CHARACTERISTIC CLASSES OF DISCRIMINANTS AND ENUMERATIVE GEOMETRY

PAOLO ALUFFI[1]

Florida State University

ABSTRACT. We compute the Euler obstruction and Mather's Chern class of the discriminant hypersurface of a very ample linear system on a nonsingular variety. Comparing the codimension-1 and 2 terms of this and other characteristic classes of the discriminant leads to a quick computation of the degrees of the loci of cuspidal and binodal sections of a very ample line bundle on a smooth variety, and of the tacnodal locus for linear systems on a surface. We also compute explicitly all terms in the Schwartz-MacPherson's classes of strata of the discriminant in the $\mathbb{P}^9$ of of cubic plane curves, and of the discriminant of $|\mathcal{O}(d)|$ on $\mathbb{P}^1$.

§0. Introduction

The *discriminant* of a linear system V on a nonsingular variety M, parametrizing singular elements of V, is an object with a rich and complicated geometry. Many enumerative results of recent interest, such as Kontsevich's beautiful recursion for the number of rational plane curves containing assortments of points, amount to the computation of the degree of selected strata of the singular locus of the discriminant of $|\mathcal{O}(d)|$ on the plane. The degree of arbitrary strata, or even an exhaustive combinatorial description of the singular locus of discriminants, seem at this point completely out of reach.

Simpler invariants of given singularities of discriminants are however well understood. For example, the *multiplicity* of a discriminant at a point X corresponding to a hypersurface on M with isolated singularities has long been known to equal the sum of the Milnor numbers of the singularities of X. More generally, the multiplicity of a discriminant at an arbitrary point can be expressed in terms of Parusiński's generalized Milnor numbers of the (not necessarily isolated) singularities of the corresponding hypersurface X, or equivalently by expressions in terms of the Segre class of the singularity *subscheme* of X (see [P1], [P2], [A-C], and [A1], p. 338-339).

[1]Supported in part by NSF grant DMS-9500843. The author is grateful to the Mittag-Leffler institute and the organizers of the year on 'enumerative geometry and its interaction with theoretical physics' for the generous hospitality under which this research was carried out.

In this note we aim to bridge from this type of information to standard enumerative results.

First, we compute the *local Euler obstruction* of a discriminant at a point X, under mild assumptions on the situation (essentially, we assume that the discriminant is a hypersurface in $\mathbb{P}^n = H^0(M,\mathcal{L})$): we show that the Euler obstruction at X equals the degree of its μ-class, introduced in [A1], and hence agrees with Parusiński's Milnor number of the singularity of X. We then obtain an explicit expression for Mather's Chern class $c_M(D)$ of a discriminant D, which corresponds to the Euler obstruction function under MacPherson's natural transformation ([MP]). Pushing this forward to the ambient projective space $\mathbb{P}^n = \mathbb{P}H^0(M,\mathcal{L})$ we find

$$c_M(D) \mapsto c(T\mathbb{P}^n) \cap \sum_{j\geq 1}(-1)^{j-1}\sum_{k=1}^{j}\binom{j-1}{k-1}\delta_k \cdot [\mathbb{P}^{n-j}]$$

where δ_k denotes the k-th rank of M in the sense of Holme (that is, the $(m-k)$-th class ρ_{m-k} in the sense of [Fulton], p. 253), explicitly

$$\delta_k = \int c_1(\mathcal{L})^{k-1} c(\mathcal{P}^1_M\mathcal{L}) \cap [M]$$

(cf. [A1], p. 340) with $c(\mathcal{P}^1_M\mathcal{L}) = c(\mathcal{L})c(T^*M\otimes\mathcal{L})$ the class of the bundle of principal parts of $\mathcal{L}$.

By comparing the degrees of the codimension-one terms of Mather's class and other characteristic classes of the discriminant, we recover formulas for the degree of the loci C, G parametrizing respectively cuspidal and binodal elements of the linear system. While formulas for these loci are not new (see for example [D-L], p. 5), the method employed here seems particularly straightforward and leads naturally to simple expressions: we find

$$\begin{aligned}\deg C &= m(\delta_1+\delta_2) + 2K\cdot c_{m-1}\\ \deg G &= \frac{\delta_1^2}{2} - \frac{3m+1}{2}(\delta_1+\delta_2) - 3K\cdot c_{m-1}\end{aligned}$$

where m denotes the dimension of M, K its canonical divisor, and $c_i = c_i(\mathcal{P}^1_M\mathcal{L})$. As another application of this circle of ideas, we obtain the (known) formula for the locus of tacnodal sections of a linear system on a surface:

$$5\delta_1 + 3\delta_2 + 26\delta_3 + 17K\cdot c_1$$

with notations as above, by comparing codimension-2 terms of suitable characteristic classes.

The lesson we draw from these examples is that it would be valuable to compute a number of 'characteristic classes' of discriminants. In principle one should aim to computing Schwartz-MacPherson's class (and not just the degree) of strata of discriminants: the good additivity properties of Schwartz-MacPherson's classes make these invariants very well-behaved, in the sense that any computation of a characteristic class yields a linear relation among these invariants. We expand on this viewpoint at the end of section 3.

More interesting applications will depend on computing several classes associated to a discriminant, such as Schwartz-MacPherson's class of its strata, or the Segre class of its singular scheme. We obtain these for the 12-ic hypersurface D of $\mathbb{P}^9$ parametrizing singular plane cubics. This allows us to compute the multiplicity and Euler obstruction of its discriminant (a degree-23,579,476,910 hypersurface of $\mathbb{P}^{293,929}$) at D—while such a result is certainly not particularly useful in itself, it should serve as a good illustration of the application of our formulas in an extreme situation.

Finally, we compute Schwartz-MacPherson's classes of strata of the discriminant of $\mathcal{O}(d)$ on $\mathbb{P}^1$, in terms of the multiplicity type of representative elements of the strata. While we are able to obtain explicit forms for these expressions, we do not know a general formula for Schwartz-MacPherson's class of the *closure* of the stratum corresponding to an arbitrary combinatorial type. We obtain such a formula for the discriminant itself, and an explicit formula for a 'weighted' Schwartz-MacPherson's class to which each substratum contributes according essentially to the order of a group of automorphisms of the corresponding combinatorial type.

I am grateful to Ragni Piene and Israel Vainsencher for several very useful comments.

§1. The Nash blow-up of a discriminant

We work over an algebraically closed field of characteristic 0.

Let M be a complete irreducible nonsingular variety, and consider a line bundle $\mathcal{L}$ on M. We assume for simplicity that $\mathcal{L}$ is very ample, giving an embedding

$$M \to \mathbb{P}H^0(M,\mathcal{L})^\vee = \mathbb{P}^{n\vee} \quad .$$

By the *discriminant* D of $\mathcal{L}$ we mean the subset of $\mathbb{P}^n$ consisting of singular sections of $\mathcal{L}$; that is, D is the dual variety $D \subset \mathbb{P}^n$ of M. The results of this note should extend to more general line bundles and linear systems, the crucial hypothesis being however that the map ν defined below is birational.

