Some characteristic numbers for nodal and cuspidal plane curves of any degree

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Abstract. Two blow-ups over the projective space $\mathbb{P}^N$ parametrizing plane curves of a given degree yield a compactification of the space of *reduced* curves used in [**2**] to obtain partial enumerative results for families of non-singular plane curves. In this paper it is shown how to employ the construction to obtain enumerative results for families of plane curves with a node or a cusp. The results recover known results for cubics, give a first modern verification of some computations of of Zeuthen's for quartics, and are new for higher degree. The heart of the computation is the derivation of key Segre classes relating the intersection calculus at the different stages of the blow-up construction.

0. Introduction. The k-th 'characteristic number' of an r-parameter family $\mathcal{F}$ of plane curves of degree d is the number of curves of $\mathcal{F}$ which are tangent *at smooth points* to k lines and contain $r-k$ points in general position in the plane.

Assume $d > 2$. Denote resp. by $S_d(k), S\ell_d(k), Sp_d(k), C_d(k), C\ell_d(k), Cp_d(k)$ the k-th characteristic number for the family of degree d:

- nodal curves;
- nodal curves with singularity on a given line;
- nodal curves with singularity at a given point;
- cuspidal curves;
- cuspidal curves with cusp on a given line;
- cuspidal curves with cusp at a given point;

then multiplicity calculations and Bézout's theorem in the projective space $\mathbb{P}^N = \mathbb{P}^{\frac{d(d-3)}{2}}$ parametrizing all plane curves of degree d yield (see Corollary 1.9 in §1)

$$\begin{aligned}
S_d(k) &= 2^{k-1}(d-1)^{k-2}(6(d-1)^4 - 6(d-1)^2k + k(k-1)) && \text{for } k < 2d-2\\
S\ell_d(k) &= 2^k(d-1)^{k-1}(3(d-1)^2 - k) && \text{for } k < 2d-3\\
Sp_d(k) &= 2^k(d-1)^k && \text{for } k < 2d-3\\
C_d(k) &= 3\cdot 2^{k-2}(d-1)^{k-2}(16(d-1)^4 - 16(d-1)^3 &&\\
&\qquad - 16(d-1)^2k + 8(d-1)k + 3k(k-1)) && \text{for } k < 2d-3\\
C\ell_d(k) &= 2^k(d-1)^{k-1}(8(d-1)(2d-3) - 3k) && \text{for } k < 2d-4\\
Cp_d(k) &= 2^{k+1}(d-1)^k && \text{for } k < 2d-4
\end{aligned}$$

In each of these cases, we compute here the next characteristic number, for which the geometry of $\mathbb{P}^N$ alone does not provide adequate information. We work in a different compactification (obtained in [**2**]) of the variety parametrizing reduced plane curves of degree d; our result is

$$S_d(2d-2) = 2^{2d-2}(d-1)^{2d-3}(3d^3 - 15d^2 + 23d - 12) + 2^{d+1}\binom{\binom{d}{2}+1}{2}$$

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$$
\begin{aligned}
S\ell_d(2d-3) &= 2^{2d-3}(d-1)^{2d-4}(3d^2-8d+6) - 2^d\binom{\binom{d}{2}+1}{2} \\
Sp_d(2d-3) &= 2^{2d-3}(d-1)^{2d-3} - 2^{d-1}\binom{d}{2} \\
C_d(2d-3) &= 3\cdot 2^{2d-4}(d-1)^{2d-5}(8d^4-56d^3 \\
&\qquad\qquad + 142d^2-161d+70) + 3\cdot 2^d\binom{\binom{d}{2}+1}{2} \\
C\ell_d(2d-4) &= 2^{2d-3}(d-1)^{2d-5}(4d^2-13d+12) - 2^d\binom{\binom{d}{2}+1}{2} \\
Cp_d(2d-4) &= 2^{2d-3}(d-1)^{2d-4} - 2^{d-1}\binom{d}{2} \qquad .
\end{aligned}
$$

For $d=3$ these results recover a few of the many known enumerative results about singular plane cubics (modern references for these are [**5**], [**6**] or [**3**]). Notice that the formulas above give for the 7-parameter family of *cuspidal cubics* the characteristic numbers $C_3(k) = 24, 60, 114, 168$ for $k = 0, 1, 2, 3$; since cuspidal cubics are self-dual, one can argue that necessarily $C_3(k) = C_3(7-k)$, so that the results in this note suffice to give a derivation of the whole list:

$$C_3(k) = 24, 60, 114, 168, 168, 114, 60, 24 \quad k = 0, \dots, 7 \quad .$$

For $d=4$, the above formulas give

	$S_4(k)$	$S\ell_4(k)$	$Sp_4(k)$	$C_4(k)$	$C\ell_4(k)$	$Cp_4(k)$
$k=0$	27	9	1	72	20	2
$k=1$	144	52	6	372	114	12
$k=2$	760	300	36	1890	648	72
$k=3$	3960	1728	216	9396	3672	432
$k=4$	20304	9936	1296	45360	$\boxed{20400}$	$\boxed{2544}$
$k=5$	101952	$\boxed{56688}$	$\boxed{7728}$	$\boxed{210960}$		
$k=6$	$\boxed{498336}$					

verifying results in [**9**] (in [**9**] *all* characteristic numbers for many families of singular quartics are presented!). The boxed numbers are the ones for which we work in a compactification other than the projective spaces parametrizing plane curves.

For $d \geq 5$ the results are new: for example, to our knowledge the number 432,016,832 of plane nodal *quintics* containing 11 points and tangent to 8 lines in general position in the plane doesn't appear elsewhere in the literature. We know of promising work in progress on similar questions that makes use of techniques originally developed by Z. Ran to compute the *degrees* (i.e., the '0-th' characteristic numbers) of varieties parametrizing families of singular plane curves. However, those techniques apparently have not yet yielded *higher* characteristic numbers for the varieties studied here.

Let $\mathbb{P}^N = \mathbb{P}^{\frac{d(d+3)}{2}}$ be the projective space parametrizing degree-d plane curves over e.g. $\mathbb{C}$, and let $F \subset \mathbb{P}^N$ parametrize a family of curves. Any enumerative problem about the family is readily translated in a problem of intersections in $\mathbb{P}^N$: the set of curves containing a given point forms a hyperplane in $\mathbb{P}^N$, and the set of curves tangent to a given line forms a hypersurface (of degree $2d-2$) in $\mathbb{P}^N$ (we call these resp. 'point-conditions' and 'line-conditions'). If $\dim F = r$, then the k-th characteristic number of the family is the number of certain *special* points in the intersection of F with k general line-conditions and $r-k$ general point-conditions: specifically, those points corresponding to curves in F that are tangent to the given lines *at smooth points*. One can check (cf. [**1**], Theorem I) that the intersection is transversal at such points; it is therefore natural to hope Bézout's theorem in $\mathbb{P}^N$ should yield information about their number.

Problems with this approach arise because the intersection of the point- and line-conditions along F may very well contain curves that don't satisfy the requirement on 'proper' tangency. For example, the intersection of the set of singular cubics with 8 general line-conditions contains the whole 4-dimensional set of non-reduced cubics, as well as points corresponding to curves tangent to 7 of the lines and having the node on the 8th, and points corresponding to curves tangent to 6 of the lines and having a node at the intersection of the remaining 2.

The first issue–the presence of non-reduced curves–is the more fundamental one. This is approached by lifting the question to another compactification of the space of reduced curves, in which non-reduced curves don't enter into play: to obtain such a compactification, one can for example resolve the rational map associating with every smooth plane curve its dual (cf. [**1**], §1). This program is executed in [**1**], [**3**] to obtain enumerative results about smooth and singular plane cubics; unfortunately, constructing such compactifications for higher degree *while mantaining control of the relevant intersection calculus* seems a very hard task. In [**2**] we show that a suitable sequence of two blow-ups at smooth centers over $\mathbb{P}^N$ produces a variety that suits our needs *as long as* the only non-reduced curves in the intersection consist of a 'double line' and a (reduced) degree-$(d-2)$ curve intersecting transversally.

In this note we use the same compactification. The limitation of the kind of non-reduced curves we can admit imposes severe restrictions on the results: for each family, our construction will only reach here the *first* characteristic number beyond the ones involving only reduced curves. The actual computation of the intersection numbers we need is performed by the same techniques of [**2**]: the missing information we have to compute here amounts essentially to Segre classes of the intersection of the centers of the blow-ups with the parameter spaces of the families (or their proper transforms).

The second issue–reduced curves that appear among the intersections because they have singularities along the given lines–is easier to handle. The main remark is that, for each configuration, the number of such curves is itself a characteristic number of *another* family. It will be easy to relate the intersection numbers we compute to the actual characteristic numbers, the only complication being that we will have to consider several families at once.

The families we treat in this note are families of nodal and of cuspidal plane curves of degree d. We see these objects as projections to $\mathbb{P}^N$ of subvarieties of $\mathbb{P}^2 \times \mathbb{P}^N$: for example, the discriminant hypersurface in $\mathbb{P}^N$ will be the projection of the bundle over $\mathbb{P}^2$ whose fiber over p is the $\mathbb{P}^{N-3}$ of curves singular at p. Similarly, the proper transforms of these objects will be projections of varieties lying in the product of the blow-ups by $\mathbb{P}^2$.

