mathbb{P}^8)$ by the Whitney formula and the exact sequence of normal bundles. Now, since $B = \check{\mathbb{P}}^2 \times C$, $c(N_B \check{\mathbb{P}}^2 \times \mathbb{P}^2) = c(N_C \mathbb{P}^2) = 1 + dh$. The formula for $c(N_{\check{\mathbb{P}}^2 \times \mathbb{P}^2} \mathbb{P}^8)$ is standard.

(iii) Recall from §2 that if $p \in \mathbb{P}^2$, P is the point-condition corresponding to p, and $F(x_0 : x_1 : x_2)$ is the (degree-d) polynomial defining C, then $\alpha \in P \iff F(\alpha(p)) = 0$: so P is defined by a degree-d equation in $\mathbb{P}^8$. ∎

We have already observed that the point-conditions are non-singular (Lemma 2.1 (ii)), so we are ready for the key computation needed to apply Proposition 3.2 to the first blow-up:

Lemma 3.4.
$$\int_B \frac{(B+i^*P)^8}{c(N_B\mathbb{P}^8)} = d(10d-9)(14d^2-33d+21)$$

Proof: By Lemma 3.3, this is

$$\int_{\check{\mathbb{P}}^2\times C} \frac{(1+dk+dh)^8(1+k)^3(1+h)^3}{(1+k+h)^9(1+dh)} \quad :$$

the statement follows by computing the coefficient of k^2h (the only term with non-zero degree, by Lemma 3.3(i)). ∎

§3.2. The second blow-up. The center of the second blow-up is a $\mathbb{P}^1$-bundle B_1 over B

$$\begin{array}{ccc} B_1 & \xrightarrow{i_1} & V_1 \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ B & \xrightarrow{i} & \mathbb{P}^8 \end{array}$$

so classes on B_1 are combinations of (the pull-backs of) k, h and $c_1(\mathcal{O}_{B_1}(-1))$; we call this latter e, and observe it is the pull-back from V_1 of the class of the exceptional divisor E_1.

Lemma 3.5.

(i) $p_{1*}e^i = \begin{cases} 0 & i=0 \\ -1 & i=1 \\ -3k+2dh-6h & i=2 \\ -6k^2+9dkh-27kh & i=3 \\ 24dk^2h-72k^2h & i=4 \end{cases}$

(ii) $c(N_{B_1}V_1) = (1+e)(1+k+dh-e)^3$

(iii) $i_1^*P_1 = dk+dh-e$

Proof: (iii) is immediate, as P is non-singular and pulls-back on B to $dk+dh$ (Lemma 3.3 (iii)).

For (i) and (ii) we need to produce $B_1 \subset E_1$ more explicitly as the projectivization of a rank-2 subbundle of $N_B\mathbb{P}^8$.

First define for any $p \in \mathbb{P}^2$ a rank-8 subbundle H_p of the trivial bundle $B\times\mathbb{C}^9$ over B: if F is a polynomial defining C, and $(k,q)\in B$, $A\in\mathbb{C}^9 = \mathrm{Hom}(\mathbb{C}^3,\mathbb{C}^3)$, say

$$((k,q),A)\in H_p \iff \sum_{i=0}^{2}\left(\frac{\partial F}{\partial x_i}\right)_q A(p)_i = 0$$

where $A(p)_i$ is the i-th coordinate of $A(p)$. So the fiber of H_p over q is the hyperplane of matrices $A\in\mathbb{C}^9$ such that $A(p)\in$ line tangent to C at q. Notice that the above equation has degree $d-1$ in the coordinates of q: thus (denoting by $\mathbb{C}^9$ the trivial bundle $B\times\mathbb{C}^9$, for short)

$$c_1\left(\frac{\mathbb{C}^9}{H_p}\right) = (d-1)h \quad .$$

Now restrict the Euler sequence for $\mathbb{P}^8$ to B via $B \stackrel{i}{\hookrightarrow} \mathbb{P}^8$: $H_p \subset \mathbb{C}^9$ determines a subbundle $\mathcal{H}_p$ of $i^*T\mathbb{P}^8$ and we have the following diagram of bundles over B (suppressing pull-back as usual)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & H_p \otimes \mathcal{O}_{\mathbb{P}^8}(1) & \longrightarrow & \mathcal{H}_p & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathbb{C}^9 \otimes \mathcal{O}_{\mathbb{P}^8}(1) & \longrightarrow & T\mathbb{P}^8 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \dfrac{\mathbb{C}^9}{H_p} \otimes \mathcal{O}_{\mathbb{P}^8}(1) & = & \dfrac{T\mathbb{P}^8}{\mathcal{H}_p} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

from which it follows

$$c\left(\frac{T\mathbb{P}^8}{\mathcal{H}_p}\right) = c\left(\frac{\mathbb{C}^9}{H_p} \otimes \mathcal{O}_{\mathbb{P}^8}(1)\right) = 1 + k + dh \quad .$$

Also, observe that each $\mathcal{H}_p$ contains TB.

