SINGULAR SCHEMES OF HYPERSURFACES

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§0. Introduction

Let Y be the singular locus of a hypersurface X in a smooth variety M, with the scheme structure defined by the Jacobian ideal of X (we will say then that Y is the singular *scheme* of X, to emphasize that the scheme structure of Y is important for our considerations). In this note we consider a class in the Chow group of Y which arises naturally in this setup, and which captures much intersection-theoretic information about the situation. The guiding question we have in mind is: which schemes Y can arise as singular schemes of hypersurfaces? We will obtain strong constraints showing for instance that the only hypersurfaces in $\mathbb{P}^N$ whose singular *schemes* are positive dimensional linear subspaces of $\mathbb{P}^N$ are quadrics, and that no (reduced) nodal curve can be the singular scheme of a hypersurface in a non-singular variety. Many more statements of this sort can be found in section 3.

A different type of application is in section 2: we show how our class relates to other invariants of the singularity of a hypersurface; the class can be used to recover results of Holme and Parusiǹski on degree and multiplicity of dual varieties, and leads naturally to a generalization of the notion of 'ranks' of a (smooth) projective variety. Also, we obtain a strengthening of Landman's parity result, and a new proof of a result of Zak on the dimension of the dual of a smooth variety. The duality results follow by applying the framework to hyperplane sections of M: the singular scheme of a section is supported on the locus of contact of the hyperplane with M, and the class can be used to measure this contact. For example, the class measures how 'general' a given section is: we show (Corollary 2.6) that if the contact *scheme* is a linear subspace $\mathbb{P}^{r-1}$, then the corresponding hyperplane is a smooth point of the dual variety of M, and the dual variety has codimension r.

The main general results are in section 1, where we prove (Corollary 1.7) that the class we introduce depends in fact only on Y and on the line bundle $\mathcal{L} = \mathcal{O}(X)|_Y$, and not on the ambient variety M (provided that Y is the singular scheme of a section of $\mathcal{L}$ in M). Also, we show that the class is well-behaved with respect to general sections (Proposition 1.3). The degree of the zero-dimensional component of the class gives the ordinary Milnor number of the singularity when Y is an isolated point, and agrees with a generalization (the "μ-number") of the Milnor number due to Parusiǹski ([P1]) for positive dimensional Y (§2.1); we call our class the "μ-class" of Y with respect to $\mathcal{L}$. We prove that the μ-class is intrinsic to Y and $\mathcal{L}$ by showing (Theorem 1.6) that (if Y is the singular scheme of a hypersurface) it can be computed in terms of $\mathcal{L}$, of the sheaf of differentials of Y, and of Fulton's canonical classes of Y ([F], Chapter 4). For example, if Y is itself non-singular, its μ-class with respect to any line bundle $\mathcal{L}$ turns out to be

$$c(T^*Y \otimes \mathcal{L}) \cap [Y] \tag{*}$$

(where T^*Y is the cotangent bundle of Y). The constraints exploited in §3 typically follow by computing the μ-class both intrinsically and with respect to a specific realization of Y as a singular scheme of a section of $\mathcal{L}$ in some ambient space. Formula (*) was inspired by a result of Parusiǹski ([P1], Proposition 1.5); however, the constraints arise from computing the codimension-1 component in the μ-class, while Parusiǹski's results only deal with the 0-dimensional component. It would be interesting to extend (if possible) other results of Parusiǹski's to the whole μ-class.

It would also be interesting to find out to what extent a "μ-class" can be defined when Y is not necessarily the singular scheme of a hypersurface. The intrinsic expression we obtain (Theorem 1.6) is not necessarily well-defined then (cf. Example 3.8); still, it might be possible to define a class for arbitrary schemes Y and line bundles $\mathcal{L}$, satisfying some functoriality property, and agreeing with the μ-class when Y is the singular scheme of a hypersurface with line bundle $\mathcal{L}$ in a smooth variety. It doesn't seem unreasonable to expect that it should be possible to define a μ-class for the singular scheme of an arbitrary subvariety of a given non-singular variety. The resulting constraints would then be rather interesting.

The μ-class introduced here is used in [A2] to compare different notions of characteristic classes for a (possibly singular) hypersurface.

Acknowledgements. I was led to consider these classes as I was working on [A-C] (of which some results are used in §2), and I want to thank F. Cukierman for collaborating with me on that project. Also, I want to thank P. Pragacz for pointing out Parusiǹski's work to me. I had helpful conversations with R. Varley while writing this note. I thank MSRI for the hospitality while working on this project, and Florida State University for a Summer Award under which part of this research was done.

§1. The μ-class of the singularity of a hypersurface.

§**1.1. Preliminaries.** Throughout this paper we will use without further mention the following notations. M will denote a smooth n-dimensional algebraic variety over an algebraically closed field of characteristic 0; $\mathcal{L}$ will be a line bundle on M, and X will be the zero-scheme of a section of $\mathcal{L}$. Typically, X will be a prime divisor of M and $\mathcal{L} = \mathcal{O}(X)$; we will refer to X as a "hypersurface" on M. The *singular scheme* of X, Sing$X$, will be the subscheme of M supported on the singular locus of X, and defined locally by the ideal $(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})$, where $x_1, \dots, x_n$ are local parameters for M, and F is the section of $\mathcal{L}$ defining X; this structure is clearly independent of the choice of local parameters.

Definition. We say Y is an 's.s.h.' (singular scheme of a hypersurface) with respect to $\mathcal{L} = \mathcal{L}|_Y$ if $Y = \mathrm{Sing} X$ for some X as above.

Note that if Y is non-singular, nothing prevents us from taking $Y = X = M$, with $X =$ zero-scheme of the zero-section of any line bundle $\mathcal{L}$ on Y. In particular, every non-singular variety is an s.s.h. with respect to any line-bundle.

One prototype situation the reader may want to keep in mind (especially for §2) is the following: M is embedded in $\mathbb{P}^N$, $\mathcal{L}$ is the hyperplane bundle, X is a hyperplane section, and Y is the 'contact locus' of the section, that is the locus where the hyperplane is tangent to M; the locus is given a scheme structure as specified above, and named the contact *scheme.* We will at times borrow the terminology arising from this situation (for example, in §2 we will systematically call 'dual variety' the variety parametrizing singular sections of $\mathcal{L}$); in general, however, we do not require $\mathcal{L}$ to be very ample.

One remark concerning this set-up is in order before we start. We will shortly introduce a class 'measuring' the singularity of X along Y, one essential ingredient of which will be the Segre class $s(Y, M)$ of Y in M. Now observe that in certain important cases (e.g., isolated singularities) there is another natural way to define a scheme structure on Y: we could take the scheme Y' defined by the (a priori) smaller ideal $(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n})$. In some contexts, this structure may be preferable to the one we chose; the remark is that this choice is essentially irrelevant:

Lemma 1.1. *With the above notations, $s(Y, M) = s(Y', M)$.*

This statement is an easy consequence of the following general fact about Segre classes, which seems of independent interest:

Lemma 1.2. *(No restriction on the characteristic here.) Let Y, Y' be subschemes of an irreducible scheme V. Suppose that for all maps $\varphi : T \to V$ of the Spec of a discrete valuation ring into V, $\varphi^{-1}(Y) = \varphi^{-1}(Y')$. Then $s(Y, V) = s(Y', V)$.*

(Notice that necessarily $Y_{\mathrm{red}} = Y'_{\mathrm{red}}$; the Segre class lives in the Chow group of this reduced scheme.)

Proof. By [Hu], Lemma (3.4), the hypothesis implies that the ideal sheaves of Y and Y' have the same integral closure in $\mathcal{O}_V$. Thus we may assume the ideal sheaf of Y is the integral closure $\mathcal{J}$ of the ideal sheaf $\mathcal{I}$ of Y'. This implies $\mathcal{I}\mathcal{J}^n = \mathcal{J}^{n+1}$ for sufficiently large n.

Now blow-up V along Y. The inverse image of Y' is defined by the ideal $\oplus_n \mathcal{I}\mathcal{J}^n$, which by the above defines the same subscheme as $\oplus_n \mathcal{J}^{n+1}$, that is the exceptional divisor. The equality of Segre classes follows then by the birational invariance of these (that is, [F], Prop. 4.2). $\square$

Lemma 1.2 implies Lemma 1.1: we just have to observe that, in characteristic 0, the order of a function at any of its zeros is at least as large as the order of its derivatives.

§**1.2. The μ-class.** This paper studies the following class arising in the situation detailed in §1.1. We will call this class "μ-class" since it generalizes most naturally Parusiǹski's "μ-number" (see §2.1).

Definition. Let Y be the singular scheme of a section of a line bundle $\mathcal{L}$ on a smooth variety M. The *μ-class of Y with respect to $\mathcal{L}$* is the class

$$\mu_{\mathcal{L}}(Y) := c(T^*M \otimes \mathcal{L}) \cap s(Y, M)$$

in the Chow group A_*Y of Y.

Here and in the following, c denotes total Chern class; s is the Segre class (in the sense of [F]); and pull-back notations are omitted when no ambiguity is feared. Notice that this definition depends a priori on the choice of M: we will prove in §1.3, Corollary 1.7, that in fact $\mu_{\mathcal{L}}(Y)$ is intrinsic to Y and $\mathcal{L}|_Y$, in the sense that if Y is realized in *two* ways as an s.s.h., say with respect to $\mathcal{L}_1$ in M_1 and to $\mathcal{L}_2$ in M_2, and $\mathcal{L}_1|_Y = \mathcal{L}_2|_Y = \mathcal{L}$, then necessarily

$$c(T^*M_1 \otimes \mathcal{L}_1) \cap s(Y, M_1) = c(T^*M_2 \otimes \mathcal{L}_2) \cap s(Y, M_2)$$

In this paper we are not defining (and we will not use) the notion of μ-class *unless* Y is a s.s.h.