Denoting by $\mathcal{V}_M$ the trivial bundle with fiber $V = H^0(M,\mathcal{L})$, we have a surjection

$$\mathcal{V}_M \xrightarrow{\rho} \mathcal{P}^1_M\mathcal{L} \to 0$$

where $\mathcal{P}^1_M\mathcal{L}$ denotes the bundle of principal parts of $\mathcal{L}$ on M.

We consider the projective subbundle $\mathcal{D} = \mathbb{P}(\ker\rho)$ of $\mathbb{P}(\mathcal{V}_M) = \mathbb{P}^n \times M$. We have observed in [A-C] that D is the image of $\mathcal{D}$ in $\mathbb{P}^n$ via the projection $\mathbb{P}^n \times M \to \mathbb{P}^n$.

Blanket hypothesis. We assume from now on that the projection

$$\nu : \mathcal{D} \to D$$

is birational, as 'in the majority of cases' (cf. [D-L], p. 4).

Note also that biduality holds, by our characteristic 0 assumption.

Lemma 1. *$\mathcal{D} \to D$ is the Nash-blow-up of D.*

Proof. The Nash-blow-up can be obtained as the closure of the image of the map

$$\begin{aligned} &D \dashrightarrow \mathbb{P}^n \times \mathbb{P}^{n\vee} \\ &X \mapsto (X, \text{tangent hyperplane to } D \text{ at } X) \end{aligned}$$

defined for all X at which D is smooth. By biduality, the tangent hyperplane to D at a smooth point X corresponds to the one point of M at which the hyperplane of $\mathbb{P}^{n\vee}$ corresponding to X is tangent to M, that is to the one singular point of X. That is, the above map factors

$$\begin{aligned} D &\dashrightarrow \mathbb{P}^n \times M \subset \mathbb{P}^n \times \mathbb{P}^{n\vee} \\ X &\mapsto (X, \operatorname{Sing} X) \end{aligned}$$

Next, $\operatorname{Sing} X$ consists of points at which sections of $\mathcal{L}$ corresponding to X vanish to first order; hence points $(X, \operatorname{Sing} X)$ are points of $\mathbb{P}(\ker \rho) = \mathcal{D}$ and the map factors

$$D \dashrightarrow \mathcal{D} \subset \mathbb{P}^n \times \mathbb{P}^{n\vee}$$

However $\dim D = \dim \mathcal{D}$, and $\mathcal{D}$ is irreducible, so $\mathcal{D}$ must be the closure of the image of this map, which is what we need. $\square$

§2. Local Euler obstruction and Mather's Chern class

The Mather-Chern class of D is obtained by pushing forward from the Nash blow-up $\mathcal{D}$ the class of the tautological bundle $\mathcal{T}$, whose fiber at $(X, \mathbb{P}H) \in \mathbb{P}^n \times \mathbb{P}^{n\vee}$ is the subspace $T\mathbb{P}H$ of $T\mathbb{P}^n = T\mathbb{P}V$.

Lemma 2. $c(\mathcal{T}) = \dfrac{c(T\mathbb{P}^n)}{c(\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^{n\vee}}(1))}$.

Proof. From the Euler sequence

$$0 \to \mathcal{O} \to H \otimes \mathcal{O}_{\mathbb{P}^n}(1) \to T\mathbb{P}H \to 0$$

we see that

$$c(\mathcal{T}) = c(\mathcal{H} \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \quad ,$$

where $\mathcal{H}$ is the (pull-back to $\mathbb{P}^n \times \mathbb{P}^{n\vee}$ of the) bundle over $\mathbb{P}^{n\vee}$ whose fiber over $\mathbb{P}H$ is the subspace $H \subset V$. This is realized by dualizing

$$0 \to \mathcal{O}_{\mathbb{P}^{n\vee}}(-1) \hookrightarrow \mathcal{V}^\vee$$

into

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathcal{O}_{\mathbb{P}^{n\vee}}(1) \to 0 \quad ;$$

tensoring by $\mathcal{O}_{\mathbb{P}^n}(1)$ shows

$$c(\mathcal{H} \otimes \mathcal{O}_{\mathbb{P}^n}(1)) = \frac{c(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^n}(1))}{c(\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^{n\vee}}(1))}$$

and we are done since $c(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^n}(1)) = c(T\mathbb{P}^n)$. $\square$

Let now $X \in D$, so that X comes from a singular section of $\mathcal{L}$. We denote by JX the singularity subscheme of X, defined by the ideal of partial derivatives of local sections of X. In [A1] we have introduced and studied the class

$$\mu_{\mathcal{L}}(JX) = c(T^*M \otimes \mathcal{L}) \cap s(JX, M) \quad .$$

THEOREM 3. *The local Euler obstruction of the discriminant D of $\mathcal{L}$ at X is the degree of the μ-class of JX with respect to $\mathcal{L}$:*

$$Eu_D\, X = \int \mu_{\mathcal{L}}(JX)$$

PROOF. As we have shown that $\mathcal{D} \xrightarrow{\nu} D$ is the Nash-blow-up of D, and computed the relevant tautological bundle in Lemma 2, we can use Gonzalez-Sprinberg—Verdier's formulation of the Euler obstruction ([Fulton], p. 78) and obtain

$$\mathrm{Eu}_D\, X = \int \frac{c(T\mathbb{P}^n)}{\mathcal{O}_{\mathbb{P}^{n\vee}}(1) \otimes \mathcal{O}_{\mathbb{P}^n}(1)} \cap s(\nu^{-1}X, \mathcal{D}) \quad .$$

Now we observe that $\nu^{-1}X = JX$, and that we computed $s(\nu^{-1}X, \mathcal{D})$ in [A-C]:

$$s(\nu^{-1}X, \mathcal{D}) = c(\mathcal{P}^1_M \mathcal{L}) \cap s(JX, M) \quad .$$

For a given X, $\mathcal{O}_{\mathbb{P}^n}(1)$ and $T\mathbb{P}^n$ are trivial once restricted to the corresponding slice $\{X\} \times \mathbb{P}^{n\vee}$, hence on its intersection $\nu^{-1}(X)$ with $\mathcal{D}$, so the class of the tautological bundle restricts to

$$\left.\frac{c(T\mathbb{P}^n)}{c(\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^{n\vee}}(1))}\right|_{\mathbb{P}^{n\vee}} = \frac{1}{c(\mathcal{O}_{\mathbb{P}^{n\vee}}(1))} = \frac{1}{c(\mathcal{L})}$$

In conclusion,

$$\mathrm{Eu}_D\, X = \int \frac{c(\mathcal{P}^1_M \mathcal{L})}{c(\mathcal{L})} \cap s(JX, M) = \int c(T^*M \otimes \mathcal{L}) \cap s(JX, M) \quad ,$$

which is the claim. □

Putting this together with Proposition 2.1 in [A1]:

COROLLARY 4. *(Over $\mathbb{C}$.) The Euler obstruction of D at X equals Parusiǹski's Milnor number of X (cf.* [P1]*).*

REMARKS. (1) The Euler obstructions of a variety and of its dual are in fact related in great generality by a Radon transform, as shown by L. Ernström ([E]); and Parusiǹski shows ([P1], Corollary 1.7) that his generalization of the Milnor number measures the difference between the Euler characteristics of X and of a general section of $\mathcal{L}$. Combining these results gives an alternative proof of Corollary 4.