Now let p_1, p_2, p_3 be non-collinear points. A matrix has image contained in a line if and only if it sends three non-collinear points to that line, thus the intersection $H_{p_1} \cap H_{p_2} \cap H_{p_3}$ is the rank-6 bundle over $B = \check{\mathbb{P}}^2 \times C$ whose fiber over $(k, q) \in B$ consists of all matrices whose image is contained in the line tangent to C at q. This is the space we used to define B_1: if we set $\mathcal{Q} = \mathcal{H}_{p_1} \cap \mathcal{H}_{p_2} \cap \mathcal{H}_{p_3}$, then

$$B_1 = \mathbb{P}\left(\frac{\mathcal{Q}}{TB}\right) \subset \mathbb{P}(N_B\mathbb{P}^8) = E_1, \quad \text{and} \quad c\left(\frac{T\mathbb{P}^8}{\mathcal{Q}}\right) = (1 + k + dh)^3 \quad .$$

Finally, the Euler sequences for E_1 and B_1 give the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \dfrac{\mathcal{Q}}{TB} \otimes \mathcal{O}_{B_1}(1) & \longrightarrow & TB_1|B & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & N_B\mathbb{P}^8 \otimes \mathcal{O}_{B_1}(1) & \longrightarrow & TE_1|B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \dfrac{T\mathbb{P}^8}{\mathcal{Q}} \otimes \mathcal{O}_{B_1}(1) & = & N_{B_1}E_1 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

(here $TB_1|B$, $TE_1|B$ denote the relative tangent bundles of B_1, E_1 over B) from which

$$c(N_{B_1}E_1) = c\left(\frac{T\mathbb{P}^8}{\mathcal{Q}} \otimes \mathcal{O}_{B_1}(1)\right) = (1+k+dh-e)^3 \quad .$$

From this discussion, it's easy to obtain (i) and (ii):

$$\begin{aligned}
\text{(i) } p_{1*}\sum_i (-1)^i e^i &= c\left(\frac{\mathcal{Q}}{TB}\right)^{-1} && \text{by [}\mathbf{Fulton}\text{], Proposition 3.1 (a)}\\
&= c\left(\frac{T\mathbb{P}^8}{\mathcal{Q}}\right) c(N_B\mathbb{P}^8)^{-1} && \text{by Whitney's formula}\\
&= \frac{(1+k+dh)^3(1+k)^3(1+h)^3}{(1+k+h)^9(1+dh)} && \text{by the above and Lemma 3.3 (ii)}\\
&= 1-3k+2dh-6h+6k^2-9dkh+27kh+24dk^2h-72k^2h \quad .
\end{aligned}$$

(ii) $c(N_{B_1}V_1) = c(N_{E_1}V_1)c(N_{B_1}E_1) = (1+e)(1+k+dh-e)^3$. ∎

Lemma 3.5 allows us to compute the term needed to apply Proposition 3.2 to the second blow-up:

LEMMA 3.6.

$$\int_{B_1} \frac{(B_1+i_1^*P_1)^8}{c(N_{B_1}V_1)} = d(2d-3)(322d^2-1257d+1233)$$

PROOF: This is

$$\int_{B_1} \frac{(1+dk+dh-e)^8}{(1+e)(1+k+dh-e)^3}$$

by Lemma 3.5 (ii) and (iii). Since the degree doesn't change after push-forwards, this is also

$$\int_B p_{1*}\frac{(1+dk+dh-e)^8}{(1+e)(1+k+dh-e)^3} \quad .$$

Computing the degree-4 term in the expansion of the fraction and applying Lemma 3.5 (i) and the projection formula, this is computed as a sum of degree-3 terms in k, h over B. Lemma 3.3 (i) is used then to obtain the stated expression. ∎

§**3.3. The third blow-up.** At this point we have to start taking flexes into account. For any $q \in C$, let $f\ell(q)$ be the order of q as a flex of C, in the sense of §2.2: so $f\ell(q) = 0$ if q is *not* a flex of C, $f\ell(q) = 1$ if q is a *simple* flex of C, and so on.