One basic fact about the μ-class is that it behaves most naturally with respect to general sections of $\mathcal{L}$, if there are enough of these:

Proposition 1.3. *Suppose $\mathcal{L}$ is generated by global sections, and let X_g be a general section of $\mathcal{L}$. Then*

$$\mu_{\mathcal{L}}(Y \cap X_g) = X_g \cdot \mu_{\mathcal{L}}(Y) \quad (= c_1(\mathcal{L}) \cap \mu_{\mathcal{L}}(Y))$$

The proof of this statement hinges on the following Lemma, which also seems of independent interest. Bertini's theorem tells us that the singular locus of $X \cap X_g$ equals set-theoretically the intersection of the singular locus of X with X_g. The following result shows that in fact the equality holds at the level of Segre classes:

Lemma 1.4. *Under the hypotheses of Proposition 1.3:*

$$s(\mathrm{Sing}(X \cap X_g), X_g) = s((\mathrm{Sing}X) \cap X_g, X_g)$$

Proof. Let F be a local equation for X, in local coordinates $x_1, \dots, x_n$ in M. For a given X_g we can choose the coordinates so that x_1 is the equation for X_g; then we have to compare the ideal of $(\mathrm{Sing}X) \cap X_g$, that is

$$\left(x_1, F, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n}\right)$$

and the ideal of $\mathrm{Sing}(X \cap X_g)$, that is

$$\left(x_1, F, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n}\right)$$

Notice the latter is included in the former; morally, we would like to say that for a general choice of X_g we can force $\frac{\partial F}{\partial x_1}$ to be contained in the second ideal, and the two would be equal.

The statement we need is however much weaker. Obtain an embedded resolution of singularities of the (support of) X in M: that is, a sequence of blow-ups at smooth centers yielding a variety $\widetilde{M} \xrightarrow{\pi} M$ such that the inverse image of X in $\widetilde{M}$ consists of the union of (multiples of) smooth divisors with normal crossing.

Claim. *For a general choice of X_g, $\pi^{-1}\left(\mathrm{Sing}(X \cap X_g)\right) = \pi^{-1}\left((\mathrm{Sing}X) \cap X_g\right)$*

For this, we argue by induction on the number r of blow-ups needed in the desingularization. If $r = 0$, that is if X consists of the union of divisors with smooth support and normal crossings, working locally we may assume X is given by $x_2^{e_2} x_3^{e_3} \cdots x_m^{e_m}$ and X_g by x_1, in which case the result is clear.

Assume then $r > 0$, and that the claim is true for $r - 1$ blow-ups. A general X_g intersects the center of the first blow-up $M_1 \xrightarrow{\pi_1} M$ transversally (by Bertini's theorem), so locally we may choose coordinates $y_1, \dots, y_n$ on M_1 in such a way that π_1 is given by

$$\begin{cases} x_1 = y_1 \\ x_2 = x_2(y_2, \dots, y_n) \\ \dots \\ x_n = x_n(y_2, \dots, y_n) \end{cases}$$

Then we have (denoting $F = \pi_1^* F$ for convenience)

$$\frac{\partial F}{\partial y_1} = \pi_1^* \frac{\partial F}{\partial x_1} \quad , \quad \frac{\partial F}{\partial y_i} = \sum_{j=2}^{n} \left(\pi_1^* \frac{\partial F}{\partial x_j} \right) \frac{\partial x_j}{\partial y_i} \quad \text{for } i > 1 \quad ,$$

and in particular

$$\left(F, \frac{\partial F}{\partial y_2}, \dots, \frac{\partial F}{\partial y_n} \right) \subset \left(F, \pi_1^* \frac{\partial F}{\partial x_2}, \dots, \pi_1^* \frac{\partial F}{\partial x_n} \right) \quad .$$

Now pull-back by the composition $\tilde{\pi} : \widetilde{M} \longrightarrow M_1$ of the remaining $r - 1$ blow-ups: this gives (writing $F = \pi^* F = \tilde{\pi}^* F$)

$$\tilde{\pi}^* \frac{\partial F}{\partial y_1} = \pi^* \frac{\partial F}{\partial x_1} \quad , \quad \left(F, \tilde{\pi}^* \frac{\partial F}{\partial y_2}, \dots, \tilde{\pi}^* \frac{\partial F}{\partial y_n} \right) \subset \left(F, \pi^* \frac{\partial F}{\partial x_2}, \dots, \pi^* \frac{\partial F}{\partial x_n} \right) \quad ;$$

and observe $\tilde{\pi}$ is a resolution of the inverse image of $\pi_1^{-1}(X)$ requiring $r-1$ blow-ups, so that by the induction hypothesis $\tilde{\pi}^* \frac{\partial F}{\partial y_1} \in \left(F, \tilde{\pi}^* y_1, \tilde{\pi}^* \frac{\partial F}{\partial y_2}, \dots, \tilde{\pi}^* \frac{\partial F}{\partial y_n} \right)$. Hence this implies that $\pi^* \frac{\partial F}{\partial x_1} \in \left(F, \pi^* x_1, \pi^* \frac{\partial F}{\partial x_2}, \dots, \pi^* \frac{\partial F}{\partial x_n} \right)$, which is the claim.

Next observe that $\pi^{-1} X_g$ = the proper transform $\widetilde{X}_g$ of X_g (since X_g is a hypersurface and its proper transforms all intersect transversally the centers of the blow-ups), so $\pi^{-1} X_g = \widetilde{X}_g \longrightarrow X_g$ is birational. By the claim we have

$$s(\pi^{-1} \mathrm{Sing}(X \cap X_g), \widetilde{X}_g) = s(\pi^{-1}((\mathrm{Sing}X) \cap X_g), \widetilde{X}_g) \quad ,$$

and the Lemma follows by applying π_* and the birational invariance of Segre classes. $\square$

Given Lemma 1.4, we can now prove Proposition 1.3:

Proof. Apply Lemma 1.4 with $Y = \mathrm{Sing}X$:

$$\begin{aligned} \mu_{\mathcal{L}}(Y \cap X_g) &= c(T^* X_g \otimes \mathcal{L}) \cap s(\mathrm{Sing}(X \cap X_g), X_g) \\ &= c(T^* X_g \otimes \mathcal{L}) \cap s((\mathrm{Sing}X) \cap X_g, X_g). \end{aligned}$$

Now we use Lemma A.3 in [A1] (with $m = 0$), and obtain

$$\mu_{\mathcal{L}}(Y \cap X_g) = c(T^* X_g \otimes \mathcal{L}) \cap X_g \cdot s(\mathrm{Sing}X, M)$$

Next, tensor

$$0 \to (N_{X_g} M)^* \to T^* M \to T^* X_g \to 0$$

by $\mathcal{L} = \mathcal{O}(X_g) = N_{X_g} M$ to realize $c(T^* X_g \otimes \mathcal{L}) = c((T^* M \otimes \mathcal{L})|_{X_g})$, and conclude

$$\begin{aligned}\mu_{\mathcal{L}}(Y \cap X_g) &= X_g \cdot c(T^* M \otimes \mathcal{L}) \cap s(\mathrm{Sing}X, M) = X_g \cdot \mu_{\mathcal{L}}(\mathrm{Sing}X) \\ &= c_1(\mathcal{L}) \cap \mu_{\mathcal{L}}(Y)\end{aligned}$$

as needed. $\square$

§**1.3.** $\mu_{\mathcal{L}}(Y)$ **is intrinsic to** Y **and** $\mathcal{L}$**.** Here we show the independence of the μ-class of Y from the specific ambient variety M (provided that Y can be realized as an s.s.h. in M): $\mu_{\mathcal{L}}(Y)$ depends only on the scheme Y and the restriction $\mathcal{L}$ of the line bundle to Y. To prove this, we will obtain (Theorem 1.6) an expression for $\mu_{\mathcal{L}}(Y)$ in terms of Fulton's intrinsic 'canonical classes' $c_*(Y)$ and of the sheaf of differentials of Y; this in turn will yield the particularly simple expression for $\mu_{\mathcal{L}}(Y)$ when Y is smooth (Corollary 1.8). The more general expression depends on the choice of a smooth scheme dominating Y; the upshot will be that this choice is irrelevant when Y is the singular scheme of a hypersurface. The expression in Theorem 1.6 is otherwise *not* necessarily well-defined; we do not know how to define a 'μ-class' of an arbitrary scheme Y with respect to a given line bundle $\mathcal{L}$. It is tempting to conjecture that there must be a reasonably functorial definition of such a class.

As a subproduct, we will also get a strong condition (Proposition 1.11) that a scheme must satisfy to be an s.s.h. In fact we do not know of any scheme satisfying this condition and which is *not* the singular scheme of a hypersurface. Question: is the condition of Proposition 1.11 in fact a criterion?

The key to our main result here is the fact that if Y is an s.s.h. in a variety M, then the operator $\dfrac{c(T^* M \otimes \mathcal{L})}{c(TM)} \cap -$ on the Chow group $A_* Y$ *does not depend on the choice of* M. To show this, we define an operator on $A_*(Y)$ in the following manner. Given any $\alpha \in A_*(Y)$, choose a non-singular variety mapping properly to Y: $Z \xrightarrow{p} Y$, together with a class $\beta \in A_*(Z)$ such that $p_*(\beta) = \alpha$: for example, we may choose Z to be a *smooth envelope* of Y (cf. [F], Example 15.1.6 and Lemma 18.3).