Some of the geometry underlying these projections is used in §1, to relate the characteristic numbers to intersection numbers in a compactification of the set of reduced curves (Theorem I). As an immediate application, the first stock of characteristic numbers is computed by applying the result to suitable intersections in $\mathbb{P}^N$ (Corollary 1.9). In §2 we exploit the blow-ups of [**2**] to obtain the Segre classes (Propositions 2.3, 2.5, 2.7); these are used in §3 to compute the relevant intersections numbers (Theorem II), and to complete the computations of the harder characteristic numbers (Theorem III).

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1. Families of singular curves. We work over an algebraically closed field of characteristic 0. The families we are going to consider are parametrized by subsets of the projective space $\mathbb{P}^N = \mathbb{P}(H^0\mathcal{O}_{\mathbb{P}^2}(d))$, $d > 2$, parametrizing degree-d plane curves. In this section we will describe these subsets as birational projections of subvarieties from $\mathbb{P}^2 \times \mathbb{P}^N$. This choice will make it relatively easy to obtain information such as the relevant degrees and multiplicities, and the relations between the characteristic numbers and intersection numbers in a suitable compactification of the family of reduced curves.

To state these relations, we need to recall some of the notations in [**1**]. For any birational map $\widetilde{V} \longrightarrow \mathbb{P}^N$, call 'point-conditions' and 'line-conditions in $\widetilde{V}$' the proper transforms of the conditions in $\mathbb{P}^N$ (defined in the introduction). We say that $\widetilde{V}$ is a 'variety of complete curves of degree d' if the intersection of all line-conditions in $\widetilde{V}$ is *empty*. Also, we denote by $\widetilde{P}$, $\widetilde{L}$ the classes of the general point- and line-condition in $\widetilde{V}$.

Consider the following subsets of $\mathbb{P}^N$:

- S: singular curves;
- $S\ell$: singular curves with singularity on a given line;
- Sp: singular curves with singularity at a given point;
- C: cuspidal curves;
- $C\ell$: cuspidal curves with cusp on a given line;
- Cp: cuspidal curves with cusp at a given point.

As in the introduction, denote the characteristic numbers of the corresponding families by $S_d(k)$, $S\ell_d(k), \dots$. In this section we will prove:

Theorem I. *Let $\widetilde{V}$ be a variety of complete curves of degree d, and denote by $\widetilde{S}, \widetilde{S\ell}$, etc. the proper transforms in $\widetilde{V}$ of $S, S\ell$, etc. Then*

$$\begin{aligned}
S_d(k) &= \widetilde{P}^{N-1-k} \cdot \widetilde{L}^k \cdot \widetilde{S} - 2kS\ell_d(k-1) - 4\binom{k}{2}Sp_d(k-2)\\
S\ell_d(k) &= \widetilde{P}^{N-2-k} \cdot \widetilde{L}^k \cdot \widetilde{S\ell} - 2kSp_d(k-1)\\
Sp_d(k) &= \widetilde{P}^{N-3-k} \cdot \widetilde{L}^k \cdot \widetilde{Sp}\\
C_d(k) &= \widetilde{P}^{N-2-k} \cdot \widetilde{L}^k \cdot \widetilde{C} - 3kC\ell_d(k-1) - 9\binom{k}{2}Cp_d(k-2)
\end{aligned}$$

$$C\ell_d(k) = \widetilde{P}^{N-3-k} \cdot \widetilde{L}^k \cdot \widetilde{C\ell} - 3kCp_d(k-1)$$
$$Cp_d(k) = \widetilde{P}^{N-4-k} \cdot \widetilde{L}^k \cdot \widetilde{Cp} \quad .$$

Remark. Basically, this says that for e.g. a configuration of k general lines and $N-1-k$ general points, curves tangent to $k-1$ lines and having a node on the k-th one 'count with multiplicity 2', and curves tangent to $k-2$ lines and with a node at the intersection of the remaining 2 'count with multiplicity 4' (a similar statement can be phrased *mutatis mutandis* for cuspidal curves). This is certainly folklore in both classical and modern enumerative geometry; we establish these results here for lack of a reference, and since we need them in the context of 'varieties of complete curves'. In a somewhat different context, such results are implicit (at least for $d = 3$) in e.g. [**6**], [**7**] (cf. Proposition 7.4 in [**7**]).

1.1. Families of nodal curves. To describe the loci $S, S\ell, Sp$, give coordinates $(x_0 : x_1 : x_2)$ to $\mathbb{P}^2$ and consider the codimension-3 subvariety $\widehat{S}$ of $\mathbb{P}^2 \times \mathbb{P}^N$ defined by

$$(p, f) \in \widehat{S} \iff \begin{cases} \dfrac{\partial f}{\partial x_0}(p) = 0 \\ \dfrac{\partial f}{\partial x_1}(p) = 0 \\ \dfrac{\partial f}{\partial x_2}(p) = 0 \end{cases} \qquad .$$

Restricting the projections $\mathbb{P}^2 \times \mathbb{P}^N \xrightarrow{p_1} \mathbb{P}^2$, $\mathbb{P}^2 \times \mathbb{P}^N \xrightarrow{p_2} \mathbb{P}^N$, gives maps $\widehat{S} \longrightarrow \mathbb{P}^2$, $\widehat{S} \longrightarrow \mathbb{P}^N$; observe that the fiber $p_1^{-1}(p) \cap \widehat{S}$ of $\widehat{S}$ over $p \in \mathbb{P}^2$ consists of all degree-d curves singular at p, while the fiber $p_2^{-1}(f) \cap \widehat{S}$ of $\widehat{S}$ over $f \in \mathbb{P}^N$ is the singular scheme of f (in $\mathbb{P}^2$). In fact $\widehat{S} \xrightarrow{p_1} \mathbb{P}^2$ is a $\mathbb{P}^{N-3}$ bundle; in particular, $\widehat{S}$ is smooth.

If $\ell \subset \mathbb{P}^2$ is a line, denote by $\widehat{S\ell}$ the inverse image $p_1^{-1}(\ell) \cap \widehat{S}$; if $p \in \mathbb{P}^2$, let $\widehat{Sp} = p_1^{-1}(p) \cap \widehat{S}$. Then clearly $S = p_2(\widehat{S})$, $S\ell = p_2(\widehat{S\ell})$, $Sp = p_2(\widehat{Sp})$, and moreover the restrictions of p_2 to $\widehat{S}$, $\widehat{S\ell}$, $\widehat{Sp}$ are birational maps.

Let now k, h resp. denote the hyperplane class in $\mathbb{P}^2$, $\mathbb{P}^N$, and their pull-backs. The definitions give immediately the total Chern classes of the normal bundles:

Lemma 1.1. *(i)* $c(N_{\widehat{S}}\mathbb{P}^2 \times \mathbb{P}^N) = (1 + (d-1)k + h)^3$*;*
(ii) $c(N_{\widehat{S\ell}}\widehat{S}) = (1 + k)$*;*
(iii) $c(N_{\widehat{Sp}}\widehat{S}) = (1 + k)^2$*;*
(iv) Also: $\quad [\widehat{S\ell}]^2 = [\widehat{Sp}], \quad [\widehat{S\ell}]^3 = 0 \quad$ *in* $\widehat{S}$*.*

All we need to compute the first characteristic numbers for $S, S\ell, Sp$ is the first part of Theorem I (which we will prove in a moment) and the degrees of $S, S\ell, Sp$. These are:

Proposition 1.2. *(i)* *deg*$(S) = 3(d-1)^2$*;*
(ii) *deg*$(S\ell) = 3(d-1)$*;*
(iii) *deg*$(Sp) = 1$*.*

PROOF: As (i) is well known, and (iii) is a triviality, we only detail (ii). Denoting the degree of a class by $\int$:

$$\begin{aligned}
\deg(S\ell) &= \int_{\mathbb{P}^N} h^{N-2} \cdot S\ell \\
&= \int_{\mathbb{P}^2\times\mathbb{P}^N} h^{N-2} \cdot \widehat{S\ell} && \text{by the projection formula} \\
&= \int_{\mathbb{P}^2\times\mathbb{P}^N} h^{N-2}(1+k)(1+(d-1)k+h)^3 && \text{by Lemma 1.1 (i) and (ii)} \\
&= \int_{\mathbb{P}^2\times\mathbb{P}^N} h^{N-2} \cdot 3(d-1)h^2k^2 = 3(d-1) && . \blacksquare
\end{aligned}$$

To prove the first part of Theorem I, let L be the line-condition in $\mathbb{P}^N$ corresponding to a general line $\ell \subset \mathbb{P}^2$, and $\widehat{L} = p_2^{-1}(L) \subset \mathbb{P}^2 \times \mathbb{P}^N$. Then $\widehat{L}$ intersects $\widehat{S}$ along $\widehat{S\ell}$ and along the closure $\widehat{L}_{\widehat{S}}$ of the subset of $\widehat{S}$ consisting of pairs (q, f) with f singular at q and tangent to ℓ *at smooth points*.

We claim that to prove the first part of Theorem I we just need to show

LEMMA 1.3. $[\widehat{L} \cap \widehat{S}] = [\widehat{L}_{\widehat{S}}] + 2[\widehat{S\ell}]$ *as cycles on* $\widehat{S}$.

