A remark on the Chern class of a tensor product

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The following elementary observation has proven useful in several enumerative geometry computations.

Let X be any algebraic scheme over a field, and let $\alpha \in K^0(X)$ be an element in the Grothendieck group of vector bundles over X. Then α has a well-defined rank $\operatorname{rk}\alpha$, and Chern classes $c_k(\alpha)$. Also, as tensor product makes $K^0(X)$ a ring, we consider $\alpha \otimes [\mathcal{L}]$, where $\mathcal{L}$ is an arbitrary line bundle on X.

Theorem. *With notations as above,*

$$c_{\operatorname{rk}\alpha+1}(\alpha) = c_{\operatorname{rk}\alpha+1}(\alpha \otimes [\mathcal{L}])$$

(So this class is independent of $\mathcal{L}$.)

Proof. Write $\alpha = [\mathcal{E}] - [\mathcal{F}]$, with $\mathcal{E}$, $\mathcal{F}$ vector bundles of rank m and $n = m - r$ respectively. We have to show that the coefficient of t^{r+1} in the formal power series

$$S(t) = \frac{c_t(\mathcal{E} \otimes \mathcal{L})}{c_t(\mathcal{F} \otimes \mathcal{L})}$$

is independent of $\ell = c_1(\mathcal{L})$.

By [1], Example 3.2.2, we have $c_t(\mathcal{E} \otimes \mathcal{L}) = (1 + \ell t)^m c_\tau(\mathcal{E})$ with $\tau = t/(1 + \ell t)$. Therefore

$$S(t) = (1 + \ell t)^r \frac{c_\tau(\mathcal{E})}{c_\tau(\mathcal{F})} = (1 + \ell t)^r \sum_{k=0}^{\infty} c_k(\alpha)\tau^k.$$

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Hence the coefficient of t^{r+1} in $S(t)$ equals that of t^{r+1} in

$$\begin{aligned}
(1+\ell t)^r &\sum_{k=0}^{r+1} c_k(\alpha)\tau^k \\
&= (1+\ell t)^r c_{r+1}(\alpha)\tau^{r+1} + (1+\ell t)^r \sum_{k=0}^{r} c_k(\alpha)\tau^k \\
&= c_{r+1}(\alpha)\frac{t^{r+1}}{1+\ell t} + \sum_{k=0}^{r} c_k(\alpha) t^k (1+\ell t)^{r-k}.
\end{aligned}$$

The coefficient of t^{r+1} in the last expression equals $c_{r+1}(\alpha)$. $\square$

References

1. W. Fulton, *Intersection theory*, Springer Verlag, Berlin, 1984.