Definition. We put

$$\mathcal{A}_{\mathcal{L}}(\alpha) = p_* \left(c([p^*(\Omega_Y \otimes \mathcal{L})] - [\mathcal{H}om(p^* \Omega_Y, \mathcal{O}_Z)]) \cap \beta \right) \quad ,$$

where $[\mathcal{F}]$ denotes the class of $\mathcal{F}$ in the Grothendieck group of Z.

Remark. Here we are using the fact that Chern classes can be defined on the group $K_0(Z)$ of coherent sheaves on Z, since Z is non-singular and therefore the group coincides with the ring $K^0(Z)$ of locally free sheaves, cf. for example [H], p. 435. Note that we can*not* a priori apply a 'projection formula' and express $\mathcal{A}_{\mathcal{L}}(\alpha)$ as the Chern class of a sheaf on Y capped with $p_* \beta = \alpha$, since Chern classes are not defined on $K_0(Y)$. In fact, for arbitrary Y the right-hand-side of the formula in the statement may change for different choices of Z and β (cf. Example 3.8). However, we claim that $\mathcal{A}$ *is* well-defined if Y is an s.s.h., and in fact we will show:

Proposition 1.5. *If Y is an s.s.h. with respect to $\mathcal{L}$ in M, then*

$$\mathcal{A}_{\mathcal{L}} = \frac{c(T^*M \otimes \mathcal{L})}{c(TM)}$$

as operators on $A_*(Y)$.

We delay the proof of this a moment, and first list a few consequences. Here we let $c_*(Y)$ denote Fulton's intrinsic class of Y ([F], Example 4.2.6): this is a class defined for any scheme that is embeddable in a non-singular variety, and which agrees with $c(TY) \cap [Y]$ when Y is itself non-singular.

Theorem 1.6. *Let Y be an s.s.h. with respect to $\mathcal{L}$; then with notations as above*

$$\mu_{\mathcal{L}}(Y) = \mathcal{A}_{\mathcal{L}}(c_*(Y))$$

There is no guarantee that the right-hand-side in this formula is well-defined unless Y is an s.s.h. It is a consequence of the theorem that the class *is* well defined if Y is the singular scheme of a hypersurface: indeed, the left-hand-side does not depend on the choice of the variety Z and the class β needed to define $\mathcal{A}_{\mathcal{L}}$. Similarly:

Corollary 1.7. *Let Y be an s.s.h. with respect to $\mathcal{L}$. Then $\mu_{\mathcal{L}}(Y)$ only depends on Y and $\mathcal{L}$.*

(That is, the right-hand-side in the definition of the μ-class in section 1 is independent of the variety M in which Y is realized as an s.s.h.)

Proof. Indeed, the right-hand-side of the expression in Theorem 1.6 does not depend on the choice of a variety M in which Y is realized as the singular scheme of a section of a line bundle restricting to $\mathcal{L}$. $\square$

If Ω_Y is locally free, we can apply the projection formula in the definition of $\mathcal{A}_{\mathcal{L}}$:

Corollary 1.8. *If Y is non-singular, then*

$$\mu_{\mathcal{L}}(Y) = c(T^*Y \otimes \mathcal{L}) \cap [Y] \quad .$$

Proof. Apply the projection formula; or simply choose $Z = Y$, $\beta = c_*(Y)$ in the definition of $\mathcal{A}_{\mathcal{L}}$ and apply Theorem 1.6 to get

$$\mu_{\mathcal{L}}(Y) = \frac{c(T^*Y \otimes \mathcal{L})}{c(TY)} \cap c_*(Y) = c(T^*Y \otimes \mathcal{L}) \cap [Y]$$

since $c_*(Y) = c(TY) \cap [Y]$ when Y is non-singular. $\square$

Remark. Corollary 1.8 can also be deduced directly from the definition of $\mu_{\mathcal{L}}$ once we have Corollary 1.7: indeed, if Y is non-singular we may then compute $\mu_{\mathcal{L}}$ by choosing (almost pathologically) $M = Y$, and $X = Y =$ the zero-locus of the zero-section of $\mathcal{L}$. Then $\text{Sing}X = Y$, and $s(Y, M) = s(Y, Y) = [Y]$, so the definition of $\mu_{\mathcal{L}}$ gives the formula in the statement.

Remark/Example. We stress again that the right-hand-side in the definition of $\mu_{\mathcal{L}}(Y)$ in section 1 computes $\mu_{\mathcal{L}}(Y)$ *only if* Y can be realized as an s.s.h. in M. It is clear that, without this assumption on M, the expression $c(T^*M\otimes\mathcal{L})\cap s(Y,M)$ cannot be controlled. For example, consider $Y=\mathbb{P}^s\subset M'=\mathbb{P}^n$ and $\mathcal{L}=\mathcal{O}(1)$: then $\mu_{\mathcal{L}}(Y)=[Y]-c_1(\mathcal{O}(1))\cap[Y]+\dots$ (for example by Corollary 1.8), while $c(T^*M'\otimes\mathcal{L})\cap s(Y,M')=[Y]-(n-s+1)c_1(\mathcal{O}(1))\cap[Y]+\dots$ *does* depend on M', and equals $\mu_{\mathcal{L}}(Y)$ only in the pathological case $n=s$. However, this just says that a proper linear subspace cannot be realized as an s.s.h. of a projective space with respect to $\mathcal{O}(1)$.

This may be the simplest possible example of the constraints that will be exploited in the applications in §2 and §3. Note that $\mathbb{P}^s$ *can* be realized as an s.s.h. with respect to to $\mathcal{O}(1)$ in a 'non-pathological' way: for example, if $(x_i),(y_i)$ are homogenous coordinates in $\mathbb{P}^2,\mathbb{P}^1$ resp., then the singular scheme of x_0y_0 in $M=\mathbb{P}^2\times\mathbb{P}^1$ is a $\mathbb{P}^1$. This reflects the fact that the dual variety of the Segre embedding of M is not a hypersurface, cf. Corollary 2.6.

Proposition 1.5 will be deduced from the following fact. Suppose $\mathcal{E}$ is a vector bundle on a scheme Y, and that for a line bundle $\mathcal{L}$ on Y we have a symmetric map

$$\mathcal{E}\otimes\mathcal{E}\xrightarrow{\varphi}\mathcal{L}$$

This induces a vector bundle map $\mathcal{E}\xrightarrow{\phi}\mathcal{F}=\mathcal{H}om(\mathcal{E},\mathcal{L})$, and the induced diagram

$$(*)\qquad\begin{array}{ccc}\mathcal{H}om(\mathcal{F},\mathcal{L}) & \xrightarrow{\phi^T} & \mathcal{H}om(\mathcal{E},\mathcal{L})\\ \Vert & & \Vert\\ \mathcal{E} & \xrightarrow{\phi} & \mathcal{F}\end{array}$$

commutes by the symmetry of φ.

Proposition 1.9. *Then for any sheaf $\mathcal{M}$ of $\mathcal{O}_Y$-modules there is an exact sequence*

$$0\to\mathcal{H}om(\mathrm{Coker}\phi,\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{M})\to\mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{M}\to\mathcal{F}\otimes_{\mathcal{O}_Y}\mathcal{M}\to\mathrm{Coker}\phi\otimes_{\mathcal{O}_Y}\mathcal{M}\to 0$$

Proof. Consider the exact sequence

$$\mathcal{E}\to\mathcal{F}\to\mathrm{Coker}\phi\to 0\quad;$$

applying $\mathcal{H}om(-,\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{M})$ and $-\otimes_{\mathcal{O}_Y}\mathcal{M}$ gives the two exact sequences

$$0\to\mathcal{H}om(\mathrm{Coker}\phi,\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{M})\to\mathcal{H}om(\mathcal{F},\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{M})\to\mathcal{H}om(\mathcal{E},\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{M})$$
$$\mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{M}\to\mathcal{F}\otimes_{\mathcal{O}_Y}\mathcal{M}\to\mathrm{Coker}\phi\otimes_{\mathcal{O}_Y}\mathcal{M}\to 0$$

Now tensor diagram (*) by $\mathcal{M}$, obtaining the commutative diagram

$$\begin{array}{ccc}\mathcal{H}om(\mathcal{F},\mathcal{L})\otimes_{\mathcal{O}_Y}\mathcal{M} & \xrightarrow{\phi^T\otimes 1} & \mathcal{H}om(\mathcal{E},\mathcal{L})\otimes_{\mathcal{O}_Y}\mathcal{M}\\ \Vert & & \Vert\\ \mathcal{E}\otimes_{\mathcal{O}_Y}\mathcal{M} & \xrightarrow{\phi\otimes 1} & \mathcal{F}\otimes_{\mathcal{O}_Y}\mathcal{M}\end{array}\quad;$$

since $\mathcal{E}$, $\mathcal{F}$ are locally free, the top row can be replaced with

$$\mathcal{H}om(\mathcal{F},\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{M})\to\mathcal{H}om(\mathcal{E},\mathcal{L}\otimes_{\mathcal{O}_Y}\mathcal{M})$$

and it is then clear that the two exact sequences link into the one given in the statement. $\square$

If Y is an s.s.h. in M, we apply the above to the Hessian $(TM\otimes TM)|_Y\to\mathcal{L}$ of the section F of which Y is a singular scheme:

Corollary 1.10. *If Y is an s.s.h. in M, then for every sheaf $\mathcal{M}$ of $\mathcal{O}_Y$-modules there is an exact sequence*

$$0 \to \mathcal{H}om(\Omega_Y, \mathcal{M}) \to TM \otimes \mathcal{M} \to T^*M \otimes \mathcal{L} \otimes \mathcal{M} \to \Omega_Y \otimes \mathcal{L} \otimes \mathcal{M} \to 0$$