(2) If the singularities of X are isolated, then Theorem 3 says that $\mathrm{Eu}_D\, X$ equals the degree of the Segre class of the singular subscheme of X in M. This however equals the sum of the Milnor numbers of the singularities of X, which is well-known to compute the multiplicity of D at X. Hence: *If the singularities of X are isolated, then $Eu_D X = m_D X$, the multiplicity of D at X.*

In general, Euler obstruction and multiplicity do not agree: for example, the discriminant of plane cubics has multiplicity 8 at a triple line X (see for example [A-C], p. 253), and Euler obstruction

$$\int \mu_{\mathcal{O}(3)}(JX) = 2 \quad .$$

Comparing Theorem 3 and the formula for the multiplicity given in [A-C] yields in fact an expression for the difference: with notations as above,

$$m_D X - e_D X = \int c_1(\mathcal{L}) \cap \mu_{\mathcal{L}}(JX) \quad .$$

For example, this shows: *If the singular locus of X is a curve, then $Eu_D X = m_D X - c_1(\mathcal{L}) \cdot \{JX\}$, where $\{JX\}$ denotes the dimension-1 term in $s(JX, M)$.*

As for Mather's Chern class $c_M(D)$ of D, let $i : D \to \mathbb{P}^n$ denote the inclusion. Then

THEOREM 5.

$$i_* c_M(D) = c(T\mathbb{P}^n) \cap \sum_{j \geq 1} (-1)^{j-1} \left(\int c(\mathcal{L})^{j-1} c(\mathcal{P}^1_M \mathcal{L}) \cap [M] \right) [\mathbb{P}^{n-j}]$$

PROOF. By Lemma 1 and 2 we have

$$c_M(D) = \nu_* c(\mathcal{T}) \cap [\mathcal{D}] = c(T\mathbb{P}^n) \cap \nu_* \frac{[\mathcal{D}]}{c(\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{O}_{\mathbb{P}^{n\vee}}(1))} \quad .$$

Tracing the Euler sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \ker \rho \otimes \mathcal{O}_{\mathbb{P}^n}(1) & \longrightarrow & T_{\mathcal{D}|M} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^n}(1) & \longrightarrow & T_{\mathbb{P}\mathcal{V}|M} & \longrightarrow & 0
\end{array}$$

gives

$$c(N_{\mathcal{D}} \mathbb{P}\mathcal{V}) = c(\mathcal{P}^1_M \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \quad ,$$

and in particular

$$[\mathcal{D}] = c_{\text{top}}(\mathcal{P}^1_M \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \cap [\mathbb{P}^n \times M]$$

as a class in $\mathbb{P}^n \times M$. Denoting by $\pi : \mathbb{P}^n \times M \to \mathbb{P}^n$ the projection and identifying $\mathcal{O}_{\mathbb{P}^{n\vee}}(1)|_M \cong \mathcal{L}$, we see then that

$$i_* c_M(D) = c(T\mathbb{P}^n) \cap \pi_* \frac{c_{\text{top}}(\mathcal{P}^1_M \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \cap [\mathbb{P}^n \times M]}{c(\mathcal{O}_{\mathbb{P}^n}(1) \otimes \mathcal{L})} \quad .$$

Now π_* kills all pull-backs from M except those in codimension $m = \dim M$. Writing $c_i = c_i(\mathcal{P}^1_M \mathcal{L})$, $\ell = c_1(\mathcal{L})$, and $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$, we are pushing forward

$$\begin{aligned}
&\left(c_m h + c_{m-1} h^2 + \cdots + h^{m+1}\right) \frac{1}{1+\ell} \cdot \frac{1}{1 + \frac{h}{1+\ell}} \\
&\qquad = \frac{1}{1+\ell} \left(c_m h + c_{m-1} h^2 + \cdots + h^{m+1}\right) \left(1 - \frac{h}{1+\ell} + \frac{h^2}{(1+\ell)^2} - \cdots\right)
\end{aligned}$$

The coefficient of h^j in this expression is

$$\sum_{i=0}^{j-1} (-1)^i \frac{c_{m+1-j+i}}{(1+\ell)^{i+1}} = \sum_{i=0}^{j-1} (-1)^i c_{m-(j-1-i)} \sum_k \binom{k+i}{i} (-\ell)^k$$

Extracting the codimension-m piece gives

$$(-1)^{j-1}\sum_{i=0}^{j-1} c_{m-(j-1-i)}\binom{j-1}{i}\ell^{j-1-i}$$
$$= (-1)^{j-1}\int (1+c_1+\cdots+c_m)\sum_{i=0}^{j-1}\binom{j-1}{i}\ell^{j-1-i}$$
$$= (-1)^{j-1}\int c(\mathcal{P}^1_M\mathcal{L})\, c(\mathcal{L})^{j-1}$$

which gives the claim. □

The coefficients of $[\mathbb{P}^{n-j}]$ in the second factor of the expression given in the Theorem are easily expressed in terms of the numbers

$$\delta_j = \int c_1(\mathcal{L})^{j-1} c(\mathcal{L})\, c(T^*M\otimes\mathcal{L})\cap[M] \quad ,$$

obtaining the formula given in the introduction.

Note that, as an immediate consequence of the Theorem, we obtain a formula for the (well-known) degree of the dual variety of M:

$$\deg D = \deg c_M(TD) = \delta_1$$

(cf. [A1], Corollary 2.4, and remember that we are assuming that the discriminant is a hypersurface).

§3. Other characteristic classes, and enumerative applications

As observed in [D-L], p. 4–5, the singular locus of D consists of the closures of two loci: the locus C consisting of $X\in D$ having a unique singular point at which the quadratic form of the defining section has rank $m-1$, and the locus G consisting of X with two non-degenerate quadratic singularities. We say that X is 'cuspidal' if it belongs to C, and 'binodal' if it belongs to G. Away from pathologies, these loci have codimension 1 in D; we will assume this is the case in the following. Also, N will denote the nonsingular locus of D, consisting of sections with a node (an ordinary double point) as unique singularity.

The first task in this section is to compare the term of codimension one in the Mather-Chern class of D and other characteristic classes; this yields at once expressions for the degrees of $\overline{C}$ and $\overline{G}$ in the ambient $\mathbb{P}^n$. Next, we similarly compare terms of codimension two to obtain the degree of the 'tacnodal' locus.