The center $B_2 \overset{i_2}{\hookrightarrow} V_2$ of the third blow-up is the disjoint union

$$B_2 = \bigcup_{f\ell(q)>0} B_2^{(q)} \quad ,$$

where each $B_2^{(q)}$ maps isomorphically to the restriction $B_1^{(q)}$ of the $\mathbb{P}^1$-bundle B_1 to $\check{\mathbb{P}}^2 \times \{q\} \subset B$. Moreover, $B_2 \cap \widetilde{E}_1 = \emptyset$ (Lemma 2.4). As h restricts to 0 on each $\check{\mathbb{P}}^2 \times \{q\}$, the intersection ring of $B_2^{(q)}$ is generated by k, e (defined as in §3.2). Also, we denote by e' the pull-back of E_2 to $B_2^{(q)}$, and by p_{20} the map $B_2^{(q)} \longrightarrow \check{\mathbb{P}}^2 \times \{q\} \cong \mathbb{P}^2$.

Lemma 3.7.

(i) $e' = e$

(ii) $p_{20*}e^i = \begin{cases} 0 & i=0 \\ -1 & i=1 \\ -3k & i=2 \\ -6k^2 & i=3 \end{cases}$

(iii) $c(N_{B_2^{(q)}}V_2) = (1+e)(1+k-2e)^3$

(iv) $i_2^*P_2 = dk - 2e$

Proof: (ii) follows from Lemma 3.5 (i), since the restriction of h to $B_2^{(q)}$ is 0.

The key observation for the other points is that $B_2^{(q)} \cap \widetilde{E}_1 = \emptyset$. Realize $B_2^{(q)} \subset \mathbb{P}(N_{B_1}V_1)$ as $\mathbb{P}(\mathcal{L})$, where $\mathcal{L}$ is a sub-line bundle of $N_{B_1}V_1$. $\widetilde{E}_1 \cap E_2$ is the exceptional divisor of the blow-up of E_1 along B_1, i.e. the projectivization of $N_{B_1}E_1$ in $N_{B_1}V_1$. That $\mathbb{P}(\mathcal{L})$ and $\mathbb{P}(N_{B_1}E_1)$ are disjoint says that $\mathcal{L} \cap N_{B_1}E_1$ is the zero-section of $N_{B_1}V_1$, and therefore

$$\mathcal{L} \cong \frac{N_{B_1}V_1}{N_{B_1}E_1} = N_{E_1}V_1 \quad \text{as bundles on } B_1^{(q)}.$$

(i) With the same notations, $\mathcal{L}$ is tautologically the universal line bundle over $\mathbb{P}(\mathcal{L})$; it must then equal the restriction to $B_2^{(q)}$ of the universal line bundle $\mathcal{O}_{E_2}(-1) \cong N_{E_2}V_2$. In other words

$$\mathcal{L} \cong N_{E_2}V_2 \quad \text{as bundles on } B_2^{(q)}.$$

Since the projection from $B_2^{(q)}$ to $B_1^{(q)}$ is an isomorphism, it follows that

$$e = c_1(N_{E_1}V_1) = c_1(\mathcal{L}) = c_1(N_{E_2}V_2) = e' \quad .$$

(iii) Call $E_2^{(q)}$ the restriction of $E_2 = \mathbb{P}(N_{B_1}V_1)$ to $B_1^{(q)}$. We have Euler sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{L}\otimes\mathcal{O}(1) & \longrightarrow & TB_2^{(q)}|B_1^{(q)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & N_{B_1}V_1\otimes\mathcal{O}(1) & \longrightarrow & TE_2^{(q)}|B_1^{(q)} & \longrightarrow & 0 \end{array}$$

and we just argued $\mathcal{L} \cong \mathcal{O}(-1)$: so

$$\begin{aligned} c(N_{B_2^{(q)}}E_2^{(q)}) &= c\left(\frac{N_{B_1}V_1}{\mathcal{L}} \otimes \check{\mathcal{L}}\right) \qquad \text{(restricted to } B_2^{(q)}) \\ &= \frac{(1+e-e')(1+k-e-e')^3}{(1+e'-e')} \\ &= (1+k-2e)^3 \quad \text{by (i);} \end{aligned}$$

next, since $N_{B_1^{(q)}}B_1$ is clearly trivial, we have $c(N_{E_2^{(q)}}E_2) = 1$; so putting $N_{B_2^{(q)}}V_2$ together:

$$c(N_{B_2^{(q)}}V_2) = c(N_{E_2}V_2)c(N_{E_2^{(q)}}E_2)c(N_{B_2^{(q)}}E_2^{(q)}) = (1+e)(1+k-2e)^3 \quad ,$$

as claimed.