Proof. Working in local coordinates, the induced symmetric morphism

$$TM|_Y \xrightarrow{\phi} T^*M \otimes \mathcal{L}|_Y$$

sends $\dfrac{\partial}{\partial x_i}$ to $\sum \dfrac{\partial^2 F}{\partial x_i \partial x_j} dx_j = d\left(\dfrac{\partial F}{\partial x_i}\right)$: since the $\left(\dfrac{\partial F}{\partial x_i}\right)$ are precisely local equations for Y, we see that the image of ϕ (up to tensoring by $\mathcal{L}$) is nothing but the image of d in the 'second exact sequence' of differentials (cf. for example [H], p. 173; notice that $dF = \sum \dfrac{\partial F}{\partial x_j} dx_j = 0$ over Y)

$$\frac{\mathcal{I}}{\mathcal{I}^2} \otimes \mathcal{L} \xrightarrow{d} T^*M \otimes \mathcal{L} \to \Omega_Y \otimes \mathcal{L} \to 0$$

where $\mathcal{I}$ is the ideal of Y in M. That is, $\mathrm{Coker}\phi = \Omega_Y \otimes \mathcal{L}$; Proposition 1.9 then gives the statement, since $\mathcal{H}om(\Omega_Y \otimes \mathcal{L}, \mathcal{M} \otimes \mathcal{L}) = \mathcal{H}om(\Omega_Y, \mathcal{M})$. $\square$

Corollary 1.10 implies Proposition 1.5: choose a Z as in the definition of $\mathcal{A}_{\mathcal{L}}$, and write the sequence with $\mathcal{M} = \mathcal{O}_Z$:

$$0 \to \mathcal{H}om(p^*\Omega_Y, \mathcal{O}_Z) \to p^*TM \to p^*T^*M \otimes \mathcal{L} \to p^*\Omega_Y \otimes \mathcal{L} \to 0$$

is exact. That is,

$$[p^*T^*M \otimes \mathcal{L}] - [p^*TM] = [p^*\Omega_Y \otimes \mathcal{L}] - [\mathcal{H}om(p^*\Omega_Y, \mathcal{O}_Z)]$$

in the Grothendieck group of Z; Proposition 1.5 follows by taking Chern classes and applying the projection formula to the left-hand-side. $\square$

To get Theorem 1.6, just apply the operator in Proposition 1.5 to $c_*(Y)$:

$$\begin{aligned}
\mu_{\mathcal{L}}(Y) &= c(T^*M \otimes \mathcal{L}) \cap s(Y, M) \\
&= \frac{c(T^*M \otimes \mathcal{L})}{c(TM)} \cap (c(TM) \cap s(Y, M)) \\
&= \frac{c(T^*M \otimes \mathcal{L})}{c(TM)} \cap c_*(Y) \\
&= \mathcal{A}_{\mathcal{L}}(c_*(Y))
\end{aligned}$$

by Proposition 1.5. Theorem 1.6 is then established. $\square$

Corollary 1.10 can also be used to obtain a strong constraint forced upon a scheme Y when this can be realized as the singular scheme of a hypersurface:

Proposition 1.11. *Let Y be the singular scheme of a section of a line bundle $\mathcal{L}$ on a non-singular variety. Then for every sheaf $\mathcal{M}$ of $\mathcal{O}_Y$-modules and all $i \geq 1$*

$$\mathcal{T}or_i^{\mathcal{O}_Y}(\Omega_Y^*, \mathcal{M}) \cong \mathcal{T}or_{i+2}^{\mathcal{O}_Y}(\Omega_Y \otimes \mathcal{L}, \mathcal{M})$$

where $\Omega_Y^* = \mathcal{H}om(\Omega_Y, \mathcal{O}_Y)$.

Proof. Apply Corollary 1.10 with with $\mathcal{M} = \mathcal{O}_Y$: the sequence

$$0 \to \Omega_Y^* \to TM \to T^*M \otimes \mathcal{L} \to \Omega_Y \otimes \mathcal{L} \to 0$$

is exact. Hence if $\ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \Omega_Y^* \to 0$ is a locally free resolution of Ω_Y^*, $\ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to TM \to T^*M \otimes \mathcal{L} \to \Omega_Y \otimes \mathcal{L} \to 0$ will be a locally free resolution of $\Omega_Y \otimes \mathcal{L}$, and the result follows by tensoring with $\mathcal{M}$ and taking homology. □

Remark. Let $Z \xrightarrow{p} Y$ be as in the definition of $\mathcal{A}_\mathcal{L}$. The exact sequence in the proof of Proposition 1.11 and easy $\mathcal{T}or$ manipulations show that (with the above notations)

$$\mathcal{A}_\mathcal{L}(\alpha) = p_*(c([p^*\Omega_Y \otimes \mathcal{L}] - [\mathcal{T}or_1^{\mathcal{O}_Y}(\Omega_Y \otimes \mathcal{L}, \mathcal{O}_Z)] + [\mathcal{T}or_2^{\mathcal{O}_Y}(\Omega_Y \otimes \mathcal{L}, \mathcal{O}_Z)] - [p^*\Omega_Y^*]) \cap \beta)$$

Now if p were a finite-$\mathcal{T}or$ morphism, then we could define as customary $p^!([\mathcal{M}]) = \sum_i (-1)^i [\mathcal{T}or_i^{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Z)]$ for any coherent sheaf $\mathcal{M}$ on Y, and get the reasonable–looking

$$\mathcal{A}_\mathcal{L}(\alpha) = p_*(c(p^![\Omega_Y \otimes \mathcal{L}] - p^![\Omega_Y^*]) \cap \beta) \quad ,$$

as all but the first few $\mathcal{T}or$s match and cancel by Proposition 1.11. This expression does not make sense in our set-up, since typically p will *not* have finite $\mathcal{T}or$; in a sense, all we are doing in this section is to make sense of this expression nevertheless.

Examples of applications of the above results will be given in sections 2 and 3. Corollary 1.8 above suffices for many applications when Y is non-singular. The following stronger statement also follows from Corollary 1.10 and is useful when Y is singular but has a non-singular component.

Corollary 1.12. *Suppose Y is an s.s.h. in a non-singular variety M, and $Z \overset{i}{\hookrightarrow} Y$ is a non-singular variety such that the inclusion i restricts to an isomorphism on a dense open subset of Z. Then*

$$[N_Z^*M \otimes \mathcal{L}] = [N_Z M] + [\mathcal{T} \otimes \mathcal{L}]$$

in the Grothendieck group of Z, where $\mathcal{T}$ is the torsion sheaf of $\Omega_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Z$ and $N_Z M$ is the normal bundle of Z in M.

Proof. By Corollary 1.10 with $\mathcal{M} = \mathcal{O}_Z$ we know

$$[T^*M \otimes \mathcal{L}] - [TM] = [i^*\Omega_Y \otimes \mathcal{L}] - [\mathcal{H}om(\Omega_Y, \mathcal{O}_Z)]$$

Now take $\mathcal{H}om(-, \mathcal{O}_Z)$ of the first exact sequence of differentials for $Z \subset Y$:

$$0 \to \Omega_Z^* \to \mathcal{H}om(\Omega_Y, \mathcal{O}_Z) \to \mathcal{H}om(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_Z) \quad ;$$

since the inclusion $Z \subset Y$ is an isomorphism on a dense open set of Z, $\mathcal{J}/\mathcal{J}^2$ is torsion and the last term of the sequence vanishes. Therefore $[\mathcal{H}om(\Omega_Y, \mathcal{O}_Z)] = [TZ]$. Next, $\mathcal{T}$ is the kernel of the natural homomorphism

$$\Omega_Y \otimes \mathcal{O}_Z \to \mathcal{H}om(\mathcal{H}om(\Omega_Y \otimes \mathcal{O}_Z, \mathcal{O}_Z), \mathcal{O}_Z) = \mathcal{H}om(\Omega_Z^*, \mathcal{O}_Z) = \Omega_Z$$

by the above; this is an epimorphism, so $[i^*\Omega_Y \otimes \mathcal{O}_Z] = [T^*Z] + [\mathcal{T}]$. Putting all together,

$$[T^*M \otimes \mathcal{L}] - [TM] = [T^*Z \otimes \mathcal{L}] + [\mathcal{T} \otimes \mathcal{L}] - [TZ]$$

and the statement follows. □

§2. Relations with other invariants, and applications to duality

Maintain the notations and definition of §1.

§2.1. Milnor numbers and generalizations. If Y is supported on a point P, then $\mu_{\mathcal{L}}(Y) = m[P]$, where m is the Milnor number of X at P: indeed, in this case $\mu_{\mathcal{L}}(Y) = s(Y, M)$, so m is the coefficient of $[P]$ in $s(Y, M)$, which agrees with the definition of Milnor number (cf. for example [F], Example 14.1.5 (d)).

Even if $\dim Y > 0$, let $|Y|$ be the support of Y, and assume Y is proper; over $\mathbb{C}$, Parusiǹski ([P1]) has introduced in the above situation an invariant, the 'μ-number' of X at $|Y|$, agreeing with the ordinary Milnor number in case Y is a point.