Indeed, suppose this has been established. Let $\widetilde{L}$ be the line-condition in $\widetilde{V}$ corresponding to ℓ. $\widetilde{L} \cap \widetilde{S}$ splits in $\widetilde{S\ell}$ and (at least) another component $\widetilde{L}_{\widetilde{S}}$ (the 'complete curves' tangent to ℓ at smooth points). The characteristic numbers are the intersection numbers of $\widetilde{P}$'s and $\widetilde{L}_{\widetilde{S}}$'s: the intersection is supported on the 'right' points, and transversal by Theorem I in [**1**]. So for example $S\ell_d(k) = [\widetilde{P} \cap \widetilde{S}]^{N-2-k} \cdot [\widetilde{L}_{\widetilde{S}}]^k \cdot [\widetilde{S\ell}]$ in $\widetilde{S}$.

Now observe that $\widehat{S}$ and $\widetilde{S}$ are birational, as they are both birational to S. Let S° be a dense open subset of S isomorphic to subsets (which we identify with S°) of $\widehat{S}$ and $\widetilde{S}$. Apply Theorem I from [**1**] to S°: general points and lines can be chosen so that the corresponding conditions in $\widetilde{V}$ meet only in S°; in computing $\widetilde{P}^{r-k} \cdot \widetilde{L}^k \cdot \widetilde{S}$ we may therefore restrict first to S°.

So we may assume $[\widetilde{L} \cap \widetilde{S}] = [\widetilde{L}_{\widetilde{S}}] + 2[\widetilde{S\ell}]$, since this equality holds after restricting to S° (as it holds on $\widehat{S}$), by Lemma 1.3. Also, we may assume $[\widetilde{S\ell}]^2 = [\widetilde{Sp}]$, $[\widetilde{S\ell}]^3 = 0$ since this holds on S°, by Lemma 1.1 (iv). Putting all together:

$$\begin{aligned}
[\widetilde{L} \cap \widetilde{S}]^k \cdot [\widetilde{Sp}] &= ([\widetilde{L}_{\widetilde{S}}] + 2[\widetilde{S\ell}])^k \cdot [\widetilde{Sp}] = [\widetilde{L}_{\widetilde{S}}]^k \cdot [\widetilde{Sp}] \\
[\widetilde{L} \cap \widetilde{S}]^k \cdot [\widetilde{S\ell}] &= ([\widetilde{L}_{\widetilde{S}}] + 2[\widetilde{S\ell}])^k \cdot [\widetilde{S\ell}] = [\widetilde{L}_{\widetilde{S}}]^k \cdot [\widetilde{S\ell}] + 2k[\widetilde{L}_{\widetilde{S}}]^{k-1} \cdot [\widetilde{Sp}] \\
[\widetilde{L} \cap \widetilde{S}]^k &= ([\widetilde{L}_{\widetilde{S}}] + 2[\widetilde{S\ell}])^k = [\widetilde{L}_{\widetilde{S}}]^k + 2k[\widetilde{L}_{\widetilde{S}}]^{k-1} \cdot [\widetilde{S\ell}] + 4\binom{k}{2}[\widetilde{L}_{\widetilde{S}}]^{k-2} \cdot [\widetilde{Sp}]
\end{aligned}$$

and the first part of Theorem I follows.

We then need to verify $[\widehat{L} \cap \widehat{S}] = [\widehat{L}_{\widehat{S}}] + 2[\widehat{S\ell}]$.

PROOF OF LEMMA 1.3: Equivalently, we can verify that $[L \cap S] = [L_S] + 2[S\ell]$ in $\mathbb{P}^N$, where L_S denotes the closure of the set of singular curves tangent at a smooth point to the line $\ell \subset \mathbb{P}^2$. To get this, we produce a curve in S and compare the restrictions of L and of L_S, $S\ell$ to it. Let $X \in S$ be a general plane curve with one node: we consider the curve $X \circ \gamma(t)$ in S obtained by translating X by elements in a 1-parameter family $\gamma(t)$ of linear transformations of the plane: we have to examine the restriction $L_\ell|_{X\circ\gamma(t)}$ of the line-condition L_ℓ corresponding to ℓ.

Now, clearly we may keep X fixed and move ℓ instead: i.e., $L_\ell|_{X\circ\gamma(t)}$ equals $L_{\ell\circ\gamma(t)^{-1}}|_X$ as divisors on the t-line. Since the line-conditions on $\mathbb{P}^2$ are just point-conditions on the dual plane $\check{\mathbb{P}}^2$, $L_{\ell\circ\gamma(t)^{-1}}|_X$ is the restriction $\check{X}|_{\ell\circ\gamma(t)^{-1}}$ of the dual $\check{X}$ of X to the curve $\ell \circ \gamma(t)^{-1}$ in $\check{\mathbb{P}}^2$. So to obtain the statement we only need to remark that (see for example [**8**, IV.6]) for X a degree-d plane curve with one node and no other singularities, $\check{X}$ consists of a simple component, giving the restriction of L_S *with multiplicity 1;* and of a multiple component, supported on the line in $\check{P}^2$ dual to the node of X, with multiplicity 2: giving the restriction of $S\ell$, *with multiplicity 2.* ∎

1.2. Families of cuspidal curves. We say that a curve is 'cuspidal' at p if it is singular at p and its tangent cone at p is a double line. $C \subset S$ is the closure of the set of cuspidal curves: i.e., the image in $\mathbb{P}^N$ of the divisor $\widehat{C}$ of $\widehat{S}$ defined by

$$(p,f) \in \widehat{C} \iff \begin{cases} \dfrac{\partial f}{\partial x_0}(p) = 0 \\ \dfrac{\partial f}{\partial x_1}(p) = 0 \\ \dfrac{\partial f}{\partial x_2}(p) = 0 \end{cases} , \quad \begin{cases} \left[\left(\dfrac{\partial^2 f}{\partial x_0 \partial x_1}\right)^2 - \dfrac{\partial^2 f}{\partial x_0^2}\dfrac{\partial^2 f}{\partial x_1^2}\right](p) = 0 \\ \left[\left(\dfrac{\partial^2 f}{\partial x_0 \partial x_2}\right)^2 - \dfrac{\partial^2 f}{\partial x_0^2}\dfrac{\partial^2 f}{\partial x_2^2}\right](p) = 0 \\ \left[\left(\dfrac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 - \dfrac{\partial^2 f}{\partial x_1^2}\dfrac{\partial^2 f}{\partial x_2^2}\right](p) = 0 \end{cases} .$$

As with $\widehat{S}$, restricting the projections gives maps $\widehat{C} \longrightarrow \mathbb{P}^2$, $\widehat{C} \longrightarrow \mathbb{P}^N$; the fiber of C over $p \in \mathbb{P}^2$ consists of a *quadric* in the $\mathbb{P}^{N-3}$ of curves singular at p, and the fiber over $f \in \mathbb{P}^N$ is what we would call the 'cuspidal scheme' of f.

Letting $\widehat{C\ell} = p_1^{-1}(\ell) \cap \widehat{C}$ and $\widehat{Cp} = p_1^{-1}(p) \cap \widehat{C}$, then $C\ell = p_2(\widehat{C\ell})$, $Cp = p_2(\widehat{Cp})$, and the restrictions of p_2 to $\widehat{C}$, $\widehat{C\ell}$, $\widehat{Cp}$ are birational morphisms.

As in §1.1, let k, h denote the hyperplane class in $\mathbb{P}^2, \mathbb{P}^N$ resp., and their pull-backs. Then we get the Chern classes:

LEMMA 1.4. *(i)* $c(N_{\widehat{C}}\widehat{S}) = (1 + 2(d-3)k + 2h)$;
(ii) $c(N_{\widehat{C\ell}}\widehat{C}) = (1 + k)$;
(iii) $c(N_{\widehat{Cp}}\widehat{C}) = (1 + k)^2$;
(iv) Also: $[\widehat{C\ell}]^2 = [\widehat{Cp}]$, $\quad [\widehat{C\ell}]^3 = 0 \quad$ *on* $\widehat{C}$.

PROOF: The only point that requires an argument is (i). Notice that, outside $\{x_0 = 0\}$, the equation for $\widehat{C}$ in $\widehat{S}$ is

$$(*) \qquad \left[\left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 - \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2}\right](p) = 0 \qquad ;$$

therefore, globally (*) defines a divisor in $\widehat{S}$ consisting of $\widehat{C}$ and of some multiple αk of the divisor $\{x_0 = 0\}$. Restricting to a curve in $\widehat{S}$ intersecting $\{x_0 = 0\}$ transversally away from $\widehat{C}$ (e.g.: $t \mapsto ((t:0:1), x_0x_1x_2^{d-2} - tx_1x_2^{d-1}))$ shows $\alpha = 2$, i.e. the divisor determined by (*) equals $\widehat{C} + 2k$. Since (*) is quadratic in the coordinates of $\mathbb{P}^N$, and of degree $2(d-2)$ in $(x_0 : x_1 : x_2)$, $\widehat{C}$ must have class $2h - 2(d-2)k - 2k = 2h - 2(d-3)k$, giving (i). ∎

To compute the first characteristic numbers for $C, C\ell, Cp$ we need the second part of Theorem I and the degrees of $C, C\ell, Cp$.

Proposition 1.5. *(i) deg*$(C) = 12(d-1)(d-2)$*;*
(ii) deg$(C\ell) = 4(2d-3)$*;*
(iii) deg$(Cp) = 2$.