(iv) Since P_1 is non-singular along B_1, P_2 restricts to $dk - e - e' = dk - 2e$ by (i). ∎

We are ready for the term needed to apply Proposition 3.2 to the third blow-up:

Lemma 3.8.

$$\int_{B_2} \frac{(B_2 + i_2^* P_2)^8}{c(N_{B_2} V_2)} = \sum_{f\ell(q)>0} (196d^2 - 960d + 1125)$$

Proof: By Lemma 3.7 (iii) and (iv), this is

$$\sum_{f\ell(q)>0} \int_{B_2^{(q)}} \frac{(1+dk-2e)^8}{(1+e)(1+k-2e)^3} = \sum_{f\ell(q)>0} \int_{\mathbb{P}^2} p_{20*} \frac{(1+dk-2e)^8}{(1+e)(1+k-2e)^3}$$

(pushing forward doesn't change degrees) and one concludes with the projection formula and Lemma 3.7 (ii). ∎

§**3.4. Further blow-ups.** Further blow-ups are necessary if there are points q on C with $f\ell(q) > 1$. We first attack the initial step.

The center $B_3 \overset{i_3}{\hookrightarrow} V_3$ of the fourth blow-up is the disjoint union

$$B_3 = \bigcup_{f\ell(q)>1} B_3^{(q)} \quad ,$$

where each $B_3^{(q)}$ is a $\mathbb{P}^1$-bundle over $B_2^{(q)}$. The intersection ring of $B_3^{(q)}$ is generated by (the pull-back of) the classes k, e of $B_2^{(q)}$, and by the class of the universal line bundle, i.e. the pull back f of E_3 from V_3. Denote by p_3 the projection $B_3^{(q)} \longrightarrow B_2^{(q)}$.

Lemma 3.9.

(i) $p_{3*} f^i = \begin{cases} 0 & i=0 \\ -1 & i=1 \\ -e & i=2 \\ -e^2 & i=3 \\ -e^3 & i=4 \end{cases}$

(ii) $c(N_{B_3^{(q)}} V_3) = (1+f)(1+k-2e-f)^3$

(iii) $i_3^* P_3 = dk - 2e - f$

Proof: (iii) is clear, as P_2 is non-singular along $B_3^{(q)}$.

For the other items, we have to produce $B_3^{(q)} \subset E_3^{(q)} = \mathbb{P}(N_{B_2^{(q)}} V_2)$ explicitly as the projectivization of a rank-2 subbundle of $N_{B_2^{(q)}} V_2$. Recall that each fiber of $B_3^{(q)}$ is spanned by two points corresponding respectively to (1) a direction transversal to E_2, and (2) a direction in E_2, transversal to the fiber of E_2. Since these two points are always distinct, $B_3^{(q)} = \mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$, where $\mathbb{P}\mathcal{L}_1$, $\mathbb{P}\mathcal{L}_2$ give the two distinguished points on each fiber. Now, $\mathcal{L}_1 \cap N_{B_2^{(q)}} E_2$ is the zero-section in $N_{B_2^{(q)}} V_2$ (the first direction is transversal to E_2); so, with $\mathcal{L}$ as in the proof of 3.7,

$$\mathcal{L}_1 \cong N_{E_2} V_2 \cong \mathcal{L} \quad .$$

Similarly, since the second direction is transversal to the *fiber* of E_2, whose normal bundle in E_2 is trivial, $\mathcal{L}_2 \cong \mathcal{O}$; and therefore we have

$$B_3^{(q)} = \mathbb{P}(\mathcal{L} \oplus \mathcal{O}) \quad .$$

(i) As in the proof of 3.5 (i),

$$p_{3*} \sum_i (-1)^i f^i = c(\mathcal{L} \oplus \mathcal{O})^{-1} = \sum_i (-1)^i e^i$$

and (i) follows by matching dimensions.