Proposition 2.1. *Over $\mathbb{C}$ and with the above notations, Parusiǹski's μ-number of X at $|Y|$ equals the degree of the zero-dimensional component of $\mu_{\mathcal{L}}(Y)$.*

Proof. The differential of the local equations of X in M determines a section of $(T^*M \otimes \mathcal{L})|_X$; over $\mathbb{C}$, this can be extended to a holomorphic section s_X of $T^*M \otimes \mathcal{L}$ over the whole of M. Parusiǹski's μ-number is defined as the contribution of $|Y|$ to the intersection number of s_X with the zero-section of $T^*M \otimes \mathcal{L}$. Now in a neighborhood of $|Y|$ we have the fiber diagram

$$\begin{array}{ccc} Y & \longrightarrow & M \\ \Big\downarrow & & \Big\downarrow{\scriptstyle s_X} \\ M & \xrightarrow{s_0} & T^*M \otimes \mathcal{L} \end{array}$$

(that is, in this set-up Y is the scheme-theoretic intersection of the two sections), so the contribution of $|Y|$ to the intersection number of the sections is ([F], Chapter 6 and 19) the degree of the zero-dimensional component of

$$c(T^*M \otimes \mathcal{L}) \cap s(Y, M) \quad ,$$

which is the claim. □

The degree of the zero-dimensional component of $\mu_{\mathcal{L}}(Y)$ generalizes slightly Parusiǹski's μ-number in the sense that it defines it over any alg. closed field of characteristic 0. Maybe more importantly, our class can be defined even if Y is not proper: in fact, for many of our applications the relevant piece of $\mu_{\mathcal{L}}(Y)$ is the codimension-1 component, which may well carry useful information when Y is not proper.

Notice that in view of Propositions 1.3 and 2.1, the 'numerical' information carried by the μ-class of Y (when Y is proper) is essentially equivalent to Parusiǹski's μ-number of $|Y|$ and of its general sections $|Y \cap X_{g_1}|$, $|Y \cap X_{g_1} \cap X_{g_2}|$, etc.

Also, taking the degree of the zero-dimensional component in Corollary 1.8 gives a computation of Parusiǹski's Milnor number recovering a special case of [P1], Prop. 1.5. Parusiǹski also studies the behavior of the number under blow-ups; it would be very interesting to extend his results to the whole μ-class (or at least to the component of codimension 1).

§2.2. Multiplicities of discriminants. View X as a point of the *discriminant* $D \subset \mathbb{P}H^0(M, \mathcal{L})$ of $\mathcal{L}$ over M, parametrizing the singular sections of $\mathcal{L}$. For example, if $\mathcal{L}$ is very ample then D is the *dual variety* of M with respect to the embedding of M in $\mathbb{P}^N = \mathbb{P}H^0(M,\mathcal{L})^*$. We will use the term 'dual variety' for the discriminant in the (slightly weaker) hypothesis that the map $M \longrightarrow \mathbb{P}^N$ is unramified (note that this hypothesis is not required in the set-up of §1); also, the linear system giving this map need not be complete. In this situation, the multiplicity of D at X was investigated in [A-C] under the further hypothesis that D is a hypersurface in $(\mathbb{P}^N)^*$. Now we are able to remove this hypothesis, and in fact to measure the codimension of D in terms of the μ-class of the singularities of X (provided of course that Sing$X$ be proper, which we will tacitly assume whenever we compute a $\int$):

Proposition 2.2. *Let X be any singular hyperplane section of M. Then the codimension of the dual variety D of M in $\mathbb{P}^{N^*}$ is the smallest integer $r \geq 1$ such that*

$$\int c_1(\mathcal{L})^{r-1}\, c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\mathrm{Sing}X) \neq 0 \quad ;$$

and for $r = codim_{\mathbb{P}^{N^*}} D$ this number equals the multiplicity of D at X.

Proof. Induction on r: for $r = 1$, the statement follows from [A-C], Proposition 1.3 and Theorem in §1. Indeed,

$$\begin{aligned} c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\mathrm{Sing}X) &= c(\mathcal{L})c(T^*M \otimes \mathcal{L}) \cap s(\mathrm{Sing}X, M) \\ &= c((T^*M \oplus \mathcal{O}) \otimes \mathcal{L}) \cap s(\mathrm{Sing}X, M) \end{aligned}$$

is the class used in [A-C] to compute the multiplicity of the 'first' discriminant.

For $r > 1$, [A-C] shows that $\int c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\mathrm{Sing}X) = 0$; next, the key observation (which we borrow from [Ho], p. 153) is that if the dual variety D of M is not a hypersurface, then the dual variety D_1 of a general hyperplane section $M \cap H$ of M is the cone over D (with the point of $(\mathbb{P}^N)^*$ corresponding to H as vertex). The multiplicity of D at X then evidently equals the multiplicity of D_1 at $X \cap H$; but D_1 has codimension codim$D - 1$, so the statement follows by induction because

$$\begin{aligned} c_1(\mathcal{L})^{r-1}\, c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\mathrm{Sing}X) &= c_1(\mathcal{L})^{r-2}\, c(\mathcal{L}) \cap (c_1(\mathcal{L}) \cap \mu_{\mathcal{L}}(\mathrm{Sing}X)) \\ &= c_1(\mathcal{L})^{r-2}\, c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\mathrm{Sing}X \cap H) \end{aligned}$$

(where we used Proposition 1.3 in the last equality). □

Remark. Proposition 2.2 says that the dual variety has codimension $\geq r$ if and only if the components of dimension i, $0 \leq i < r-1$ of $c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\mathrm{Sing}X)$ vanish, for all sections X. That is, if the dual variety has codimension $\geq r$ then for all sections X

$$c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\mathrm{Sing}X) = A_{\dim \mathrm{Sing}X} + \cdots + A_r + A_{r-1}$$

with A_j a class in dimension j, $j = r-1, \ldots, \dim \mathrm{Sing}X$ (depending on X).

With this in mind, we can use Proposition 2.2 to recover and 'algebraize' an expression of Parusiński's for the multiplicity of the dual variety (Formula 2 in [P2]):

Corollary 2.3. *If the dual variety D has codimension r, then its multiplicity at X is*

$$\int (-1)^{r-1}\mu_{\mathcal{L}}(\mathrm{Sing}X) + c_1(\mathcal{L})^r \cap \mu_{\mathcal{L}}(\mathrm{Sing}X)$$

Proof. By the remark,

$$\mu_{\mathcal{L}}(\mathrm{Sing}X) = c(\mathcal{L})^{-1} \cap (A_{\dim \mathrm{Sing}X} + \cdots + A_{r-1})$$

(with $A_{r-1} \neq 0$), and therefore

$$\int (-1)^{r-1}\mu_{\mathcal{L}}(\mathrm{Sing}X) = \sum_{i=r-1}^{\dim \mathrm{Sing}X} (-1)^{i-r+1} c_1(\mathcal{L})^i \cap A_i$$

while

$$\int c_1(\mathcal{L})^r c(\mathcal{L})^{-1} \cap (A_{\dim \mathrm{Sing}X} + \cdots + A_{r-1}) = \sum_{i=r}^{\dim \mathrm{Sing}X} (-1)^{i-r} c_1(\mathcal{L})^i \cap A_i \quad .$$

Hence

$$\int (-1)^{r-1}\mu_{\mathcal{L}}(\mathrm{Sing}X) + c(\mathcal{L})^r \cap \mu_{\mathcal{L}}(\mathrm{Sing}X) = \int c_1(\mathcal{L})^{r-1} \cap A_{r-1} \quad ,$$

which equals the multiplicity of the dual variety at X by Proposition 2.2. □

To see that this implies Parusiński's formula, use Proposition 1.3 to write the second summand as the μ-class of the intersection of X with r general sections, then apply Proposition 2.1.

§**2.3. Ranks and Holme's Theorem.** Proposition 2.2 was stated with an eye to the notion of *ranks* (see for example [K], §4). In [Ho], A. Holme extended to the case of arbitrary codimension of D the well-known expression for the degree of the dual variety in terms of the degrees of the Chern classes of M. Holme defines

$$\delta_s = \sum_{i=s}^{n} \binom{i+1}{s+1} e_{n-i}$$

where e_j denotes the degree of $c_j(T^*M)$ with respect to the embedding determined by $\mathcal{L}$. The number δ_i is the 'i-th rank' of M.

Claim. *The i-th rank is given by*

$$\delta_i = \int c_1(\mathcal{L})^{i-1}\, c(\mathcal{L}) \cap \mu_{\mathcal{L}}(M)$$

Indeed, $\mu_{\mathcal{L}}(M) = c(T^*M \otimes \mathcal{L}) \cap [M]$ since M is non-singular (Corollary 1.8); the claim follows then from a standard Chern class computation, which we leave to the reader.

Holme's result (Theorem 1.7 in [Ho]) can then be derived as a particular case of Proposition 2.2:

Corollary 2.4. *The codimension of the dual variety is the smallest integer $r \geq 1$ such that $\delta_r \neq 0$, and for $r = codim_{\mathbb{P}^{N*}} D$ this number equals the degree of D.*

Proof. By Proposition 2.2, this is computing the multiplicity of the dual variety at the zero-section; which amounts to computing the multiplicity at the origin of the cone in $\mathbb{A}^{N+1^*}$ over $D \subset \mathbb{P}^{N^*}$; that is, to computing the degree of D (cf. Corollary 1.1 in [A-C]). $\square$

In view of this observation, the numbers

$$\int c_1(\mathcal{L})^{i-1}\, c(\mathcal{L}) \cap \mu_{\mathcal{L}}(\mathrm{Sing} X)$$

should be seen as an extension to arbitrary sections of $\mathcal{L}$ of the notion of ranks, to which they specialize for the zero-section.