Proof: These follow immediately from Lemma 1.4. For example:

$$\begin{aligned}
\deg(C) &= \int_{\mathbb{P}^N} h^{N-2} \cdot C \\
&= \int_{\mathbb{P}^2 \times \mathbb{P}^N} h^{N-2} \cdot \widehat{C} \qquad\qquad \text{by the projection formula} \\
&= \int_{\mathbb{P}^2 \times \mathbb{P}^N} h^{N-2} (1 + 2(d-3)k + 2h)(1 + (d-1)k + h)^3 \\
&= \int_{\mathbb{P}^2 \times \mathbb{P}^N} h^{N-2} \cdot (6(d-1)^2 + 6(d-1)(2d-3))h^2k^2 \\
&= 12(d-1)(d-2) \qquad . \blacksquare
\end{aligned}$$

The argument to show the second part of Theorem I is entirely analogous to the argument for the first part, detailed in §1.1. If now we denote by $\widehat{L}_{\widehat{C}}$ the closure of the subset of $\widehat{C}$ consisting of pairs (q, f) with f cuspidal at q and tangent to a line $\ell \subset \mathbb{P}^2$ at a smooth point, the key computation is:

Lemma 1.6. $[\widehat{L} \cap \widehat{C}] = [\widehat{L}_{\widehat{C}}] + 3[\widehat{C\ell}]$.

Proof: By the same argument as in the proof of Lemma 1.3, we just need to remark that the dual of a degree-d plane curve with one cusp (and no other singularities) consists of a simple component (that accounts for $[\widehat{L}_{\widehat{C}}]$) and of the line dual to the cusp, with multiplicity 3 (accounting for $3[\widehat{C\ell}]$). ∎

1.3. Characteristic numbers, I. The information collected in §1.1,2 suffices to compute the characteristic numbers of $S, S\ell, \dots$ for configurations involving only *reduced* curves. Indeed, $\mathbb{P}^N$ is isomorphic to a variety of complete curves outside of the set of non-reduced curves (this point is made more formal in [**2**], Lemma I, for characteristic numbers of non-singular curves. We don't repeat the argument here, leaving the straightforward adjustments to the reader).

All we need to spot the right configurations is a dimension count from [**2**]:

Lemma 1.7. *For* $j > N - 2d + 1$ *and* $P_1, \dots, P_j$ *general point-conditions in* $\mathbb{P}^N$, $P_1 \cap \dots \cap P_j$ *meets* $S, S\ell, C, C\ell$ *only at points corresponding to reduced curves; also,* $P_1 \cap \dots \cap P_{j-1}$ *meets* Sp, Cp *only at points corresponding to reduced curves.*

Proof: This follows from Lemma 1.1 in [**2**] and Remark 1, §1 in [**1**], since the set of non-reduced curves is contained in $S, S\ell, C, C\ell$ and cut in codimension 1 by Sp, Cp. ∎

Proposition 1.8. *Let $\widetilde{V}$ be a variety of complete curves of degree d. Denote by $\widetilde{P}, \widetilde{L}$ resp. the classes of the general point- and line-conditions in $\widetilde{V}$; also, denote by $\widetilde{S}, \widetilde{S\ell}, \dots$ the proper transforms of $S, S\ell, \dots$. Then*

$$\begin{aligned}
\widetilde{P}^{N-1-k}\cdot\widetilde{L}^k\cdot\widetilde{S} &= 3(d-1)^2(2d-2)^k && \text{for } k<2d-2\\
\widetilde{P}^{N-2-k}\cdot\widetilde{L}^k\cdot\widetilde{S\ell} &= 3(d-1)(2d-2)^k && \text{for } k<2d-3\\
\widetilde{P}^{N-3-k}\cdot\widetilde{L}^k\cdot\widetilde{Sp} &= (2d-2)^k && \text{for } k<2d-3\\
\widetilde{P}^{N-2-k}\cdot\widetilde{L}^k\cdot\widetilde{C} &= 12(d-1)(d-2)(2d-2)^k && \text{for } k<2d-3\\
\widetilde{P}^{N-3-k}\cdot\widetilde{L}^k\cdot\widetilde{C\ell} &= 4(2d-3)(2d-2)^k && \text{for } k<2d-4\\
\widetilde{P}^{N-4-k}\cdot\widetilde{L}^k\cdot\widetilde{Cp} &= 2(2d-2)^k && \text{for } k<2d-4
\end{aligned}$$

Proof: In the specified ranges, we can choose point-conditions to avoid the locus of non-reduced curves, by Lemma 1.7. Therefore the intersection numbers can be computed in $\mathbb{P}^N$, where they are given by Bézout's Theorem: the degree of the line-conditions in $\mathbb{P}^N$ is $(2d-2)$, and the degrees of $S, S\ell, \dots$ are computed in Propositions 1.2 and 1.5. ∎

The first results listed in the introduction follow now immediately from Proposition 1.8 and Theorem I:

Corollary 1.9.

$$\begin{aligned}
S_d(k) &= 2^{k-1}(d-1)^{k-2}(6(d-1)^4-6(d-1)^2k+k(k-1)) && \text{for } k<2d-2\\
S\ell_d(k) &= 2^k(d-1)^{k-1}(3(d-1)^2-k) && \text{for } k<2d-3\\
Sp_d(k) &= 2^k(d-1)^k && \text{for } k<2d-3\\
C_d(k) &= 3\cdot 2^{k-2}(d-1)^{k-2}(16(d-1)^4-16(d-1)^3\\
&\qquad -16(d-1)^2k+8(d-1)k+3k(k-1)) && \text{for } k<2d-3\\
C\ell_d(k) &= 2^k(d-1)^{k-1}(8(d-1)(2d-3)-3k) && \text{for } k<2d-4\\
Cp_d(k) &= 2^{k+1}(d-1)^k && \text{for } k<2d-4
\end{aligned}$$

2. Segre classes. To apply Theorem I to the first cases not covered by the formulas in Corollary 1.9, we need to evaluate the intersection products

$$\begin{matrix}
\widetilde{P}^{N-2d+1}\cdot\widetilde{L}^{2d-2}\cdot\widetilde{S} & & \widetilde{P}^{N-2d+1}\cdot\widetilde{L}^{2d-3}\cdot\widetilde{C}\\
\widetilde{P}^{N-2d+1}\cdot\widetilde{L}^{2d-3}\cdot\widetilde{S\ell} & , & \widetilde{P}^{N-2d+1}\cdot\widetilde{L}^{2d-4}\cdot\widetilde{C\ell}\\
\widetilde{P}^{N-2d}\cdot\widetilde{L}^{2d-3}\cdot\widetilde{Sp} & & \widetilde{P}^{N-2d}\cdot\widetilde{L}^{2d-4}\cdot\widetilde{Cp}
\end{matrix}$$

(notations as in Theorem I) *in a variety of complete curves.* As in §1.3, we will compute these products in a variety isomorphic to a variety of complete curves along an open set containing the intersection points of a general choice of conditions. Following the notations of [**2**], denote by $B \subset \mathbb{P}^N$ the set of curves $c\mu^2$ consisting of a degree-$(d-2)$ curve c and of the double line supported on a line μ. Also, denote by B° the open subset of B formed by curves $c\mu^2$ with c reduced and transversal to μ. The analogue of Lemma 1.7 in the new situation is:

Lemma 2.1. *For $j = N - 2d + 1$ and $P_1, \dots, P_j$ general point-conditions in $\mathbb{P}^N$, $P_1 \cap \cdots \cap P_j$ meets $S, S\ell, C, C\ell$ at points corresponding to either reduced curves or curves in B°. The same conclusion applies to the intersection of $P_1 \cap \cdots \cap P_{j-1}$ and Sp, Cp.*

Proof: As for Lemma 1.7, this follows from Lemma 1.1 in [**2**] and Remark 1, §1 in [**1**] (also, cf. Lemma 1.3 in [**2**]). ∎

Lemma 2.1 gives us the prescription to fulfill to compute the products $\widetilde{P}^{N-2d+1} \cdot \widetilde{L}^{2d-2} \cdot \widetilde{S}$, etc. above: the products may be computed in any variety $\widetilde{V} \xrightarrow{\pi} \mathbb{P}^N$ such that $\pi^{-1}(B^\circ)$ is disjoint from the intersection of all line-conditions in $\widetilde{V}$. Indeed, such a $\widetilde{V}$ is isomorphic to a variety of complete curves along an open subset containing $\pi^{-1}(B^\circ)$, and general conditions won't intersect in the complement of this open set, by Lemma 2.1.

Such a variety is the variety obtained in [**2**], §3, by the following procedure.[1]

Let $V_1 \xrightarrow{\pi_1} \mathbb{P}^N$ be the blow-up of $\mathbb{P}^N$ along B. B is smooth along B° (cf. Lemma 1.1 in [**2**]), so the fiber $\pi_1^{-1}(c\mu^2)$ over a $c\mu^2 \in B^\circ$ is the $\mathbb{P}^{2d-2}$ consisting of all normal directions to B in $\mathbb{P}^N$ centered at $c\mu^2$. Those directions determined by lines $c\mu^2 + tk\mu$ in $\mathbb{P}^N$ (k being a degree-$(d-1)$ curve) define a $\mathbb{P}^{d-3}$ in $\pi_1^{-1}(c\mu^2)$, and a $\mathbb{P}^{d-3}$-bundle B_1° over B° as $c\mu^2$ moves in B°. We let B_1 be the closure of B_1° in V_1. Next, let $V_2 \xrightarrow{\pi_2} V_1$ be the blow-up of V_1 along B_1. It follows from Proposition 3.4 in [**2**] that $\pi_2^{-1}\pi_1^{-1}(B^\circ)$ is disjoint from the intersection of all line-conditions in V_2: V_2 is therefore a variety satisfying our requirement.