First, recall that there exists a natural transformation c_* from the functor of constructible functions, with push-forward defined by Euler characteristic of the fibers, to Chow groups ([MP]), such that the image $c_*(1_V)$ of the characteristic function of a nonsingular variety V evaluates its total Chern class $c(TV)\cap[V]$. In general, one defines

$$c_{\mathrm{SM}}(V) = c_*(1_V) \quad ,$$

for possibly singular V; $c_{\mathrm{SM}}(V)$ is known as Schwartz-MacPherson's Chern class of V. A subproduct of the construction of c_* is that Mather's Chern class is the image of the local Euler obstruction function:

$$c_M(V) = c_*(\mathrm{Eu}_V) \quad .$$

By Theorem 3 we have

$$\mathrm{Eu}_D\, X = \begin{cases} & \dots \\ 2 & G \\ 2 & C \\ 1 & N \end{cases} \quad ,$$

where we indicate the value of the function for X =the general point of the listed locus. Therefore

$$\mathrm{Eu}_D\, X = 1_D + 1_{\overline{C}} + 1_{\overline{G}} + \dots \quad ,$$

and it follows that, under our hypotheses:

$$c_{\mathrm{SM}}(D) = c_M(D) - [\overline{C}] - [\overline{G}] + \text{higher codimension terms}$$

Now Theorem 5 gives an expression for the right-hand-side; independent computations of $c_{\mathrm{SM}}(D)$ will then yield relations involving $[\overline{C}]$ and $[\overline{G}]$. Here is one such computation:

Proposition 6. *Denote by $c_F(D)$ the class of the virtual tangent bundle of D. Then*

$$c_{\mathrm{SM}}(D) = c_F(D) + 2[\overline{C}] + [\overline{G}] + \textit{higher codimension terms} \quad ,$$

that is

$$i_* c_{\mathrm{SM}}(D) \equiv c(T\mathbb{P}^n) \cap \left(\delta_1\, [\mathbb{P}^{n-1}] - \delta_1^2\, [\mathbb{P}^{n-2}] + \cdots\right) + 2[\overline{C}] + [\overline{G}]$$

up to terms of higher codimension.

Proof. This follows from the main result in [A3] (in fact Lemma 3 in [A2] suffices for degree computations). The difference between c_{SM} and c_F is measured by a twist of $s(JD, \mathbb{P}^n)$; as the twist does not affect the top-dimensional term, it suffices to show that

$$s(JD, \mathbb{P}^n) = 2[\overline{C}] + [\overline{G}] + \cdots \quad ,$$

and this follows from the well-known fact that the discriminant is itself cuspidal along C, while it has two nonsingular branches along G.

Next, we have observed in §2 that D has degree δ_1 in $\mathbb{P}^n$. Therefore

$$\begin{aligned} c_F(D) &= c(T\mathbb{P}^n) \cap \frac{\delta_1\, [\mathbb{P}^{n-1}]}{1 + \delta_1\, h} \\ &= c(T\mathbb{P}^n) \cap \left(\delta_1\, [\mathbb{P}^{n-1}] - \delta_1^2\, [\mathbb{P}^{n-2}] + \cdots\right) \quad . \end{aligned}$$

from which the stated expression follows. □

Comparing the two expressions obtained for $c_{\mathrm{SM}}(D)$:

$$c_M(D) - c_F(D) = 3[\overline{C}] + 2[\overline{G}] + \text{higher codimension terms} \quad ,$$

and pushing forward to $\mathbb{P}^n$ we obtain the simple relation

COROLLARY 7.
$$3[\overline{C}]+2[\overline{G}]=(\delta_1^2-\delta_1-\delta_2)[\mathbb{P}^{n-2}]$$
where the δ_i are defined in the introduction. As Ragni Piene kindly pointed out to me, this is nothing but Plücker's formula for a general plane section of the discriminant.

To get expressions for $[C]$ and $[G]$ individually we need more information. For this, push-forward the characteristic function of $\mathcal{D}$ to obtain

$$\nu_*(1_{\mathcal{D}})(X)=\begin{cases} & \dots \\ 2 & G \\ 1 & C \\ 1 & N \end{cases}$$

(since general cuspidal sections have one singular point, and binodal sections have two), that is
$$\nu_*(1_{\mathcal{D}})(X)=1_D+1_G+\dots \quad .$$
Applying MacPherson's natural transformation then gives (since $\mathcal{D}$ is nonsingular)

$$c_{\mathrm{SM}}(D)=\nu_*c(T\mathcal{D})\cap[\mathcal{D}]-[\overline{G}]+\cdots$$

and combining with the above:

$$\begin{aligned}[\overline{C}] &\equiv c_M(D)-\nu_*c(T\mathcal{D})\cap[\mathcal{D}] \quad , \\ [\overline{G}] &\equiv \frac{1}{2}\left(3\,\nu_*c(T\mathcal{D})\cap[\mathcal{D}]-2\,c_M(D)-c_F(D)\right)\end{aligned}$$

up to higher codimension terms. To obtain explicit formulas we need to compute $\nu_*c(T\mathcal{D})\cap[\mathcal{D}]$:

PROPOSITION 8. *The push-forward to $\mathbb{P}^n$ of $\nu_*c(T\mathcal{D})\cap[\mathcal{D}]$ is given by*

$$c(T\mathbb{P}^n)\cap\left(\delta_1\,[\mathbb{P}^{n-1}]-(2K\cdot c_{m-1}+(m+1)(\delta_1+\delta_2))[\mathbb{P}^{n-2}]+\cdots\right)$$

(up to higher codimension terms), where $m=\dim M$, K is the canonical divisor of M, and $c_i=c_i(\mathcal{P}^1_M\mathcal{L})\cap[M]$.

PROOF. As we have seen in the proof of Theorem 5,

$$c(N_{\mathcal{D}}\mathbb{P}\mathcal{V})=c(\mathcal{P}^1_M\mathcal{L}\otimes\mathcal{O}_{\mathbb{P}^n}(1)) \quad ;$$

from this it follows that the push-forward of $\nu_*c(T\mathcal{D})\cap[\mathcal{D}]$ is given by

$$c(T\mathbb{P}^n)\cap\nu_*\left(c(TM)\cdot\frac{c_{\mathrm{top}}(\mathcal{P}^1_M\mathcal{L}\otimes\mathcal{O}(1))\cap[\mathbb{P}^n\times M]}{c(\mathcal{P}^1_M\mathcal{L}\otimes\mathcal{O}(1))}\right)$$

and the statement follows by evaluating the first two terms in this expression. This is a straightforward computation, which we leave to the reader. □

From Proposition 8 and the expressions obtained above, we get

Corollary 9.

$$\deg \overline{C} = m(\delta_1 + \delta_2) + 2\,K \cdot c_{m-1}$$
$$\deg \overline{G} = \frac{\delta_1^2}{2} - \frac{3m+1}{2}(\delta_1 + \delta_2) - 3\,K \cdot c_{m-1}$$

as stated in the introduction.