§**2.4. Landman-Ein parity Theorem.** A remarkable result of Landman states that if M is not a linear subspace of $\mathbb{P}^N$, and the dual of M is not a hypersurface, then the codimension of the dual variety of M is congruent to $\dim M + 1 \bmod 2$: in other words, if the dual is not a hypersurface then the contact locus of a general hyperplane must have even codimension in M. Landman's approach was via Picard-Lefschetz theory; later this was recovered among a wealth of beautiful results in Ein's papers on small duals ([E]). Assume M is embedded in $\mathbb{P}^N$, $\mathcal{L} = \mathcal{O}(1)$, and Y is the contact locus of a general tangent hyperplane H with M; then it is easily seen that Y is a linear subspace of $\mathbb{P}^N$, and Ein proves ([E] I, Theorem 2.2)

$$N_Y M \cong \mathcal{H}om(N_Y M, \mathcal{O}(1)) \quad (= N_Y^* M \otimes \mathcal{O}(1)) \quad , \quad \text{implying}$$
$$c(N_Y M) = c(N_Y^* M \otimes \mathcal{O}(1)) \quad ;$$

and this Chern class equality implies easily Landman's parity result. In Ein's set-up this is significant when the dual variety is not a hypersurface (else Y is a point!).

Our observation now is that Corollary 1.12 gives a generalization of the Chern class equality, and from this we ought to derive a strengthening of Landman's result. The upshot is that one does not need to assume the section to be general, or the contact locus to be a linear subspace, or the dual variety to be small, or indeed even that $\mathcal{L}$ be necessarily very ample. The precise statement is

Proposition 2.5. *Let Y be a s.s.h. with respect to $\mathcal{L}$ in a smooth variety M. Assume that Y is pure-dimensional and contains a complete curve C such that*

(i) Y is non-singular in a neighborhood of C and

(ii) the $\mathcal{L}$-degree of C is odd.

Then $\dim Y \equiv \dim M \mod 2$.

Proof. We can apply Corollary 1.12 to the non-singular neighborhood Z of C, then restrict to C and compare c_1's: C avoids the torsion of Ω_Y, so Corollary 1.12 gives

$$c_1(N_Z M) \cap [C] = (-c_1(N_Z M) + (\mathrm{codim}_Y M)\, c_1(\mathcal{L})) \cap [C]$$

from which

$$(\mathrm{codim}_Y M) \cdot \left(\int c_1(\mathcal{L}) \cap [C] \right) = 2 \int c_1(N_Z M) \cap [C] \quad ,$$

hence $(\mathrm{codim}_Y M)\cdot(\int c_1(\mathcal{L})\cap[C])$ is even, hence $\mathrm{codim}_Y M$ is even. $\square$

This implies Landman's result: say $M\subset\mathbb{P}^N$ and $\mathcal{L}=\mathcal{O}(1)$, and consider a general hyperplane section; the contact locus (that is, Y) is then a linear subspace $\mathbb{P}^k$. If the dual of M is not a hypersurface then $k>0$, and we are in the hypotheses of Proposition 2.5 (with $C=$ a line): the codimension of the contact locus must be even.

Proposition 2.5 says that this holds for any hyperplane and regardless of the dimension of the dual (so the result is significant even when the dual variety *is* a hypersurface), so long as the contact scheme has positive dimension and is non-singular for example in the neighborhood of a line. To see that hypothesis (ii) is necessary, consider the Veronese surface in $\mathbb{P}^5$: hyperplane sections correspond to conics in $\mathbb{P}^2$; a hyperplane corresponding to a 'double line' is tangent to the surface along a conic in $\mathbb{P}^5$, which is codimension 1 in the surface (and violates (ii)). This example is somewhat typical: a quadric in $\mathbb{P}^n$ may be singular in any codimension—that is, there is no restriction on the codimension of the singular locus of hyperplane sections of a 2-Veronese embedding. Proposition 2.5 shows that the situation is much more constrained for e.g., the codimension of the singular locus of odd-degree hypersurfaces of $\mathbb{P}^n$: if the singular scheme Y of an odd-degree hypersurface of $\mathbb{P}^n$ is non-singular in the neighborhood of a curve of odd degree, then Y must have even codimension in $\mathbb{P}^n$. For example, it follows that no hypersurface of odd degree in $\mathbb{P}^{2k}$ may have singularity of 'transverse type A1' along a non-singular curve of odd degree.

Another curious remark concerning the codimension of the dual variety is the following: as noted above, it is well known that the contact locus of a general hyperplane with a variety is a linear subspace $\mathbb{P}^{r-1}$, where $r=\mathrm{codim} D$ (cf. for example [K], p.340) . Corollary 1.8 yields a 'converse' to this:

Corollary 2.6. *Suppose the contact scheme Y of a hyperplane H with M is a linear space $\mathbb{P}^{r-1}$; then the dual variety of M has codimension r, and is smooth at H.*

(Recall that by contact *scheme* we mean the contact locus endowed of the scheme structure that naturally makes it an s.s.h.: that is, Y is the singular scheme of $H\cap M$ in M)

The surprising element in Corollary 2.6 is that we are *not* assuming the hyperplane to be general. The dimension of the contact scheme of a hyperplane need not be equal to the dimension of the contact locus of a general hyperplane; the claim here is that if the contact scheme is a linear space, then its dimension *must* equal the dimension of the general contact. For example, if the general contact is a simple point (that is, the dual is a hypersurface) then *no hyperplane can touch M* ('scheme-theoretically') *along a linear subspace* of positive dimension.

Proof. If $Y\cong\mathbb{P}^{r-1}$ and $\mathcal{L}=\mathcal{O}(1)$, then by Corollary 1.8

$$c(\mathcal{L})\cap\mu_{\mathcal{L}}(Y)=c(\mathcal{O}(1))c(T^*\mathbb{P}^{r-1}\otimes\mathcal{O}(1))\cap[\mathbb{P}^{r-1}]=[\mathbb{P}^{r-1}]\quad,$$

and the result follows immediately from Proposition 2.2. $\square$

More generally, Proposition 2.2 shows (by an analogous argument) that if the contact scheme of a variety with a given hyperplane is smooth, then the *defect of* *the variety equals the defect of the contact scheme:* the last corollary works because the defect of a linear space equals its dimension. We do not know if this is true for singular contact loci, or indeed what the appropriate statement would be in that case.

As a concrete example of the situation of Corollary 2.6, consider the Segre embedding of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, where $n_1 \geq \cdots \geq n_r$. In [A-C], p. 257 we have observed that if $n_1 > m_1 = \sum_{i>1} n_i$, then there are hyperplane sections touching $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ along a $\mathbb{P}^{n_1 - m_1}$. It follows then immediately from Corollary 2.6 and without further computations (for example, without having to prove that such sections are 'general') that the dual variety of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ must have codimension $n_1 - m_1 + 1$.

§**2.5. Zak's Theorem.** As a last application of the results in §1.3 to duality, we will show that

Proposition 2.7. *The dual variety of a smooth nonlinear variety $M \subset \mathbb{P}^N$ has dimension $\geq \dim M$.*

Proof. This is usually derived as a consequence of Zak's beautiful theorem on tangencies (see for example [F-L], §7). To obtain the result in our set-up, let Y be the contact scheme of the general tangent hyperplane H with M (note $M \not\subset H$, as M is non-linear), and let $\mathcal{L} = \mathcal{O}(1)$ be the hyperplane bundle. Then $Y \cong \mathbb{P}^{r-1}$, where r is the codimension of the dual variety, so by the remark preceding Corollary 2.3

$$c(\mathcal{L}) \cap \mu_{\mathcal{L}}(Y) = [Y] \quad .$$

We have to show $r \leq N - n$ (where as usual we denote $n = \dim M$), that is $\dim Y < N - n$; we will assume $\dim Y \geq N - n$ and derive a contradiction. By taking general hyperplane sections we may and will assume $\dim Y = N - n$, that is $Y = \mathbb{P}^{N-n}$.

We are going to derive a contradiction by computing in two ways the component of dimension zero $\{s(Y, M)\}_0$ of the Segre class of Y in M. The first observation is that since Y is an s.s.h. in M with respect to $\mathcal{L}$:

$$[Y] = c(\mathcal{L}) \cap \mu_{\mathcal{L}}(Y) = c(\mathcal{L})c(T^*M \otimes \mathcal{L}) \cap s(Y, M) = c(\mathcal{P}^1\mathcal{L}) \cap s(Y, M) \quad ,$$

where $\mathcal{P}^1\mathcal{L}$ is the bundle of principal parts of $\mathcal{L}$. Now say $(\mathbb{P}^N)^* = \mathbb{P}V$, and let $\mathcal{V}$ be the trivial bundle with fiber V; then we have the exact sequence

$$0 \to W \to \mathcal{V} \to \mathcal{P}^1\mathcal{L} \to 0$$

with W a rank-$(N-n)$ bundle. Restrict this to Y; if $v \in \mathcal{V}$ is the section determining H, $v \mapsto 0$ in $\mathcal{P}^1\mathcal{L}|_Y$ (by definition of Y!), so v gives a non-vanishing section of $W|_Y$. Therefore $c_{N-n}(W) \cap [Y] = 0$, and since $\dim Y = N - n$ it follows

$$\begin{aligned} 0 &= \{c(W) \cap [Y]\}_0 = \{c(W)c(\mathcal{P}^1\mathcal{L}) \cap s(Y, M)\}_0 = \{c(\mathcal{V}) \cap s(Y, M)\}_0 \\ &= \{s(Y, M)\}_0 \end{aligned} \tag{1}$$

On the other hand, consider the projection from a hyperplane $Z = \mathbb{P}^{N-n-1}$ of $Y \cong \mathbb{P}^{N-n}$ to a $\mathbb{P}^n \subset \mathbb{P}^N$. This induces a regular map π from the blow-up $\widetilde{M}$ of M along Z to $\mathbb{P}^n$, of which $\widetilde{Y} = B\ell_Z Y \cong Y$ is a fiber. We claim this map is generically finite: for this, it suffices to pick a general $p \in M$ and show that the intersection of the span S of p and Z with M consists of Z and finitely many points. But otherwise $S \cap M$ would contain a curve hitting Z (since Z is a hyperplane in S) at some point p'; $S \cap M$ would be singular at p', which would imply that the tangent space to M at p' contains S, and in particular p. This cannot be, because we chose p general and in particular not in H, while at all points of Z the tangent space to M must be contained in H (because Z is in the contact locus Y of H and M).