Let then $\widetilde{V}$ be V_2, $\widetilde{P}, \widetilde{L}$ be the classes of the general point- and line-conditions in $\widetilde{V} = V_2$, etc.: by the above discussion, this switch in notation won't affect the result of computing $\widetilde{P}^{N-2d+1} \cdot \widetilde{L}^{2d-2} \cdot \widetilde{S}$, etc.

In the rest of this section we will get the main ingredients needed to compute these intersection products: i.e., an information amounting to certain terms in the Segre classes $s(B \cap S, S), s(B \cap S\ell, S\ell), \dots$ and terms in corresponding classes of loci in V_1. In §3 we will use these results to compute the intersection products listed at the beginning of this section; these in turn (by Theorem I) will give the characteristic numbers.

In fact, to optimize the computations, we will obtain here the classes in a different form. For $W \subset V$ non-singular varieties, and $X \subset V$ a subscheme, we denote by $W \circ X$ the class $c(N_W V) \cap s(W \cap X, X)$ (this is the 'full intersection class' of [**1**], §2). In §2.1 below we will compute relevant terms in the classes $B^\circ \circ S$, $B^\circ \circ S\ell$, etc. These are classes in $B^\circ \cap S$, $B^\circ \cap S\ell$ etc.; however, the terms we will compute here will extend uniquely to classes of B, (since their codimension will be lower than

[1] In [**2**] we had a blanket assumption $d > 3$; however, this construction and the results we will quote from [**2**] work for $d = 3$ as well.

the codimension of the complement of B°), therefore we will *write these classes as classes of* B, and denote them by $B\circ S$, $B\circ S\ell$, etc. for short. Similar considerations and choice of notations apply to the classes $B_1\circ S_1$, $B_1\circ S\ell_1$, etc. (denoting by $S_1, S\ell_1$, etc. the proper transforms of $S, S\ell$, etc. in V_1), which we will compute in §2.2, and to classes $\widehat{B}\circ\widehat{S}$, etc.

We should mention that only a small portion of the information encoded in the above classes is needed for our computations. For example, as S is the discriminant hypersurface and $B\subset S$, the information we obtain here about $B\circ S$ is basically just the multiplicity of the discriminant along the set of curves containing a double line.

2.1. Classes in $\mathbb{P}^N$. Call B the locus of curves containing a double line (as above). B is the image of a map $\mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2\longrightarrow\mathbb{P}^N$, where $\check{\mathbb{P}}^2$ parametrizes the double line and $\mathbb{P}^{\frac{(d-2)(d+1)}{2}}$ parametrizes the residual degree-$(d-2)$ curve; B° is identified via this map with an open subset of $\mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2$ (cf. Lemma 1.1 in [**2**]). Let now $\widehat{B}=\mathbb{P}^2\times B$ be the inverse image $p_2^{-1}(B)$ in $\mathbb{P}^2\times\mathbb{P}^N$; so $\widehat{B}^\circ=p_2^{-1}(B^\circ)$ can be identified with an open subset of $\mathbb{P}^2\times\mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2$.

We denote by k,h resp. the classes of the hyperplane in $\mathbb{P}^2,\mathbb{P}^N$ (and their pull-backs). The Chow ring of $\mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2$ is generated by the pull-backs ℓ, m of the hyperplane classes from the factors, with obvious relations: so h pulls-back to $\ell+2m$. The classes of B that we will consider will be push-forward of classes by $\mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2\longrightarrow B$; classes in $\widehat{B}$ will be push-forward of classes by $\mathbb{P}^2\times\mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2\longrightarrow\widehat{B}$. To ease the exposition we will suppress push-forward and pull-back notations, so that e.g. classes in $\widehat{B}$ will be denoted simply as polynomials in k,ℓ,m (unless we fear ambiguity).

—*Nodal curves.*

Recall the notations of §1: we have described the discriminant $S\subset\mathbb{P}^N$ as the projection to $\mathbb{P}^N$ of a codimension-3 smooth subvariety $\widehat{S}$ of $\mathbb{P}^2\times\mathbb{P}^N$; similarly, $S\ell, Sp$ are projections of subvarieties $\widehat{S\ell},\widehat{Sp}$ of $\widehat{S}$.

Lemma 2.2. *With the above notations:*

(i) $B\circ S=$ coefficient of k^2 in $\widehat{B}\circ\widehat{S}$

(ii) $B\circ S\ell=$ coefficient of k^1 in $\widehat{B}\circ\widehat{S}$

(iii) $B\circ Sp=$ coefficient of k^0 in $\widehat{B}\circ\widehat{S}$

Proof: (i) follows from the birational invariance of Segre classes ([**4**], Proposition 4.2): since p_2 maps $\widehat{S}$ birationally to S, $s(B\cap S,S)=p_{2*}s(\widehat{B}\cap\widehat{S},\widehat{S})$; then the projection formula gives (i), since the only terms that don't vanish after pushing forward via p_2 are the terms multiplying k^2, and $N_{\widehat{B}^\circ}\mathbb{P}^2\times\mathbb{P}^N$ is the pull-back of $N_{B^\circ}\mathbb{P}^N$.

(ii), (iii) follow by the same argument, after remarking that $s(\widehat{B}^\circ\cap\widehat{S\ell},\widehat{S\ell})=k\cdot s(\widehat{B}^\circ\cap\widehat{S},\widehat{S})$, $s(\widehat{B}^\circ\cap\widehat{Sp},\widehat{Sp})=k^2\cdot s(\widehat{B}^\circ\cap\widehat{S},\widehat{S})$ (cf. Lemma 1.1 (ii), (iii), and observe that $\widehat{S\ell},\widehat{Sp}$ cut properly (in $\widehat{S}$) the support of the cone of $\widehat{B}^\circ\cap\widehat{S}$ in $\widehat{S}$). ∎

The highest dimensional terms in the classes for nodal curves are given by:

Proposition 2.3.
$B \circ S = 2(2d-3)[B] + \dots$
$B \circ S\ell = [B] + \dots$
$B \circ Sp = m + \dots$

Proof: By Lemma 2.2, we need to show that, discarding all but the highest dimensional terms involving powers of k:

$$\widehat{B} \circ \widehat{S} = m + k + 2(2d-3)k^2 + \dots$$

Now, since $\widehat{B}^\circ, \widehat{S}$ and $\mathbb{P}^2 \times \mathbb{P}^N$ are non-singular,

$$\begin{aligned}\widehat{B} \circ \widehat{S} &= c(N_{\widehat{B}^\circ}\mathbb{P}^2 \times \mathbb{P}^N) s(\widehat{B}^\circ \cap \widehat{S}, \widehat{S}) \\ &= c(N_{\widehat{S}}\mathbb{P}^2 \times \mathbb{P}^N) s(\widehat{B}^\circ \cap \widehat{S}, \widehat{B}^\circ) \\ &= (1 + 3(d-1)k + \dots) s(\widehat{B}^\circ \cap \widehat{S}, \widehat{B}^\circ)\end{aligned}$$

(this follows from [**4**], Example 4.2.6. The class $c(N_{\widehat{S}}\mathbb{P}^2 \times \mathbb{P}^N)$ was computed in Lemma 1.1). Regarding $s(\widehat{B}^\circ \cap \widehat{S}, \widehat{B}^\circ)$, pull-back the equations for $\widehat{S}$ via $\mathbb{P}^2 \times \mathbb{P}^{\frac{(d-2)(d+1)}{2}} \times \check{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2 \times \mathbb{P}^N$. In codimension ≤ 2 we find $\widehat{B}^\circ \cap \widehat{S}$ is supported on a divisor of $\widehat{B}^\circ$, consisting of pairs $(p, c\mu^2)$ with $p \in \mu$, and has an embedded component supported on the set of pairs $(p, c\mu^2)$ with $p \in c \cap \mu$. The reader will easily verify that the classes of these loci are $m+k$, $(m+k)(\ell + (d-2)k)$ resp., so that

$$\begin{aligned}s(\widehat{B}^\circ \cap \widehat{S}, \widehat{B}^\circ) &= (m+k) - (m+k)^2 + \dots + (m+k)(\ell + (d-2)k) + \dots \\ &= (m+k) + (m+k)(\ell - m + (d-3)k) + \dots\end{aligned}$$

Thus

$$\begin{aligned}\widehat{B} \circ \widehat{S} &= (1 + 3(d-1)k + \dots)((m+k) + (m+k)(\ell - m + (d-3)k) + \dots) \\ &= (m+k) + (2(2d-3)k^2 + \dots) + \dots\end{aligned}$$

as claimed. ∎

—*Cuspidal curves.*

Again as in §1, the set C of cuspidal curves is the projection to $\mathbb{P}^N$ of a divisor $\widehat{C}$ (whith class $2(d-3)k + 2h$) of $\widehat{S}$. $C\ell, Cp$ are projection of subvarieties $\widehat{C\ell}, \widehat{Cp}$ of $\widehat{C}$.