(ii) Another pair of Euler sequences: on $B_3^{(q)}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & (\mathcal{L} \oplus \mathcal{O}) \otimes \mathcal{O}(1) & \longrightarrow & TB_3^{(q)}|B_2^{(q)} & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & N_{B_2^{(q)}} V_2 \otimes \mathcal{O}(1) & \longrightarrow & TE_3^{(q)}|B_2^{(q)} & \longrightarrow & 0
\end{array}$$

Since $c_1(\mathcal{O}(1)) = -f$ and E_3 is the disjoint union of the $E_3^{(q)}$:

$$\begin{aligned}
c(N_{B_3^{(q)}} E_3) &= c(N_{B_3^{(q)}} E_3^{(q)}) \\
&= c\left(\frac{N_{B_2^{(q)}} V_2}{\mathcal{L} \oplus \mathcal{O}} \otimes \mathcal{O}(1) \right) \\
&= (1 + k - 2e - f)^3
\end{aligned}$$

(the Chern roots of $N_{B_2^{(q)}} V_2$ are $e, k - 2e, k - 2e, k - 2e, 0$ by Lemma 3.7 (iii)). Finally:

$$c(N_{B_3^{(q)}} V_3) = c(N_{E_3} V_3) c(N_{B_3^{(q)}} E_3) = (1 + f)(1 + k - 2e - f)^3$$

as stated. ∎

Lemma 3.9 describes the situation at the fourth blow-up. The next blow-ups are built on this in the sequence described in §2.4: the center $B_j \overset{i_j}{\hookrightarrow} V_j$ of the $(j+1)$-st blow-up ($j \geq 3$) is the disjoint union

$$B_j = \bigcup_{f\ell(q) > j-2} B_j^{(q)} \quad ,$$

where each $B_j^{(q)}$ maps isomorphically down to $B_3^{(q)}$, and is disjoint from $\widetilde{E}_{i-1}$ (Lemma 2.11). The intersection ring of each $B_j^{(q)} \cong B_3^{(q)}$ is then generated by k, e, f, and the relations stated in Lemma 3.9 (i) hold, for the projection $p_{j2} : B_j^{(q)} \to B_2^{(q)}$. Denote by f_j the pull-back of E_j to $B_j^{(q)}$; Lemma 3.9 can be extended to all stages in the sequence:

LEMMA 3.9 (CONTINUED). *For* $3 \le j \le f\ell(q)+1$

$(i)_j$ $f_j = f$

$(ii)_j$ $c(N_{B_j^{(q)}}V_j) = (1+f)(1+k-2e-(j-2)f)^3$

$(iii)_j$ $i_j^* P_j = dk - 2e - (j-2)f$

PROOF: For $j = 3$ this is given by Lemma 3.9. So it suffices to show that, for $3 \le j \le f\ell(q)$, $(i)_j, (ii)_j, (iii)_j$ imply $(i)_{j+1}, (ii)_{j+1}, (iii)_{j+1}$. Consider then $B_{j+1}^{(q)} = \mathbb{P}(\mathcal{L}_{j+1}) \subset \mathbb{P}(N_{B_j^{(q)}}V_j)$. So f_{j+1} is the class of $\mathcal{O}_{B_{j+1}^{(q)}}(-1)$, i.e. of $\mathcal{L}_{j+1}$. Since $B_{j+1}^{(q)} \cap \widetilde{E}_j = \emptyset$ (Lemma 2.11 (iv)), we get by the usual argument

$$f_{j+1} = c_1(\mathcal{L}_{j+1}) = c_1(N_{E_j}V_j) = f_j \quad :$$

and $f_j = f$ by $(i)_j$; so $f_{j+1} = f$, giving $(i)_{j+1}$.

$(iii)_{j+1}$ follows then from $(iii)_j$ and $(i)_{j+1}$, since P_j is non-singular along B_j.