Example. For $M = \mathbb{P}^m$ and $\mathcal{L} = \mathcal{O}(d)$ (that is, the d-th Veronese embedding of $\mathbb{P}^n$) we have

$$\delta_1 = (m+1)(d-1)^m \quad , \quad \delta_2 = \binom{m+1}{2} d(d-1)^{m-1}$$
$$\deg K = -(m+1) \quad , \quad \deg c_{m-1} = \binom{m+1}{2}(d-1)^{m-1}$$

giving

$$\deg C = \frac{m\,(m+1)\,(m+2)}{2}(d-2)\,(d-1)^{m-1}$$
$$\deg G = \frac{(d-1)^{2m}(m+1)^2}{2} - \frac{(d-1)^{m-1}(m+1)\,\big((m+2)(3m+1)d - 2(3m^2+6m+1)\big)}{4}$$

Specializing further, for $m = 2$ we get the degrees of the loci of cuspidal, resp. binodal plane curves of degree d:

$$\deg C = 12\,(d-1)(d-2)$$
$$\deg G = 3\,(d-1)(d-2)\frac{3\,d^2 - 3\,d - 11}{2}$$

which are of course well known (cf. for example [DF-I], p. 86–88, for a very concrete derivation of this last formula).

As another application of the same philosophy, we consider the locus of *tacnodal* sections of a linear system on a nonsingular surface S. We assume (as is generically the case) that in codimension 2 in the discriminant one finds only curves with three kinds of singularities: one node and one cusp; or three nodes; or one tacnode. Call curves with such singularities *NC*, *NNN*, and *TAC* respectively. Keeping the notations as above, the local Euler obstruction and image of $1_{\mathcal{D}}$ have values:

$$\mathrm{Eu}_D\, X = \begin{cases} & \cdots \\ 3 & TAC \\ 3 & NNN \\ 3 & NC \\ 2 & G \\ 2 & C \\ 1 & N \end{cases} \quad , \quad \nu_* 1_{\mathcal{D}}(X) = \begin{cases} & \cdots \\ 1 & TAC \\ 3 & NNN \\ 2 & NC \\ 2 & G \\ 1 & C \\ 1 & N \end{cases}$$

We will also consider the push-forward $\gamma_* 1_{C'}$, where C' is the quadric bundle in $\mathcal{D}$ parametrizing curves with a cusp or worse, and $\gamma = \nu_{C'}$:

$$\gamma_* 1_{C'}(X) = \begin{cases} & \cdots \\ 1 & TAC \\ 0 & NNN \\ 1 & NC \\ 0 & G \\ 1 & C \\ 0 & N \end{cases}$$

and observe that therefore

$$\mathrm{Eu}_D X - \nu_* 1_{\mathcal{D}}(X) - \gamma_* 1_{C'}(X) = \begin{cases} & \cdots \\ 1 & X \in TAC \\ 0 & X \in NNN \\ 0 & X \in NC \\ 0 & X \in G \\ 0 & X \in C \\ 0 & X \in N \end{cases}$$

From this:

$$c_M(D) - \nu_* c(T\mathcal{D}) - \gamma_* c_{\mathrm{SM}}(C') = [\overline{TAC}] + \text{ higher codimensional terms}$$

We carry this out explicitly for $\mathbb{P}^2$ and $\mathcal{O}(d)$, for $d \geq 3$. By Lemma 1.4 in [A4], C' has class $2(d-3)k + 2h$ in $\mathcal{D} \subset \mathbb{P}^n \times \mathbb{P}^2$, where h, k denote the pull-backs of the hyperplanes from the factors of the product. The singular locus of C' projects to the set of curves with a triple point, so it does not affect the codimension 2 term of $\gamma_* c_{\mathrm{SM}}(C')$, which is then computed by

$$\gamma_* \left(\frac{(1+h)^{\binom{d+2}{2}}(1+k)^3 [C']}{(1+(d-1)k+h)^3 (1+2(d-3)k+2h)} + \cdots \right)$$
$$= 12(d-2)(d-1)h^2 + 2(3d^4 - 65d^2 + 168d - 120)h^3 + \cdots$$

Mather's class and $\nu_* c(T\mathcal{D})$ are given by Theorem 5 and (the next term in) Proposition 8: we get:

$$3(d-1)^2 h + \frac{3}{2}(d-1)d(d^2+2d-5)h^2 + \frac{1}{8}d(3d^5 + 12d^4 - 24d^3 - 66d^2 + 125d - 42)h^3 + \ldots$$

for Mather's class and

$$3(d-1)^2 h + \frac{3}{2}(d-1)(d^3+2d^2-13d+16)h^2 + \frac{1}{8}(d^6+4d^5-24d^4-22d^3+255d^2-398d+192)h^3 + \ldots$$

for the push-forward of $c(T\mathcal{D})$. The difference yields

$$[\overline{TAC}] = 2\left(25\, d^2 - 96\, d + 84\right)\, h^3 \quad .$$

Summarizing, we have proved

Corollary 10. *For $d \geq 3$, the degree of the closure of the locus of tacnodal degree-d plane curves is*
$$2\left(25\,d^2 - 96\,d + 84\right) \quad .$$

It is in fact not much harder to do the same for very ample line bundles $\mathcal{L}$ on a smooth surface S, under the blanket assumptions on the discriminant mentioned above. In this case and keeping the above notations,

$$c_1(\mathcal{O}(C')) = -2c_1(TS) + 2\ell + 2h = 2c_1(\mathcal{P}^1_S\mathcal{L}) - 2\ell + 2h \quad ,$$

so C' has class $2(c_1 - \ell + h)$. Curves with multiplicity≥ 3 occur in codimension 3, so MacPherson's class of C' up to codimension 2 agrees with the push forward of the class of its virtual tangent bundle,

$$c(T\mathcal{D})\,\frac{2(c_1 - \ell + h)}{1 + 2(c_1 - \ell + h)} \quad .$$

Computing from this gives

$$\begin{aligned}\gamma_*(c_{\mathrm{SM}}(C')) = c(T\mathbb{P}^n)\,\big(2(\delta_1 + \delta_2 + K\cdot c_1)&[\mathbb{P}^{n-2}] \\ &-(10\delta_1 + 14\delta_2 + 28\delta_3 + 26K\cdot c_1)[\mathbb{P}^{n-3}] + \dots\big)\end{aligned}$$

At the same time, Theorem 5 and Proposition 8 give, for a surface:

$$\begin{aligned}\gamma_*(c_{\mathrm{SM}}(\mathcal{D})) = c(T\mathbb{P}^n)\,\big(\delta_1[\mathbb{P}^{n-1}] - (3\delta_1 + 3\delta_2 + 2K\cdot c_1)&[\mathbb{P}^{n-2}] \\ &+(6\delta_1 + 13\delta_2 + 3\delta_3 + 9K\cdot c_1)[\mathbb{P}^{n-3}] + \dots\big)\end{aligned}$$

and

$$c_M(D) = c(T\mathbb{P}^n)\,\big(\delta_1[\mathbb{P}^{n-1}] - (\delta_1 + \delta_2)[\mathbb{P}^{n-2}] + (\delta_1 + 2\delta_2 + \delta_3)[\mathbb{P}^{n-3}] + \dots\big)$$

Taking $c_M(D) - \nu_* c(T\mathcal{D}) - \gamma_* c_{\mathrm{SM}}(C')$ gives

$$(5\,\delta_1 + 3\,\delta_2 + 26\,\delta_3 + 17\,K\cdot c_1)[\mathbb{P}^{n-3}] + \dots$$

from which we read the degree of the locus of tacnodal curves, given in the introduction.