Lemma 2.4.
(i) $B \circ C =$ *coefficient of* k^2 *in* $\widehat{B} \circ \widehat{C}$
(ii) $B \circ C\ell =$ *coefficient of* k^1 *in* $\widehat{B} \circ \widehat{C}$
(iii) $B \circ Cp =$ *coefficient of* k^0 *in* $\widehat{B} \circ \widehat{C}$

Proof: As in Lemma 2.2, these follow from the birational invariance of Segre classes. ∎

The highest dimensional terms in the classes for cuspidal curves are now given by

PROPOSITION 2.5.
$B \circ C = 6(d-2)[B] + \dots$
$B \circ C\ell = [B] + \dots$
$B \circ Cp = m + \dots$

PROOF: By Lemma 2.4, we need to compute the highest dimensional terms involving powers of k in $\widehat{B} \circ \widehat{C}$. Now we observe that $\widehat{C}$ contains $\widehat{B} \cap \widehat{S}$ scheme-theoretically (another coordinate computation); by Lemma 1.4, the class of $\widehat{C}$ in $\widehat{S}$ is $2(d-3)k+h$, restricting on $\widehat{B}$ to $2(d-3)k + \ell + 2m$; we have then

$$s(\widehat{B}^\circ \cap \widehat{C}, \widehat{C}) = (1 + 2(d-3)k + \ell + 2m)s(\widehat{B}^\circ \cap \widehat{S}, \widehat{S})$$

and therefore

$$\begin{aligned}\widehat{B} \circ \widehat{C} &= (1 + 2(d-3)k + \ell + 2m)\widehat{B} \circ \widehat{S} \\ &= (1 + 2(d-3)k + \ell + 2m)(m + k + 2(2d-3)k^2 + \dots) \\ &= m + k + (2(d-3) + 2(2d-3))k^2 + \dots \\ &= m + k + 6(d-2)k^2 + \dots\end{aligned}$$

from which the statement follows. ∎

2.2. Classes in V_1. $\pi_1 : V_1 \longrightarrow \mathbb{P}^N$ is the blow-up of $\mathbb{P}^N$ along B, E is the exceptional divisor. For $c\mu^2 \in B^\circ$, the fiber $\pi_1^{-1}(c\mu^2)$ consists of the $\mathbb{P}^{2d-2}$ of normal directions to B in $\mathbb{P}^N$ centered at $c\mu^2$. The directions determined by lines $c\mu^2 + tk\mu$ in $\mathbb{P}^N$ determine, as $c\mu^2$ varies in B°, a subvariety B_1° of E; and we let B_1 be the closure of B_1° in V_1.

Denote by $S_1, S\ell_1$, etc. the proper transforms of S, S_1, etc. in V_1; also, denote by $\widehat{S}_1, \widehat{S\ell}_1$, etc. the proper transforms of $\widehat{S}, \widehat{S\ell}$, etc. via the map $\mathbb{P}^2 \times V_1 \xrightarrow{\text{id.}\times\pi_1} \mathbb{P}^2 \times \mathbb{P}^N$, i.e. the blow-up of $\mathbb{P}^2 \times \mathbb{P}^N$ along $\widehat{B}$. Finally, let $\widehat{B}_1^\circ = \mathbb{P}^2 \times B_1^\circ, \widehat{B}_1 = \mathbb{P}^2 \times B_1$.

LEMMA 2.6.
$B_1 \circ S_1 =$ coeff. of k^2 in $\widehat{B}_1 \circ \widehat{S}_1$
$B_1 \circ S\ell_1 =$ coeff. of k^1 in $\widehat{B}_1 \circ \widehat{S}_1$
$B_1 \circ Sp_1 =$ coeff. of k^0 in $\widehat{B}_1 \circ \widehat{S}_1$
$B_1 \circ C_1 =$ coeff. of k^2 in $\widehat{B}_1 \circ \widehat{C}_1$
$B_1 \circ C\ell_1 =$ coeff. of k^1 in $\widehat{B}_1 \circ \widehat{C}_1$
$B_1 \circ Cp_1 =$ coeff. of k^0 in $\widehat{B}_1 \circ \widehat{C}_1$

PROOF: These follow again from the birational invariance of Segre classes, as in Lemmas 2.2, 2.4. ∎

Since B_1° is a projective bundle over B°, classes of B_1° can be expressed in terms of those of B° and of the class of the universal line bundle on B_1°: this is the restriction of the class of the exceptional divisor, which we denote e. This time we need the highest dimensional terms *involving powers of e*. These are

PROPOSITION 2.7.
$B_1 \circ S_1 = 2(2d-3)(1-e) + \dots$
$B_1 \circ S\ell_1 = (1-e)^2 + \dots$

$B_1 \circ Sp_1 = m(1-e)^2 + \dots$
$B_1 \circ C_1 = 6(d-2)(1-e)^2 + \dots$
$B_1 \circ C\ell_1 = (1-e)^3 + \dots$
$B_1 \circ Cp_1 = m(1-e)^3 + \dots$

Proof: By Lemma 2.6, we have to show

$$\widehat{B}_1 \circ \widehat{S}_1 = m + k - 2em - 2ek + 2(2d-3)k^2 + e^2m + e^2k - 2(2d-3)ek^2 + \dots$$
$$\widehat{B}_1 \circ \widehat{C}_1 = m + k - 3em - 3ek + 6(d-2)k^2 + 3e^2m + 3e^2k \\ - 12(d-2)ek^2 - e^3m - e^3k + 6(d-2)e^2k^2 + \dots$$

(omitting all but the highest dimensional terms involving monomials e^ik^j).

Computing these classes is a little tricky. Let $\widetilde{\mathbb{P}^2 \times \mathbb{P}^N}$ be the blow-up of $\mathbb{P}^2 \times \mathbb{P}^N$ along the incidence correspondence $I = \{(p, c\mu^2) \in \widehat{B} : p \in c \cap \mu\}$ (recall $\widehat{B}^\circ \cap \widehat{S}$ has an embedded component along this locus). Let $\widetilde{\mathbb{P}^2 \times V_1}$ be the blow-up of $\mathbb{P}^2 \times V_1$ along $(\mathrm{id.} \times \pi_1)^{-1}(I)$. By the universal property of blow-ups, $\widetilde{\mathbb{P}^2 \times V_1}$ is also the blow-up of $\widetilde{\mathbb{P}^2 \times \mathbb{P}^N}$ along the proper transform of $\widehat{B}$:

$$\begin{tikzcd}
\widetilde{\mathbb{P}^2 \times V_1} \arrow[r, "\text{Blow-up } (\mathrm{id.}\times\pi_1)^{-1}(I)"] \arrow[d, "\text{Blow-up proper transf. of } \widehat{B}"'] & \mathbb{P}^2 \times V_1 \arrow[d, "\mathrm{id.}\times\pi_1"] \\
\widetilde{\mathbb{P}^2 \times \mathbb{P}^N} \arrow[r, "\text{Blow-up } I"] & \mathbb{P}^2 \times \mathbb{P}^N
\end{tikzcd}$$

Notice that the bottom map blows-up each $\mathbb{P}^2 \times c\mu^2 \subset \widehat{B}^\circ$ at the finite set of points $c \cap \mu$. The proper transform of $\widehat{S}$ in $\widetilde{\mathbb{P}^2 \times \mathbb{P}^N}$ cuts each of these blown-up $\mathbb{P}^2$ along the proper transform of μ and along the exceptional divisors. By chasing the above diagram, one concludes that, above $\widehat{B}^\circ$, $\widehat{S}_1$ intersects $\widehat{E} = \mathbb{P}^2 \times E$ in two irreducible components, whose closures we denote $\widehat{E}_1$, $\widehat{E}_2$: $\widehat{E}_1$ dominates the support of $\widehat{B}^\circ \cap \widehat{S}$, $\widehat{E}_2$ dominates I, i. e. the embedded component in $\widehat{B}^\circ \cap \widehat{S}$. One can also see, again working in coordinates, that $\widehat{S}_1$ intersects $\widehat{B}_1^\circ$ precisely along the divisor of $\widehat{B}_1^\circ$ mapping to the support of $\widehat{B}^\circ \cap \widehat{S}$ (this time without embedded components). So, with our convention of omitting pull-back notations:

$$s(\widehat{B}_1^\circ \cap \widehat{S}, \widehat{B}_1^\circ) = (m+k) - (m+k)^2 + \dots$$

Next, we compute the first Chern class of the normal bundle to $\widehat{S}_1$ in $\widetilde{\mathbb{P}^2 \times V_1}$ ($\widehat{S}_1$ is regularly embedded in low codimension). We leave to the reader to chase the above diagram and verify that: if b denotes the codimension of $\widehat{B}$ in $\mathbb{P}^2 \times \mathbb{P}^N$, then $c_1(T\mathbb{P}^2 \times V_1) = c_1(T\mathbb{P}^2 \times \mathbb{P}^N) - (b-1)\widehat{E}$ restricts on $\widehat{S}$ to $c_1(T\mathbb{P}^2 \times \mathbb{P}^N) - (b-1)\widehat{E}_1 - 2(b-1)\widehat{E}_2$; while $c_1(T\widehat{S}_1) = c_1(T\widehat{S}) - (b-3)\widehat{E}_1 - (2b-5)\widehat{E}_2$. So

$$c_1(N_{\widehat{S}_1}\mathbb{P}^2 \times V_1) = c_1(N_{\widehat{S}}\mathbb{P}^2 \times \mathbb{P}^N) - 2\widehat{E}_1 - 3\widehat{E}_2 = c_1(N_{\widehat{S}}\mathbb{P}^2 \times \mathbb{P}^N) - 2\widehat{E} + \widehat{E}_2$$