Summing up, π is a quasi-finite map $\widetilde{M} \to \mathbb{P}^n$ of which $\widetilde{Y} \cong Y$ is a fiber, say over $q \in \mathbb{P}^n$. By [F], Prop. 4.2 (a), $s(\widetilde{Y}, \widetilde{M})$ must push-forward to a multiple of $s(q, \mathbb{P}^n) = [q]$, and in particular $\{s(\widetilde{Y}, \widetilde{M})\}_0 \neq 0$. We are almost done: $\widetilde{Y} \cong Y \cong \mathbb{P}^{N-n}$ identifies the exceptional divisor in $\widetilde{Y}$ with the hyperplane Z, so

$$s(\widetilde{Y}, \widetilde{M}) = c(N_{\widetilde{Y}}\widetilde{M})^{-1} \cap [\widetilde{Y}] = c(N_Y M \otimes \mathcal{O}(-1))^{-1} \cap [Y]$$

(by [F], p. 437); by Corollary 1.12, $c(N_Y M \otimes \mathcal{O}(-1)) = c(N_Y^* M)$; and finally

$$\begin{aligned}\{s(Y, M)\}_0 = \{c(N_Y M)^{-1} \cap [Y]\}_0 &= \pm\{c(N_Y^* M)^{-1} \cap [Y]\}_0 = \pm\{s(\widetilde{Y}, \widetilde{M})\}_0 \\ &\neq 0\end{aligned}$$

contradicting (1). □

§3. Examples

We list here a few examples illustrating how the results in §1 pose strong constraints on what schemes may appear as singular schemes of hypersurfaces in a given variety. We do not have any specific guiding principle in mind in choosing these examples, other than they seem amusing and we find it somewhat surprising that they do not belong to the classical literature or to the folklore.

The idea for most of the examples is to equate the expressions for $\mu_{\mathcal{L}}$ obtained from the intrinsic computations of §1.3 and from a given realization of Y as a singular scheme of a hypersurface. This is quickly done in a large class of examples by way of Corollary 1.12: suppose Y is a singular scheme of a hypersurface X in a smooth variety M, and let $Z \subset Y$ be a smooth subvariety of Y such that the inclusion is an isomorphism away from a subset of Z of codimension ≥ 2. Suppose $\dim Z = m$, $\dim M = n$, and let K_Z, K_M denote the canonical divisors of Z, M.

Claim. *Then* $\quad 2(K_Z - K_M) = (n - m)X$

(as divisors of Z). This follows immediately from Corollary 1.12, once one observes that the torsion term is supported in codimension ≥ 2.

Note that any smooth hypersurface is itself the singular scheme of its 'double': the formula in the claim then reduces to the usual adjunction formula, and is automatically satisfied. Similarly, smooth codimension-2 complete intersections are clearly singular schemes of the union of two hypersurfaces defining them, and again the above relation is just adjunction in disguise. Question: are there any *singular* complete intersections that are s.s.h.? (This seems unlikely.)

For the examples that follow, we specialize the ambient variety to $M = \mathbb{P}^n$, a projective space, and $\mathcal{O}(X) = \mathcal{O}(d)$. The Claim then says then that if, outside of a subscheme of codimension ≥ 2, $Y \subset \mathbb{P}^N$ is smooth and a component of the singular scheme of a hypersurface X of degree d, then necessarily

$$(*) \qquad 2K_Y = (d(n-m) - 2(n+1))H$$

where K_Y and H denote respectively the canonical class and the hyperplane section of Y.

Example 3.1. *Suppose a linear subspace $Y = \mathbb{P}^m \subset \mathbb{P}^n$, $0 < m < n$, is a connected component of the singular scheme of a hypersurface X. Then X is necessarily a quadric.*

Indeed $K_Y = -(m+1)H$ in this case, and $m > 0$, so (*) gives

$$-2(m+1) = d(n-m) - 2(n+1) \quad , \quad \text{that is} \quad (d-2)(n-m) = 0 \quad ;$$

so $d = 2$ since $m < n$. For example, the union of two disjoint lines cannot be the singular scheme of a hypersurface in $\mathbb{P}^n$: by the above, the hypersurface would necessarily be a quadric, contradiction. Similarly, a $\mathbb{P}^2$ "with embedded points" cannot be a component of the singular scheme of a hypersurface in $\mathbb{P}^n$: indeed, the embedded points are ignored in codimension 1, so the hypersurface would be a quadric by the above, while the singular scheme of a quadric is reduced (cf. also Example 3.8 below). Singular schemes of hypersurfaces, even when supported on a non-singular variety, may well have embedded components: the classic Whitney umbrella $x^2 = yz^2$ is singular along a line, and the singular scheme has an embedded component at the pinch point. Question: are embedded components of an s.s.h. necessarily in codimension 1? $\square$

Example 3.2. *The only Veronese embeddings*

$$\mathbb{P}^m \overset{\mathcal{O}(r)}{\hookrightarrow} \mathbb{P}^n \quad , \quad n = \binom{m+r}{r} - 1$$

that are singular schemes of hypersurfaces are the linear ones (cf. Example 3.1), the nonsingular conic $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, and the Veronese surface $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

(This latter is the singular scheme of its chordal variety, a divisor of degree 3. Interpreting $\mathbb{P}^5$ as parametrizing 3×3 symmetric matrices, the Veronese surface is the set of rank–1 matrices, and its chordal variety is the set of rank ≤ 2 symmetric matrices.) To see these are the only cases, assume the r-th Veronese embedding of $\mathbb{P}^m$ in $\mathbb{P}^n$ is a connected component of the singular scheme of a hypersurface of degree d, and apply (*) above to compute

$$d = 2 + \frac{2(m+1)(r-1)}{r(n-m)} \quad , \quad \text{that is} \quad d = 2 + \frac{2(r-1)}{\binom{m+r}{r-1} - r} \quad .$$

Now $\binom{m+r}{r-1}$ is increasing in m, so

$$\binom{m+r}{r-1} - r \leq 2(r-1) \implies \binom{1+r}{r-1} - r \leq 2(r-1) \implies r \leq 4 \quad .$$

Since d is an integer, $r = 3$ or 4 are excluded; $r = 1$ gives the linear embeddings; for $r = 2$, the formula gives $d = 2 + \frac{2}{m}$: so necessarily $m = 1$ or 2, which is the claim. $\square$

Example 3.3. *The only Segre embeddings*

$$\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \hookrightarrow \mathbb{P}^n \quad , \quad n = (n_1+1)\cdots(n_r+1) - 1$$

($r \geq 2$, $n_i \geq 1$) that are singular schemes of hypersurfaces are

$$\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3 \quad \textit{and} \quad \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$$

(The second is the embedding of the set of rank–1 matrices in the $\mathbb{P}^8$ of 3×3 matrices, and is the singular scheme of its chordal, that is the degree-3 hypersurface consisting of matrices of rank ≤ 2.) To see that these are the only possibilities, apply (*) to get

$$-2((n_1+1)h_1+\cdots+(n_r+1)h_r) = (d(n-n_1-\cdots-n_r)-2(n+1))(h_1+\cdots+h_r) \quad ,$$

where h_i denotes the pull-back of the hyperplane from the i-th factor. Taking the coefficient of h_i gives

$$n(d-2) = dn_1 + \cdots + (d-2)n_i + \cdots + dn_r$$

for every i: thus $n_1 = \cdots = n_r$, and

$$((n_1+1)^r - 1)(d-2) = (rd-2)n_1 \quad .$$

This gives

$$d = 2 + \frac{2(r-1)}{\sum_{j=2}^r \binom{r}{j} n_1^{j-1}} \quad :$$

the denominator on the right is then necessarily $\leq 2(r-1)$; on the other hand, it is clearly increasing in n_1 for $r \geq 2$ and $n_1 \geq 1$. It follows that the above can be an integer only if $2^r - 1 - r \leq 2(r-1)$, which implies (as $r \geq 2$) $r = 2$ or 3. For $r = 2$ the only possibilities are $n_1 = 1$ and 2, giving the above two cases. For $r = 3$, one finds that necessarily $n_1 = 1$, thus ruling out all cases but $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^7$; but this would imply $d = 3$, and a degree 3 hypersurface which is singular along $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ would contain its chordal variety: however, this is easily seen to be the whole of $\mathbb{P}^7$, so this case cannot occur. □

Now we specialize further, and assume Y is a smooth curve of degree r and genus g. Taking degrees in (*) gives

$$2(2g-2) = r(d(n-1) - 2(n+1)) \quad ,$$

that is (for $n > 1$)

$$d = 2 + \frac{4(r+g-1)}{r(n-1)} \quad .$$

In particular, n is bounded above for any given genus $g > 0$: in other words, *hypersurfaces of $\mathbb{P}^n$ whose singular scheme is a smooth curve of low positive genus are a phenomenon belonging to low dimension.* For genus 0:

Example 3.4. *Let Y be a smooth rational curve of degree r in $\mathbb{P}^n$, which is a connected component of the singular scheme of a hypersurface of degree d. Then the only possibilities are:*

(1) $r = 1$, *any* n, $d = 2$*;*
(2) $r = 2$, $(n, d) = (2, 4)$ *or* $(3, 3)$*;*
(3) $r = 4$, $n = 4$, $d = 3$.