For example, for a degree-d surface in $\mathbb{P}^3$ one computes

$$\delta_1 = (d-1)^2 d \quad , \quad \delta_2 = (d-1)d \quad , \quad \delta_3 = d \quad , \quad K\cdot c_1 = (d-4)(d-1)d$$

from which the degree of the tacnodal locus (that is, the number of hyperplanes intersecting S into a tacnodal curve) turns out to be

$$2\,d\,(d-2)\,(11d - 24) \quad .$$

Singularities of surface sections are studied in [V2].

In view of enumerative applications such as the simple ones presented above, it would be valuable to have explicit expressions for several characteristic classes associated with discriminants, in the form of their push-forward to the ambient projective space. By *characteristic class* here we loosely mean the image of a geometrically defined constructible function on D via MacPherson's transformation. We are equally imprecise on the notion of stratum in posing the following problem:

Compute the image $c_*(1_Y)$ of the characteristic functions of strata Y of the discriminant.

Giving a stratification of the discriminant amounts to classifying singularities of sections of a line bundle; the 'right' classification depends on the context. (For example, for second-order singularities a stratification was introduced by J. Roberts, cf. [R] and [V1].) Note that the information of $c_*(1_Y)$ contains in particular the information of degree of Y. The computation of this piece of information alone for the locus Y of sections with arbitrarily prescribed singularity 'type' is completely out of reach at present, so it should seem even more unreasonable to ask for the whole information carried by $c_*(1_Y)$. On the other hand, every computation of a characteristic class provides with a relation among the $c_*(1_Y)$, as dictated by the corresponding constructible function, so it is relatively easy to give partial answers to the problem in the form of nontrivial combinations of $c_*(1_Y)$.

Summarizing, we have seen how to get such an expression for Mather's Chern class; similarly one can write out all terms in the class $\nu_* c(T\mathcal{D}) \cap [\mathcal{D}]$ of Proposition 8. These two classes correspond via MacPherson's natural transformation c_* respectively to the Euler obstruction function (computed in Theorem 3), and to the constructible function mapping $X \in D$ to the Euler characteristic of the singular locus of X, and hence provide us with information on selected combinations of $c_*(1_Y)$. As we have seen, this information yields immediately enumerative applications.

In the next two sections we answer in full the problem given above, for the discriminant of *plane cubic curves,* and for the discriminant of $\mathcal{O}(d)$ on $\mathbb{P}^1$.

§4. Characteristic classes of the discriminant of plane cubics

The discriminant D of $\mathcal{O}(3)$ on $M = \mathbb{P}^2$ is a hypersurface of $\mathbb{P}^9$ parametrizing *singular* plane cubic curves. In this case $\delta_1 = 12$, $\delta_2 = 18$, $\delta_3 = 9$, and $\delta_i = 0$ for $i \geq 4$; according to Theorem 5, the push-forward to $\mathbb{P}^9$ of Mather's Chern class is then:

$$12[\mathbb{P}^8]+90[\mathbb{P}^7]+297[\mathbb{P}^6]+567[\mathbb{P}^5]+693[\mathbb{P}^4]+567[\mathbb{P}^3]+315[\mathbb{P}^2]+117[\mathbb{P}^1]+27[\mathbb{P}^0] \quad .$$

This class corresponds to the Euler obstruction function via MacPherson's natural transformation c_*. In this section we will compute the push-forward to $\mathbb{P}^9$ of the image via c_* of the characteristic functions of all strata of D, parametrizing different kinds of singular cubics. Names for these (open) strata will be:

- I: the set of triple lines;
- X: the set of unions of a line and a double (distinct) line;
- S: the set of stars of three distinct lines through a point;
- T: the set of unions of three nonconcurrent lines;
- P: the set of unions of a nonsingular conic and a tangent line;
- G: the set of unions of a nonsingular conic and a transversal line;
- C: the set of cuspidal cubics;
- N: the set of nodal cubics.

Abusing notations, we denote by $c_*(I)$, etc. the push-forward to $\mathbb{P}^9$ of the corresponding classes $c_*(1_I)$, etc. We will prove:

THEOREM 11.

$$\begin{aligned}
c_*(I) &= 9[\mathbb{P}^2]+9[\mathbb{P}^1]+3[\mathbb{P}^0]\\
c_*(X) &= 24[\mathbb{P}^4]+54[\mathbb{P}^3]+42[\mathbb{P}^2]+18[\mathbb{P}^1]+6[\mathbb{P}^0]\\
c_*(S) &= 15[\mathbb{P}^5]+18[\mathbb{P}^4]+6[\mathbb{P}^3]\\
c_*(T) &= 15[\mathbb{P}^6]+30[\mathbb{P}^5]+33[\mathbb{P}^4]+21[\mathbb{P}^3]+9[\mathbb{P}^2]+3[\mathbb{P}^1]+[\mathbb{P}^0]\\
c_*(P) &= 42[\mathbb{P}^6]+87[\mathbb{P}^5]+96[\mathbb{P}^4]+60[\mathbb{P}^3]+24[\mathbb{P}^2]+6[\mathbb{P}^1]\\
c_*(G) &= 21[\mathbb{P}^7]+21[\mathbb{P}^6]+21[\mathbb{P}^5]+21[\mathbb{P}^4]+21[\mathbb{P}^3]+12[\mathbb{P}^2]+3[\mathbb{P}^1]\\
c_*(C) &= 24[\mathbb{P}^7]+42[\mathbb{P}^6]+57[\mathbb{P}^5]+60[\mathbb{P}^4]+48[\mathbb{P}^3]+24[\mathbb{P}^2]+6[\mathbb{P}^1]\\
c_*(N) &= 12[\mathbb{P}^8]
\end{aligned}$$

We note that these classes show several features which seem hardly 'random'. We have no conceptual explanation for most of these features. However, the very simple expression for $c_*(N)$:

$$c_*(N) = 12\,[\mathbb{P}^8]$$

must be due to the fact that N is an orbit under PGL(3) with finite stabilizer.

PROOF OF THEOREM 11. We consider naive parametrizations for the closures of the loci listed above; each parametrization gives a class in $A_*\mathbb{P}^9$ (obtained by pushing-forward Schwartz-MacPherson's class of the parameter space), and a constructible function (obtained by pushing-forward the characteristic function of the parameter space). Every parametrization gives then a specific linear combination of the classes $c_*(I)$, $c_*(X)$, etc., and 8 independent such combinations suffice to determine the individual classes.