(cf. this computation for $d = 3$ in [**3**]; there $b = 5$). Now, on $\widehat{B}_1$, $c_1(N_{\widehat{S}}\mathbb{P}^2 \times \mathbb{P}^N)$ pulls-back to $3(d-1)k + 3\ell + 6m$ (Lemma 1.1), $\widehat{E}$ restricts to e, and $\widehat{E}_2$ restricts to $\ell + (d-2)k$ (cf. the proof of Proposition 2.3), so

$$\begin{aligned} c_1(N_{\widehat{S}_1}\mathbb{P}^2 \times V_1) \text{ restricts to } & 3(d-1)k + 3\ell + 6m - 2e + \ell + (d-2)k \\ & = (4d-5)k - 2e + 4\ell + 6m \quad . \end{aligned}$$

Applying again [**4**], Example 4.2.6 (as in Proposition 2.3) gives the first terms of $\widehat{B}_1 \circ \widehat{S}_1$:

$$\begin{aligned} \widehat{B}_1 \circ \widehat{S} &= c(N_{\widehat{B}_1^\circ}\mathbb{P}^2 \times V_1) s(\widehat{B}_1^\circ \cap \widehat{S}_1, \widehat{S}_1) \\ &= c(N_{\widehat{S}_1}\mathbb{P}^2 \times V_1) s(\widehat{B}_1^\circ \cap \widehat{S}_1, \widehat{B}_1^\circ) \\ &= (1 + (4d-5)k - 2e + 4\ell + 6m + \dots)((m+k) - (m+k)^2 + \dots) \\ &= (m+k) + (m+k)(2(2d-3)k - 2e + 4\ell + 5m) + \text{terms in cod.} \geq 3 \end{aligned}$$

Finally, we observe that the only term in codimension ≥ 3 in $\widehat{B}_1 \circ \widehat{S}_1$ is $\widehat{B}_1 \cdot \widehat{S}_1$ (as defined in [**4**]; see the Lemma in [**1**], §2); i.e., the pull-back to $\widehat{B}_1$ of the class of $\widehat{S}_1$. This latter can be obtained by applying [**4**], Theorem 6.7; in terms of 'full intersection classes' (and omitting pull-backs as usual):

$$[\widehat{S}_1] = [\widehat{S}] - E \cdot \text{cod. 2 terms in } \frac{\widehat{B} \circ \widehat{S}}{1+E}$$

(see the Claim in the proof of Theorem II in [**1**], §2, with $r = 1$). By Lemma 1.1, (i), $\widehat{S}$ has class $((d-1)k + h)^3$; so

$$\begin{aligned} \widehat{B}_1 \cdot \widehat{S}_1 &= ((d-1)k + \ell + 2m)^3 - e \cdot \text{ cod. 2 terms in } \frac{m + k + 2(2d-3)k^2 + \dots}{1+e} \\ &= e^2 m + e^2 k - 2(2d-3)ek^2 + \cdots \end{aligned}$$

omitting all but the highest dimensional terms involving $e^i k^j$. Putting all together (and omitting irrelevant terms):

$$\widehat{B}_1 \circ \widehat{S}_1 = m + k - 2em - 2ek + 2(2d-3)k^2 + e^2 m + e^2 k - 2(2d-3)ek^2 + \dots$$

as claimed.

To get $\widehat{B}_1 \circ \widehat{C}_1$ we proceed as in Proposition 2.5: one checks that $\widehat{C}_1$ contains $\widehat{B}_1^\circ \cap \widehat{S}_1 = \widehat{B}_1^\circ$; and since $\widehat{C}$ contains $\widehat{B}^\circ \cap \widehat{S}$ and is generically smooth along it, the class of the divisor $\widehat{C}_1$ in $\widehat{S}_1$ must be $2(d-3)k + h - e$ (cf. Lemma 1.4). So, arguing as in Proposition 2.5,

$$\widehat{B}_1 \circ \widehat{C}_1 = (1 + 2(d-3) - e + \dots)\widehat{B}_1 \circ \widehat{S}_1 \quad ,$$

which gives the result stated at the beginning of the proof. ∎

3. Characteristic numbers, II. After Propositions 2.3, 2.5, 2.7, computing the intersection numbers

$$\begin{array}{lcl} \widetilde{P}^{N-2d+1}\cdot\widetilde{L}^{2d-2}\cdot\widetilde{S} & & \widetilde{P}^{N-2d+1}\cdot\widetilde{L}^{2d-3}\cdot\widetilde{C} \\ \widetilde{P}^{N-2d+1}\cdot\widetilde{L}^{2d-3}\cdot\widetilde{S\ell} & , & \widetilde{P}^{N-2d+1}\cdot\widetilde{L}^{2d-4}\cdot\widetilde{C\ell} \\ \widetilde{P}^{N-2d}\cdot\widetilde{L}^{2d-3}\cdot\widetilde{Sp} & & \widetilde{P}^{N-2d}\cdot\widetilde{L}^{2d-4}\cdot\widetilde{Cp} \end{array}$$

is a rather straightforward procedure, given the details of the blow-up construction (as in [**2**], §4). The main tool is a formula from [**1**]:

Proposition. *Let $B\subset V$ be smooth varieties, $X_1,\dots,X_n$ subvarieties of V, $\widetilde{V}\to V$ the blow-up of V along B, and $\widetilde{X}_1,\dots,\widetilde{X}_n$ the proper transforms of $X_1,\dots,X_n$ in $\widetilde{V}$. If the codimensions of the X_i's add to the dimension of V, then*

$$\widetilde{X}_1\cdot\ldots\cdot\widetilde{X}_n = X_1\cdot\ldots\cdot X_n - \int_B \frac{\prod_j (B\circ X_j)}{c(N_B V)} \tag{*}$$

([**1**], Theorem II). As seen in [**2**], §4, this extends to our case: each of the two blow-up restricts to the situation of the proposition on a dense open set containing all the intersection points. In fact, we can use for B in (*) (as in [**2**], §4.2,3) suitable varieties mapping birationally onto the centers of the blow-ups: $\mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2$ for the first blow-up, and a variety we called $\overline{\mathbb{P}(\mathcal{G})}$ in [**2**], §4.2, for the second. Concerning $\overline{\mathbb{P}(\mathcal{G})}$, we only need to remark that there is a surjection $\overline{\mathbb{P}(\mathcal{G})}\xrightarrow{p}\mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2$, making the diagram

$$\begin{array}{ccc} \overline{\mathbb{P}(\mathcal{G})} & \longrightarrow & V_1 \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ \mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2 & \longrightarrow & \mathbb{P}^N \end{array}$$

commutative; if we denote by e the pull-back of the exceptional divisor from V_1 to $\overline{\mathbb{P}(\mathcal{G})}$, then

Lemma 3.1. *$p_*e^j = 0$ for $j < d-3$; $p_*e^{d-3} = (-1)^{d-1}$.*

Proof: See [**2**], Lemma 4.1. ∎

We apply (*) to the two blow-ups giving the variety $\widetilde{V}$ of §2:

Proposition 3.2. *For $X = S, S\ell, \dots$; $X_1 = S_1, S\ell_1, \dots$ the proper transform of X in V_1; $\widetilde{X} = \widetilde{S}, \widetilde{S\ell}, \dots$ the proper transform of X_1 in $\widetilde{V}$; P, L, P_1, L_1, $\widetilde{P}, \widetilde{L}$ resp. point- and line-conditions in $\mathbb{P}^N$, V_1, $\widetilde{V}$; and $c = codim_{\mathbb{P}^N} X$,*

$$P_1^{N-k-c}\cdot L_1^k\cdot X_1 = P^{N-k-c}\cdot L^k\cdot X - \int_{\mathbb{P}^{\frac{(d-2)(d+1)}{2}}\times\check{\mathbb{P}}^2} (\ell+2m)^{N-k-c} B\circ X$$

$$\widetilde{P}^{N-k-c}\cdot\widetilde{L}^k\cdot\widetilde{X} = P_1^{N-k-c}\cdot L_1^k\cdot X_1 - \int_{\overline{\mathbb{P}(\mathcal{G})}} (\ell+2m)^{N-k-c}\frac{(1-e)^{k-d+1}}{(1+e)} B_1\circ X_1$$

where $k \leq 2d-2$ *for* $X = S$, $k \leq 2d-3$ *for* $X = S\ell, Sp, C$, *and* $k \leq 2d-4$ *for* $X = C\ell, Cp$.