Finally, we use the Euler sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{L}_{j+1} \otimes \mathcal{O}(1) & \longrightarrow & TB_{j+1}^{(q)}|B_j^{(q)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & N_{B_j^{(q)}}V_j \otimes \mathcal{O}(1) & \longrightarrow & TE_{j+1}^{(q)}|B_j^{(q)} & \longrightarrow & 0
\end{array}$$

to get (since E_{j+1} is the disjoint union of the $E_{j+1}^{(q)}$)

$$\begin{aligned}
c(N_{B_{j+1}^{(q)}}E_{j+1}) &= c(N_{B_{j+1}^{(q)}}E_{j+1}^{(q)}) \\
&= c\left(\frac{N_{B_j^{(q)}}V_j}{\mathcal{L}_{j+1}} \otimes \mathcal{O}(1)\right) \\
&= \frac{(1+f-f)(1+k-2e-(j-2)f-f)^3}{(1+f-f)} \qquad \text{by } (ii)_j \\
&= (1+k-2e-(j-1)f)^3 \qquad ,
\end{aligned}$$

so

$$c(N_{B_{j+1}^{(q)}}V_{j+1}) = c(N_{E_{j+1}}V_{j+1})c(N_{B_{j+1}^{(q)}}E_{j+1}) = (1+f)(1+k-2e-(j-1)f)^3 \quad ,$$

i.e. $(ii)_{j+1}$. ∎

We get then the key term to apply Proposition 3.2 to the j-th blow up in the sequence. In fact, we can cover Lemma 3.8 as well in one statement:

LEMMA 3.10. *For* $j \ge 2$

$$\begin{aligned}
\int_{B_j} \frac{(B_j + i_j^* P_j)^8}{c(N_{B_j}V_j)} = \sum_{f\ell(q) > j-2} & 30j^4 - 96(d-1)j^3 \\
& + 12(d-1)(7d-11)j^2 + 84(d-1)^2 j - 7(2d-3)(22d-39).
\end{aligned}$$

Proof: For $j = 2$, this is Lemma 3.8. For $j \geq 3$, by Lemma 3.9 this is

$$\sum_{f\ell(q)>j-2} \int_{B_j^{(q)}} \frac{(1+dk-2e-(j-2)f)^8}{(1+f)(1+k-2e-(j-2)f)^3} \quad .$$

If p_{j2} denotes the projection $B_j^{(q)} \longrightarrow B_2^{(q)}$, (and p_{20} is the map $B_2^{(q)} \longrightarrow \check{\mathbb{P}}^2 \times \{q\} \cong \mathbb{P}^2$, as in §3.3), this can be computed as

$$\sum_{f\ell(q)>j-2} \int_{\mathbb{P}^2} p_{20*} p_{j2*} \frac{(1+dk-2e-(j-2)f)^8}{(1+f)(1+k-2e-(j-2)f)^3} \quad ,$$

which is evaluated by using the projection formula, 3.9 (i) and 3.7 (ii). ∎

§**3.5. The predegree of $\overline{O_C}$.** Computing the predegree of $\overline{O_C}$ is now a straightforward application of Proposition 3.2 and Lemmas 3.4, 3.6 and 3.10: by Proposition 3.2

$$\widetilde{P}^8 = P^8 - \sum_{j\geq 0} \int_{B_j} \frac{(B_j - i_j^* P_j)^8}{c(N_{B_j} V_j)}$$

(where $B_0 = B$, etc.), and the terms in the summation have been computed in sections 3.1–3.4. This gives

Proposition 3.11. *The predegree of O_C is*

$$\begin{aligned} d^8 - d(10d-9)(14d^2-33d+21) - d(2d-3)(322d^2-1257d+1233) \\ - \sum_{j\geq 2} \sum_{\substack{q\in C \\ f\ell(q)>j-2}} (30j^4 - 96(d-1)j^3 + 12(d-1)(7d-11)j^2 \\ + 84(d-1)^2 j - 7(2d-3)(22d-39)). \end{aligned}$$

This result can be given in handier forms. For example:

Theorem III(a). *The predegree of O_C is*

$$\begin{aligned} d(d-2)(d^6+2d^5+4d^4+8d^3-1356d^2+5280d-5319) - \sum_{q\in C} f\ell(q)(f\ell(q)-1) \\ \big(6f\ell(q)^3 + (75-24d)f\ell(q)^2 + (28d^2-240d+393)f\ell(q) + 196d^2 - 960d + 1125\big) \end{aligned}$$

Proof: Invert the order of the summations in Proposition 3.11, then use the fact that $\sum_{q\in C} f\ell(q) = 3d(d-2)$ (the number of flexes of C, counted with multiplicity). ∎

Or, in another form:

Theorem III(b). *Denote by $f_C^{(r)}$ the sum $\sum_{q\in C} f\ell(q)^r$. Then the predegree of O_C is*

$$\begin{aligned} d^8 - 8d(98d^3 - 492d^2 + 843d - 486) - (168d^2 - 720d + 732) f_C^{(2)} \\ - (28d^2 - 216d + 318) f_C^{(3)} - (69 - 24d) f_C^{(4)} - 6 f_C^{(5)}. \end{aligned}$$