We will denote the parametrizing spaces by adding a $'$ to the letter of the corresponding locus in D, and the parametrization map by the corresponding lower case letter. Also, we will write $c_{\mathrm{SM}}(\,)$ for the push-forward to $\mathbb{P}^9$ of Chern-Schwartz-MacPherson's class of parameter spaces. Here are most of the notations in one diagram:

$$\begin{tikzcd}
C' \subset N' \arrow[dr, "c"] & N' = \mathcal{D} \subset \mathbb{P}^9 \times \mathbb{P}^2 \arrow[d, "n"] & I' \cong \mathbb{P}^2 \arrow[dl, "i"'] \\
G' \cong \mathbb{P}^5 \times \mathbb{P}^2 \arrow[r, "g"] & \mathbb{P}^9 & X' \cong \mathbb{P}^2 \times \mathbb{P}^2 \arrow[l, "x"'] \\
P' \subset G' \arrow[ur, "p"] & T' \cong \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \arrow[u, "t"] & S' \subset T' \arrow[ul, "s"']
\end{tikzcd}$$

•$I' = \overline{I} \cong \mathbb{P}^2$ and $c(T\mathbb{P}^2) \cap [\mathbb{P}^2]$. Pushing forward to $\mathbb{P}^9$ via the 3rd Veronese embedding gives:

$$c_{\mathrm{SM}}(I') = 9[\mathbb{P}^2] + 9[\mathbb{P}^1] + 3[\mathbb{P}^0] \quad .$$

The corresponding constructible function is

$$i_* 1_{I'}(p) = 1 \quad p \in I \quad ,$$

0 on other loci.

•$\overline{X}$ is dominated by the nonsingular $X' = \mathbb{P}^2 \times \mathbb{P}^2$. Therefore $c_{\mathrm{SM}}(X')$ is the push-forward to $\mathbb{P}^9$ of the class of the tangent bundle of $\mathbb{P}^2 \times \mathbb{P}^2$, giving

$$c_{\mathrm{SM}}(X') = 24[\mathbb{P}^4] + 54[\mathbb{P}^3] + 51[\mathbb{P}^2] + 27[\mathbb{P}^1] + 9[\mathbb{P}^0] \quad .$$

The parametrizing map x is a bijection, hence

$$x_* 1_{X'}(p) = \begin{cases} 1 & p \in I \\ 1 & p \in X \end{cases}$$

• Next, we consider locus $S' \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ of concurrent (ordered) lines, and the generically 6-to-1 map

$$S' \xrightarrow{s} \overline{S} \quad .$$

Listing Euler characteristics of preimages, we see that

$$s_* 1_{S'}(p) = \begin{cases} 1 & p \in I \\ 3 & p \in X \\ 6 & p \in S \end{cases}$$

To compute $c_{\mathrm{SM}}(S')$ we will use Proposition IV.6 in [A3]. In natural coordinates $((a_0 : a_1 : a_2), (b_0 : b_1 : b_2), (c_0 : c_1 : c_2))$, S' is given by

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix} = 0 \quad ;$$

so S' is of type $(1,1,1)$, and it is easily checked that the singular scheme of S' is the small diagonal $\mathbb{P}^2 \overset{\Delta}{\hookrightarrow} \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Applying Proposition IV.6 from [A3]:

$$c_{\mathrm{SM}}(S') = \frac{(1+h_1)^3(1+h_2)^3(1+h_3)^3}{1+h_1+h_2+h_3} \cap [S'] + \frac{(1+h)^3}{1+3h} \cap [\mathbb{P}^2]$$

with hopefully evident notations. Pushing forward to $\mathbb{P}^9$ (the hyperplane pulls back to $h_1+h_2+h_3$ and $3h$), this gives

$$c_{\mathrm{SM}}(S') = 90[\mathbb{P}^5] + 180[\mathbb{P}^4] + 198[\mathbb{P}^3] + 135[\mathbb{P}^2] + 63[\mathbb{P}^1] + 21[\mathbb{P}^0] \quad .$$

• Moving on to triangles, we have the map

$$T' = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{t} \overline{T}$$

which gives

$$t_* 1_{T'}(p) = \begin{cases} 1 & p \in I \\ 3 & p \in X \\ 6 & p \in S \\ 6 & p \in T \end{cases}$$

As is easily seen, $c(T\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2)$ pushes forward to

$$c_{\rm SM}(T')=90[\mathbb{P}^6]+270[\mathbb{P}^5]+378[\mathbb{P}^4]+324[\mathbb{P}^3]+189[\mathbb{P}^2]+81[\mathbb{P}^1]+27[\mathbb{P}^0]\quad.$$

• Conics union tangent lines: we consider the analogous locus $P'\subset\mathbb{P}^5\times\mathbb{P}^2$, and the obvious map $P'\xrightarrow{p}\overline{P}$. We find

$$p_*1_{P'}(p)=\begin{cases}1 & p\in I\\ 2 & p\in X\\ 3 & p\in S\\ 1 & p\in P\end{cases}$$

To compute $c_{\rm SM}(P')$, we again use [A3]. Straightforward computations give that P' is a divisor of type $(2,2)$ in $\mathbb{P}^5\times\mathbb{P}^2$, with singular scheme the locus of pairs (c,ℓ) where ℓ is a component of c. This locus is nonsingular, and isomorphic to $\mathbb{P}^2\times\mathbb{P}^2$ embedded in $\mathbb{P}^5\times\mathbb{P}^2$ by

$$(\ell_1,\ell_2)\mapsto(\ell_1\ell_2,\ell_2)\quad.$$

Proposition IV.6 from [A3] then gives

$$c_{\rm SM}(P')=\frac{(1+h)^6(1+k)^3}{1+2h+2k}\cap[P']-\frac{(1+h_1)^3(1+h_2)^3}{1+2h_1+4h_2}\cap[\mathbb{P}^2\times\mathbb{P}^2]$$

with evident notations, and explicitly (the hyperplane from $\mathbb{P}^9$ pulls back to $(h+k)$ and (h_1+2h_2)):

$$c_{\rm SM}(P')=42[\mathbb{P}^6]+132[\mathbb{P}^5]+198[\mathbb{P}^4]+186[\mathbb{P}^3]+117[\mathbb{P}^2]+51[\mathbb{P}^1]+15[\mathbb{P}^0]\quad.$$

• The set $\overline{G}$ of conics union arbitrary lines is dominated in the evident way by $\mathbb{P}^5\times\mathbb{P}^2$:

$$G'=\mathbb{P}^5\times\mathbb{P}^2\xrightarrow{g}\overline{G}$$

with

$$g_*1_{G'}(p)=\begin{cases}1 & p\in I\\ 2 & p\in X\\ 3 & p\in S\\ 3 & p\in T\\ 1 & p\in P\\ 1 & p\in G\end{cases}$$

Pushing forward $c(T\mathbb{P}^5\times\mathbb{P}^2)$ to $\mathbb{P}^9$ gives

$$c_{\rm SM}(G')=21[\mathbb{P}^7]+108[\mathbb{P}^6]+243[\mathbb{P}^5]+318[\mathbb{P}^4]+270[\mathbb{P}^3]+156[\mathbb{P}^2]+63[\mathbb{P}^1]+18[\mathbb{P}^0]\quad.$$