Proof: This are just the results obtained applying (*) and omitting terms that give no contribution in the specified range. For example, applying (*) to the second blow-up gives really

$$\begin{aligned}\widetilde{P}^{N-k-c} \cdot \widetilde{L}^k \cdot \widetilde{X} = P_1^{N-k-c} \cdot L_1^k \cdot X_1 - \int_{\mathbb{P}(\mathcal{G})} (\ell + 2m)^{N-k-c} \\ \cdot \frac{(1 + (2d-2)\ell + (4d-4)m - e)^k (1 + \ell + m - e)^{\binom{d+1}{2}}}{(1+e)(1 + \ell + 2m - e)^{\binom{d+2}{2}}} B_1 \circ X_1\end{aligned}$$

(see [**2**] §4.2); however, in the specified range the fraction contributes terms in codimension $d-3$; and by Lemma 3.1 and the projection formula, the only monomial in ℓ, m, e that can have non-zero degree in codimension $d-3$ is e^{d-3}; so ℓ and m can be discarded in the fraction, and one gets the second formula as stated. ∎

Now the computation of the intersection numbers is a straightforward application of Propositions 1.2, 1.5, 2.3, 2.5, 2.7. and 3.2. As an illustration, we trace the computation for the locus of cuspidal curves:

—by Proposition 1.5,

$$\begin{aligned}P^{N-2d+1} \cdot L^{2d-3} \cdot C &= (2d-2)^{2d-3} \cdot 12(d-1)(d-2) \\ &= 3 \cdot 2^{2d-1}(d-1)^{2d-2}(d-2);\end{aligned}$$

—by Proposition 2.5 and the first formula in Proposition 3.2,

$$\begin{aligned}P_1^{N-2d+1} \cdot L_1^{2d-3} \cdot C_1 &= 3 \cdot 2^{2d-1}(d-1)^{2d-2}(d-2) - \int (\ell + 2m)^{N-2d+1} B \circ X \\ &= 3 \cdot 2^{2d-1}(d-1)^{2d-2}(d-2) - \int (\ell + 2m)^{N-2d+1}(6(d-2) + \dots) \\ &= 3 \cdot 2^{2d-1}(d-1)^{2d-2}(d-2) - 24(d-2)\binom{\binom{d}{2}+1}{2}\end{aligned}$$

—by Proposition 2.7 and the second formula in Proposition 3.2,

$$\begin{aligned}\widetilde{P}^{N-2d+1} \cdot \widetilde{L}^{2d-3} \cdot \widetilde{C} = 3 \cdot 2^{2d-1}(d-1)^{2d-2}(d-2) - 24(d-2)\binom{\binom{d}{2}+1}{2} \\ - \int_{\mathbb{P}(\mathcal{G})} (\ell + 2m)^{N-2d+1} \frac{(1-e)^{d-4}}{(1+e)} (6(d-2)(1-e)^2 + \dots);\end{aligned}$$

since the term of degree $d-3$ in $\dfrac{(1-e)^{d-2}}{(1+e)}$ is $(-1)^{d-1}(2^{d-2}-1)e^{d-3}$, Lemma 3.1

and the projection formula give

$$\begin{aligned}
\widetilde{P}^{N-2d+1} \cdot \widetilde{L}^{2d-3} \cdot \widetilde{C} &= 3 \cdot 2^{2d-1}(d-1)^{2d-2}(d-2) - 24(d-2)\binom{\binom{d}{2}+1}{2} \\
&\quad - 6(d-2)(2^{d-2}-1)\int_{\mathbb{P}^{\frac{(d-2)(d+1)}{2}} \times \check{\mathbb{P}}^2} (\ell + 2m)^{N-2d+1} \\
&= 3 \cdot 2^{2d-1}(d-1)^{2d-2}(d-2) - 24(d-2)\binom{\binom{d}{2}+1}{2} \\
&\quad - 6(d-2)(2^{d-2}-1) \cdot 4\binom{\binom{d}{2}+1}{2} \\
&= 3 \cdot 2^{2d-1}(d-1)^{2d-2}(d-2) - 3 \cdot 2^{d+1}(d-2)\binom{\binom{d}{2}+1}{2}
\end{aligned}$$

This procedure, applied to all loci, gives the list:

Theorem II.

$$\widetilde{P}^{N-2d+1} \cdot \widetilde{L}^{2d-2} \cdot \widetilde{S} = 3 \cdot 2^{2d-2}(d-1)^{2d} - 2^{d+1}(2d-3)\binom{\binom{d}{2}+1}{2}$$

$$\widetilde{P}^{N-2d+1} \cdot \widetilde{L}^{2d-3} \cdot \widetilde{S\ell} = 3 \cdot 2^{2d-3}(d-1)^{2d-2} - 2^{d}\binom{\binom{d}{2}+1}{2}$$

$$\widetilde{P}^{N-2d} \cdot \widetilde{L}^{2d-3} \cdot \widetilde{Sp} = 2^{2d-3}(d-1)^{2d-3} - 2^{d-1}\binom{d}{2}$$

$$\widetilde{P}^{N-2d+1} \cdot \widetilde{L}^{2d-3} \cdot \widetilde{C} = 3 \cdot 2^{2d-1}(d-1)^{2d-2}(d-2) - 3 \cdot 2^{d+1}(d-2)\binom{\binom{d}{2}+1}{2}$$

$$\widetilde{P}^{N-2d+1} \cdot \widetilde{L}^{2d-4} \cdot \widetilde{C\ell} = 2^{2d-2}(2d-3)(d-1)^{2d-4} - 2^{d}\binom{\binom{d}{2}+1}{2}$$

$$\widetilde{P}^{N-2d} \cdot \widetilde{L}^{2d-4} \cdot \widetilde{Cp} = 2^{2d-3}(d-1)^{2d-4} - 2^{d-1}\binom{d}{2}$$

By the discussion in the beginning of §2, these are the intersection numbers of the loci in *any varieties of complete curves* $\widetilde{V}$, so we can proceed and apply Theorem I from §1 to conclude the computation of the characteristic numbers. Taking again cuspidal curves as an example,

$$Cp_d(2d-5) = 2^{2d-4}(d-1)^{2d-5}$$

by Corollary 1.9, so

$$\begin{aligned}
C\ell_d(2d-4) &= \widetilde{P}^{N-2d+1} \cdot \widetilde{L}^{2d-4} \cdot \widetilde{C\ell} - 3(2d-4)Cp_d(2d-5) \\
&= 2^{2d-2}(2d-3)(d-1)^{2d-4} - 2^{d}\binom{\binom{d}{2}+1}{2} - 3 \cdot 2^{2d-3}(d-1)^{2d-5}(d-2) \\
&= 2^{2d-3}(d-1)^{2d-5}(4d^2-13d+12) - 2^{d}\binom{\binom{d}{2}+1}{2}
\end{aligned}$$

by Theorem I and Theorem II; and therefore

$$\begin{aligned}
C_d(2d-3) &= \widetilde{P}^{N-2d+1}\cdot\widetilde{L}^{2d-3}\cdot\widetilde{C} - 3(2d-3)C\ell_d(2d-4) \\
&\quad - 9\binom{2d-3}{2}Cp_d(2d-5) \\
&= 3\cdot 2^{2d-1}(d-1)^{2d-2}(d-2) - 3\cdot 2^{d+1}(d-2)\binom{\binom{d}{2}+1}{2} \\
&\quad - 3(2d-3)\left[2^{2d-3}(d-1)^{2d-5}(4d^2-13d+12) - 2^d\binom{\binom{d}{2}+1}{2}\right] \\
&\quad - 9\binom{2d-3}{2}\left[2^{2d-4}(d-1)^{2d-5}\right] \\
&= 3\cdot 2^{2d-4}(d-1)^{2d-5}(8d^4-56d^3+142d^2-161d+70) + 3\cdot 2^d\binom{\binom{d}{2}+1}{2}
\end{aligned}$$

by Theorems I and II again.

This procedure gives

Theorem III.

$$\begin{aligned}
S_d(2d-2) &= 2^{2d-2}(d-1)^{2d-3}(3d^3-15d^2+23d-12) + 2^{d+1}\binom{\binom{d}{2}+1}{2} \\
S\ell_d(2d-3) &= 2^{2d-3}(d-1)^{2d-4}(3d^2-8d+6) - 2^d\binom{\binom{d}{2}+1}{2} \\
Sp_d(2d-3) &= 2^{2d-3}(d-1)^{2d-3} - 2^{d-1}\binom{d}{2} \\
C_d(2d-3) &= 3\cdot 2^{2d-4}(d-1)^{2d-5}(8d^4-56d^3 \\
&\qquad\qquad + 142d^2-161d+70) + 3\cdot 2^d\binom{\binom{d}{2}+1}{2} \\
C\ell_d(2d-4) &= 2^{2d-3}(d-1)^{2d-5}(4d^2-13d+12) - 2^d\binom{\binom{d}{2}+1}{2} \\
Cp_d(2d-4) &= 2^{2d-3}(d-1)^{2d-4} - 2^{d-1}\binom{d}{2}
\end{aligned}$$

as stated in the introduction.

References

[**1**] Aluffi, P., *The enumerative geometry of plane cubics I: smooth cubics*, Trans. Amer. Math. Soc. **317** (1990), 501–539.

[**2**] Aluffi, P., *Two characteristic numbers for smooth plane curves of any degree*, Trans. Amer. Math. Soc. **329** (1992), 73–96.

[**3**] Aluffi, P., *The enumerative geometry of plane cubics II: nodal and cuspidal cubics*, Math. Ann. **289** (1991), 543–572.

[**4**] Fulton, W., "Intersection Theory," Springer Verlag, 1984.

[**5**] Kleiman, S., Speiser, R., *Enumerative geometry of cuspidal plane cubics*, Vancouver Proc., Canad. Math. Soc. Conf. Proc. **6** (1986), 227–268.

[**6**] Kleiman, S., Speiser, R., *Enumerative geometry of nodal plane cubics*, Algebraic Geometry – Sundance 1986 (1988), 156–196, Springer Lecture Notes 1311.
[**7**] Kleiman, S., Speiser, R., *Enumerative geometry of nonsingular plane cubics*, in "Algebraic geometry: Sundance 1988," Contemp. Math., 116, 1991, pp. 85–113.
[**8**] Walker, R.J., "Algebraic Curves," Princeton Univ. Press, 1950.
[**9**] Zeuthen, H.G., *Almindelige Egenskaber ved Systemer af plane Kurver*, Kongelige Danske Videnskabernes Selskabs Skrifter - Naturvidenskabelig og Mathematisk **10** (1873), 287–393.

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