By Theorem III(B), if C is smooth then *the predegree of O_C depends only on the degree d of C and on the four numbers $f_C^{(2)}$, $f_C^{(3)}$, $f_C^{(4)}$ and $f_C^{(5)}$.*

If C only has simple flexes, then $f\ell(q) = 0$ or 1 for all $q \in C$, so Theorem III(A) gives

Corollary. *If all flexes of C are simple, then the predegree of O_C is*

$$d(d-2)(d^6 + 2d^5 + 4d^4 + 8d^3 - 1356d^2 + 5280d - 5319)$$
$$= d^8 - 1372d^4 + 7992d^3 - 15879d^2 + 10638d \quad .$$

Denoting this polynomial in d by $P(d)$, we remark that it gives the degree of the orbit closure of the general smooth plane curve of degree $d \geq 4$ (indeed, such a curve C has no non-trivial automorphisms, so by Lemma 3.1 the degree of $\overline{O_C}$ equals the predegree).

Remark. Denoting by $f_k(d)$ the (negative) contribution to the predegree arising from a flex of order k on a curve of degree d, we have, as an immediate consequence of Theorem III(A):

$$f_k(d) = -k(k-1)((28k+196)d^2 - (24k^2+240k+960)d + (6k^3+75k^2+393k+1125)).$$

E.g., $f_2(d) = -6(84d^2 - 512d + 753)$ and $f_3(d) = -6(280d^2 - 1896d + 3141)$. It is an easy calculus exercise to show that $f_k(d) < 0$ whenever $d \geq k+2 \geq 4$. This proves that the predegree is maximal for a curve with only simple flexes.

§**3.6. Examples.** It is a consequence of Lemma 3.1 that the predegree of the orbit of a smooth plane curve is divisible by the order of its PGL(3)-stabilizer. This cuts both ways. On the one hand, each curve with non-trivial automorphisms provides us with a non-trivial check of the formulas above. On the other hand, these formulas might help in determining which automorphism groups of smooth plane curves occur. We illustrate this below.

Consider, for $d \geq 3$, the Fermat curve $x^d + y^d + z^d$. Its $3d$ flexes have order $d-2$, so the predegree of its orbit is $P(d) + 3d \cdot f_{d-2}(d)$. So for each d this *number* is divisible by $6d^2$, the order of the stabilizer. This implies that in the ring $\mathbb{Z}[d]$ the *polynomial* $P(d) + 3d \cdot f_{d-2}(d)$ is divisible by d^2 and that the quotient polynomial takes values divisible by 6. Indeed

$$P(d) + 3d \cdot f_{d-2}(d) = d^2(d-2)(d^5 + 2d^4 - 26d^3 - 7d^2 + 192d - 192) \quad .$$

Dividing this by $6d^2$, we get the degree of the orbit closure of the Fermat curve, i.e., of the trisecant variety to the d-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^N$, as mentioned in the introduction.

Here is a similar example for all $d \geq 5$: the curve $x^{d-1}y + y^{d-1}z + z^{d-1}x$. The points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$ are flexes of order $d-3$; counted with multiplicity, $3(d^2 - 3d + 3)$ flexes remain. The group D of diagonal matrices with entries $(1, \zeta, \zeta^{2-d})$, where ζ is a $(d^2 - 3d + 3)$-rd root of unity, acts on the latter flexes without fixed points; so either there is one orbit of flexes of order 3, or one orbit of flexes of order 2 and one orbit of simple flexes, or, finally, three orbits of simple flexes. Now one uses the automorphism $\sigma\colon (x:y:z) \mapsto (y:z:x)$ to exclude the first two possibilities; moreover, one verifies that the automorphism group G of the curve is the semidirect product of D and $<\sigma>$ (and that the simple flexes form one G-orbit). So the degree of the orbit closure is

$$\frac{P(d)+3f_{d-3}(d)}{3(d^2-3d+3)} = \frac{1}{3}(d^6+3d^5+6d^4-21d^3-1354d^2+5463d-5508) \quad .$$