All other possibilities are excluded by the above relation among r, d, n when $g = 0$. To see the listed cases arise:

(1) clear (cf. Example 3.1);
(2) for $n = 2$, any smooth plane conic is the singular scheme of the "double conic" supported on it; for $n = 3$, the union of a quadric and a plane intersecting transversally is a cubic surface whose singular scheme is the conic of intersection;
(3) interpret $\mathbb{P}^4$ as the space of quartic forms in two variables x, y, or equivalently as the space parametrizing 4-tuples of points on $\mathbb{P}^1$; then consider the closure of the orbit of the 4-tuple given by $x^4 - y^4$ under the action of the linear group on $\mathbb{P}^1$. This is a degree-3 hypersurface which is singular precisely along the rational normal curve of "quadruple points", cf. [A-F], Proposition 4.3.

For example, twisted cubics can*not* be connected components of the singular scheme of a hypersurface in any projective space. Note however that the tangential surface of the twisted cubic in $\mathbb{P}^3$ *is* singular precisely along the twisted cubic; the scheme structure of the singular scheme is non-reduced in this case. $\square$

Example 3.5. *For $g = 1$, the only possibilities are*

(1) $n = 2$, $d = 6$*;*
(2) $n = 3$, $d = 4$*;*
(3) $n = 5$, $d = 3$.

All other possibilities are ruled out by the above relation. To realize the cases listed here:

(1) any smooth plane cubic is the singular scheme of its double;
(2) transversal intersections of a plane and a smooth cubic surface, or of two smooth quadrics are examples;
(3) this can be realized as follows: let Y be the image of a smooth plane cubic in $\mathbb{P}^5$ by Veronese; the chordal variety of Y is then a nonic, complete intersection of two cubic hypersurfaces; a general combination of these latter is a cubic hypersurface whose singular scheme is Y. $\square$

Example 3.6. *No smooth curve of genus* 2 *is the singular scheme of a hypersurface in a projective space.*

Indeed, the above condition imposes $r \,|\, 4$: but $r = 1, 2, 4$ cannot occur for a smooth curve of genus 2 in any $\mathbb{P}^n$, cf. [H], p.354. $\square$

We end with a few examples illustrating the situation when Y itself is singular. For these, the main tools are Proposition 1.11 and Theorem 1.6.

Example 3.7. *The planar triple point $Y = Spec \dfrac{k[x, y]}{(x^2, xy, y^2)}$ is not the singular scheme of any hypersurface in any non-singular variety.*

Indeed, it is easy to compute minimal resolutions of Ω_Y and $\mathcal{H}om(\Omega_Y, \mathcal{O}_Y)$; tensoring with $k = \dfrac{\mathcal{O}_Y}{(x,y)}$ gives the complexes

$$\cdots \longrightarrow k^{24} \xrightarrow{0} k^{12} \xrightarrow{0} k^6 \xrightarrow{0} k^3 \xrightarrow{0} k^2 \xrightarrow{\cong} \Omega_Y \otimes k \longrightarrow 0$$
$$\cdots \longrightarrow k^{16} \xrightarrow{0} k^8 \xrightarrow{0} k^4 \xrightarrow{\cong} \mathcal{H}om(\Omega_Y, \mathcal{O}_y) \otimes k \longrightarrow 0$$

from which $\mathcal{T}or_3(\Omega_Y, k) \cong k^{12}$, $\mathcal{T}or_1(\mathcal{H}om(\Omega_Y, \mathcal{O}_Y), k) \cong k^8$. By Proposition 1.11, Y cannot be the singular scheme of a hypersurface. Any curvilinear multiple point $\operatorname{Spec}\dfrac{K[x]}{(x^m)}$ of course *is* an s.s.h. $\square$

Example 3.8. *Let Y be the subscheme of $\mathbb{P}^3_{x:y:z:w}$ defined by the ideal (x^2, xy, xz) (that is, a plane with an embedded point). Then Y cannot be the singular scheme of a hypersurface in a non-singular variety.*

This can be seen again by explicitly computing the Tors and applying Proposition 1.11. Or in fact one can compute the right-hand-side of the statement of Theorem 1.6, first using (in the definition of $\mathcal{A}_{\mathcal{L}}$) $Z = Y_{\text{red}} = \mathbb{P}^2$ and then using $Z =$ the blow-up of Y_{red} at the support of the embedded point. The computation is a little laborious; we get

$$[\mathbb{P}^2] - 3[\mathbb{P}^1] + 5[\mathbb{P}^0]$$

in the first case, and

$$[\mathbb{P}^2] - 3[\mathbb{P}^1] + 3[\mathbb{P}^0]$$

in the second. This shows the right-hand-side in Theorem 1.6 is not well-defined for this scheme, and it follows that the plane with an embedded point cannot be the singular scheme of a hypersurface. $\square$

Example 3.9. *A nodal curve (with its reduced structure) cannot be the singular scheme of a hypersurface.*[1]

This follows by applying Proposition 1.11 with $\mathcal{M}$ = the skyscraper sheaf k supported on the node. As a prototype situation (to which one may reduce by taking completions at the singular point) to evaluate the $\mathcal{T}or$, consider the union C_2 of 2 lines intersecting at a point with independent tangent directions; in fact it is no more work to consider the union C_n of n such lines, given for example by the ideal $(x_i x_j)_{i<j}$ in $\mathbb{A}^n_{x_1,\dots,x_n}$:

Claim. $n \neq 3 \implies$ *C_n is not a singular scheme of a hypersurface.*

Indeed, compute minimal resolutions of Ω_Y and $\mathcal{H}om(\Omega_Y, \mathcal{O}_Y)$; after tensoring with k these become

$$\cdots \longrightarrow k^{\binom{n}{2}(n-2)(n-1)} \xrightarrow{0} k^{\binom{n}{2}(n-2)} \xrightarrow{0} k^{\binom{n}{2}} \xrightarrow{0} k^n \xrightarrow{\cong} \Omega_Y \otimes k \longrightarrow 0$$
$$\cdots \longrightarrow k^{n(n-1)^3} \xrightarrow{0} k^{n(n-1)^2} \xrightarrow{0} k^{n(n-1)} \xrightarrow{0} k^n \xrightarrow{\cong} \mathcal{H}om(\Omega_Y, \mathcal{O}_Y) \otimes k \longrightarrow 0$$

[1] I am grateful to Robert Varley for showing to me that this can also be proved in a more elementary way.

Hence

$$\mathcal{T}or_{i+2}(\Omega_Y, k) \cong k^{\binom{n}{2}(n-2)(n-1)^i}\ , \mathcal{T}or_i(\mathcal{H}om(\Omega_Y, \mathcal{O}_y), k) \cong k^{n(n-1)^i} \quad i \geq 0 \quad ,$$

and the statement follows by Proposition 1.11.

Notice that for $n = 3$, C_n *is* a singular scheme of a hypersurface: the union of three coordinate planes (cf. also Example 3.10 below). Notice also that a nodal curve may be the singular scheme of a hypersurface if one allows an embedded point at the node (for example, the singular scheme of the union of a smooth quadric and a tangent plane in $\mathbb{P}^3$). □

The claim shows that the union of *three* curves meeting at a point with independent tangent directions is the only example of an s.s.h. of this kind. Even in this case the situation, at least in projective space, is tremendously constrained:

Example 3.10. *Let Y be the the union of three curves meeting at exactly one point with independent tangent directions, considered with the reduced structure. Suppose Y is the singular scheme of a hypersurface of degree d in $\mathbb{P}^n$, and assume its components are smooth rational curves. Then $n = d = 3$, and Y is necessarily the union of three lines, realized as the singular scheme of the union of three general planes in $\mathbb{P}^3$.*

To see this, consider one component C and apply Corollary 1.12 to the embedding of C in Y. To evaluate the relevant torsion sheaf we may work locally, therefore we may assume for a moment that $Y = \operatorname{Spec}(k[x, y, z]/(xy, xz, yz))$ and that C has ideal (y, z) in Y. Tensoring the standard presentation of Ω_Y by $\mathcal{O}_C$ gives

$$\mathcal{O}_C^3 \xrightarrow{\begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & x & 0 \end{pmatrix}} \mathcal{O}_C^3 \longrightarrow \Omega_Y \otimes \mathcal{O}_C \longrightarrow 0$$

showing that the torsion of $\Omega_Y \otimes \mathcal{L} \otimes \mathcal{O}_C$ consists of a k^2 concentrated at the singular point.

Going back to the projective situation, assuming C has degree r, applying Corollary 1.12 and taking degrees now yields

$$r(2 + d + 2n - dn) = 2 \quad :$$

therefore $r = 1$ or 2. If $r = 2$, this relation forces $(d, n) = (5, 2)$ or $(3, 4)$; but these cannot occur: the first because three curves in $\mathbb{P}^2$ cannot have independent tangent directions; for the second, the other two components of Y would also have to be conics (by the same relation), and if there were a cubic threefold in $\mathbb{P}^4$ singular along 3 conics, then its general hyperplane section would be a cubic surface in $\mathbb{P}^3$ with 6 isolated singularities; there is no such surface ([G-H], p. 644).

Thus necessarily $r = 1$, so C is a line; the same relation now easily shows that necessarily $n = 3$ and $d = 3$, and it follows also that the other two components must be lines. So Y consists of three lines, and is the singular locus of a cubic surface X in $\mathbb{P}^3$. The general hyperplane section of X is a cubic curve in the plane, with three singular points, so it must be the union of three general lines: it follows that X is the union of three general planes, as claimed. □

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