Next we list, for some small values of d, the numbers (and their factorizations) we get from the corollary to Theorem III:

d	$P(d)$	$P(d)$ factored
3	216	$2^3 \cdot 3^3$
4	14280	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 17$
5	188340	$2^2 \cdot 3 \cdot 5 \cdot 43 \cdot 73$
6	1119960	$2^3 \cdot 3^3 \cdot 5 \cdot 17 \cdot 61$
7	4508280	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 1789$
8	14318256	$2^4 \cdot 3 \cdot 317 \cdot 941$
9	38680740	$2^2 \cdot 3^6 \cdot 5 \cdot 7 \cdot 379$
10	92790480	$2^4 \cdot 3 \cdot 5 \cdot 59 \cdot 6553$

So for $d=3$ we get 216 for the predegree of the orbit of any smooth plane cubic curve. This gives the well-known numbers 12, resp. 6, resp. 4 for the degree of the orbit closure of a smooth plane cubic with $j \neq 0, 1728$, resp. $j=1728$, resp. $j=0$. Note that the group of projective automorphisms of a smooth cubic contains the 9 translations over points of order dividing 3 as a normal subgroup. The quotient can be identified with the automorphisms that fix a given flex. Thus there exist 18, resp. 36, resp. 54 projective automorphisms when $j \neq 0, 1728$, resp. $j=1728$, resp. $j=0$.

For $d=4$ we get 14280 for the predegree of the orbit of a smooth plane quartic with only simple flexes. An example of such a curve is the Klein curve $x^3y+y^3z+z^3x$; it has 168 automorphisms, so the degree of its orbit closure is $14280/168=85$.

If a smooth quartic has n hyperflexes (i.e., flexes of order 2), the predegree of its orbit equals $14280-294n$. E.g., the degree of the orbit closure of the Fermat quartic is 112, as there are 12 hyperflexes and 96 automorphisms. As another example, consider the curve $x^4+xy^3+yz^3$. It has 1 hyperflex and 9 automorphisms, so the degree of its orbit closure is $(14280-294)/9=1554$. In fact, in [**Vermeulen**] there is a complete list of the automorphism groups that occur for a quartic with a given number of hyperflexes. The implied congruence conditions are equivalent to requiring that $P(4)$ be divisible by 168 and that $P(4)+28f_2(4)$ be divisible by 2016. (This follows already from the existence of the 3 quartics above.)

In the other direction, these formulas give non-trivial information on the automorphism groups of plane curves. Consider smooth plane curves of degree d with only simple flexes. The least common multiple of the orders of the stabilizers of these curves divides $P(d)$. Now it is well-known that a smooth curve of positive genus $g(=\binom{d-1}{2})$ cannot have an automorphism of prime order $p>2g+1(=d^2-3d+3)$.

Using the Hurwitz formula one also excludes the cases $(d, g, p) = (4, 3, 5)$, $(6, 10, 17)$ and $(10, 36, 59)$. Looking at the table above, we conclude then that said l.c.m. divides 216, 168, 60, 1080, 2520, 48, 102060, 240 respectively for d equal to 3, 4, 5, 6, 7, 8, 9, 10 respectively.

These bounds seem to be pretty good: by the above, the actual l.c.m. equals 108 (resp. 168) for $d = 3$ (resp. 4); it's not unreasonable to expect that the bound is sharp for $d = 5$, 8 and 10 (perhaps there even exist curves with automorphism groups of this order); the Valentiner sextic has only simple flexes and 360 automorphisms (cf. [**BHH**]), so the bound for $d = 6$ is sharp if and only if there exists a sextic with only simple flexes and with 27 dividing the order of its stabilizer. Finally, for $d = 9$ the bound is probably not optimal.

References

[**Aluffi1**] Aluffi, P., *The enumerative geometry of plane cubics I: smooth cubics*, Trans. Amer. Math. Soc. **317** (1990), 501–539.

[**Aluffi2**] Aluffi, P., *Two characteristic numbers for smooth plane curves of any degree*, Trans. Amer. Math. Soc. **329** (1992), 73–96.

[**Aluffi-Faber**] Aluffi, P., Faber, C., *Linear orbits of d-tuples of points in* $\mathbb{P}^1$, J. Reine Angew. Math. **445** (1993), 205–220.

[**BHH**] Barthel, G., Hirzebruch, F., Höfer, Th., "Geradenkonfigurationen und Algebraische Flächen," Vieweg, 1987.

[**Fulton**] Fulton, W., "Intersection Theory," Springer Verlag, 1984.

[**Vermeulen**] Vermeulen, A.M., "Weierstrass points of weight two on curves of genus three," Thesis, Universiteit van Amsterdam, 1983.

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