• Next we work in the bundle $N'=\mathcal{D}\subset\mathbb{P}^9\times\mathbb{P}^2$ introduced in §1, realizing the Nash-blow-up of $D=\overline{N}$. Note that the fiber of $\mathcal{D}$ over $p\in\mathbb{P}^2$ is the $\mathbb{P}^6$ of cubics singular at p. Cuspidal (or worse) curves determine a quadric bundle $C'\subset\mathcal{D}$ over $\mathbb{P}^2$ surjecting onto $\overline{C}$:

$$C'\xrightarrow{c}\overline{C}\quad.$$

As the fiber of the projection $\mathcal{D} \to \mathbb{P}^9$ over $Y \in \mathbb{P}^9$ consists of the singular locus of Y, we see

$$c_* 1_{C'}(p) = \begin{cases} 2 & p \in I \\ 2 & p \in X \\ 1 & p \in S \\ 1 & p \in P \\ 1 & p \in C \end{cases}$$

(2 =Euler characteristic of $\mathbb{P}^1$ =singular locus of a triple line, etc.). Now the singular subscheme of C' is easily seen to be the $\mathbb{P}^3$-bundle $\mathcal{R}$ whose fiber over p consists of cubics with a triple point (or worse) at p. Applying again Proposition IV.6 from [A3] gives

$$c_{\mathrm{SM}}(C') = \frac{c(T\mathcal{D})}{c(N_{C'}\mathcal{D})} \cap [C'] - \frac{c(T\mathcal{R})}{c(N_{C'}\mathcal{D})} \cap [\mathcal{R}]$$

(where push-forward to $\mathbb{P}^9$ is understood). We use [A4] (Lemma 1.4) to see that C' is a divisor in $\mathcal{D}$ of class $2h$, where h is the pull-back of the hyperplane from $\mathbb{P}^9$. As for $\mathcal{R}$, it is the complete intersection of six divisors of class $(h+k)$ in $\mathbb{P}^9 \times \mathbb{P}^2$ (the vanishing of the 6 second partials). Together with the computation of the class of $\mathcal{D}$ from §2 (proof of Theorem 5), we get

$$\frac{(1+h)^{10}(1+k)^3}{(1+h+2k)^3(1+2h)}[C'] - \frac{(1+h)^{10}(1+k)^3}{(1+h+k)^6(1+2h)}[\mathcal{R}]$$

and pushing down to $\mathbb{P}^9$

$$c_{\mathrm{SM}}(C') = 24[\mathbb{P}^7] + 84[\mathbb{P}^6] + 159[\mathbb{P}^5] + 222[\mathbb{P}^4] + 222[\mathbb{P}^3] + 150[\mathbb{P}^2] + 66[\mathbb{P}^1] + 18[\mathbb{P}^0] \quad .$$

- Finally, we dominate $D = \overline{N}$ with $N' = \mathcal{D}$:

$$N' = \mathcal{D} \xrightarrow{n} D = \overline{N} \quad .$$

Computing Euler characteristics we see

$$n_* 1_{\mathcal{D}}(p) = \begin{cases} 2 & p \in I \\ 2 & p \in X \\ 1 & p \in S \\ 3 & p \in T \\ 1 & p \in P \\ 2 & p \in G \\ 1 & p \in C \\ 1 & p \in N \end{cases}$$

Also, we have already computed $c_{\mathrm{SM}}(N')$ =push-forward of $c(T\mathcal{D}) \cap [\mathcal{D}]$ in general in §3, Proposition 8. For $M = \mathbb{P}^2$ and $\mathcal{L} = \mathcal{O}(3)$, this gives

$$12[\mathbb{P}^8] + 66[\mathbb{P}^7] + 171[\mathbb{P}^6] + 291[\mathbb{P}^5] + 363[\mathbb{P}^4] + 327[\mathbb{P}^3] + 201[\mathbb{P}^2] + 81[\mathbb{P}^1] + 21[\mathbb{P}^0] \quad .$$

Summarizing, we have obtained

$$\begin{pmatrix} i_*1_{I'} \\ x_*1_{X'} \\ s_*1_{S'} \\ t_*1_{T'} \\ p_*1_{P'} \\ g_*1_{G'} \\ c_*1_{C'} \\ n_*1_{N'} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 6 & 6 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 3 & 1 & 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1_I \\ 1_X \\ 1_S \\ 1_T \\ 1_P \\ 1_G \\ 1_C \\ 1_N \end{pmatrix}$$

from which, inverting the matrix and applying c_*:

$$\begin{pmatrix} c_*(1_I) \\ c_*(1_X) \\ c_*(1_S) \\ c_*(1_T) \\ c_*(1_P) \\ c_*(1_G) \\ c_*(1_C) \\ c_*(1_N) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -1 & 1 & 0 & 0 \\ -\frac{1}{3} & -1 & \frac{1}{3} & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 2 & -2 & -1 & 1 \end{pmatrix} \begin{pmatrix} c_{\mathrm{SM}}(I') \\ c_{\mathrm{SM}}(X') \\ c_{\mathrm{SM}}(S') \\ c_{\mathrm{SM}}(T') \\ c_{\mathrm{SM}}(P') \\ c_{\mathrm{SM}}(G') \\ c_{\mathrm{SM}}(C') \\ c_{\mathrm{SM}}(N') \end{pmatrix},$$

which gives the statement. $\square$

The data collected in Theorem 11 can now be assembled to compute any characteristic class of the strata. For example, by Theorem 3 (using for example Remark (2) following Corollary 4 in §2) we have

$$\mathrm{Eu}_D(p) = \begin{cases} 2 & p \in I \\ 3 & p \in X \\ 4 & p \in S \\ 3 & p \in T \\ 3 & p \in P \\ 2 & p \in G \\ 2 & p \in C \\ 1 & p \in N \end{cases}$$

from which

$$c_M(D) = 2c_*(I) + 3c_*(X) + 4c_*(S) + 3c_*(T) + 3c_*(P) + 2c_*(G) + 2c_*(C) + c_*(N)$$

with the same result as listed in the beginning of the section.

Applying to

$$1_D = 1_I + 1_X + 1_S + 1_T + 1_P + 1_G + 1_C + 1_N$$

we obtain

Corollary 12. *The push-forward to $\mathbb{P}^9$ of the Schwartz-MacPherson class of the discriminant of plane cubics is*

$$12[\mathbb{P}^8]+45[\mathbb{P}^7]+120[\mathbb{P}^6]+210[\mathbb{P}^5]+252[\mathbb{P}^4]+210[\mathbb{P}^3]+120[\mathbb{P}^2]+45[\mathbb{P}^1]+10[\mathbb{P}^0]$$

The remarkable near-symmetry of this expression is inherited from the total Chern class of $\mathbb{P}^9$:

$$c(T\mathbb{P}^9)\cap[\mathbb{P}^9]=[\mathbb{P}^9]+10[\mathbb{P}^8]+45[\mathbb{P}^7]+120[\mathbb{P}^6]+210[\mathbb{P}^5]\\+252[\mathbb{P}^4]+210[\mathbb{P}^3]+120[\mathbb{P}^2]+45[\mathbb{P}^1]+10[\mathbb{P}^0]$$

from which we also obtain a class for the locus $\mathbb{P}^9-D$ of *nonsingular* cubics:

$$c_*(1_{\mathbb{P}^9-D})=[\mathbb{P}^9]-2[\mathbb{P}^8]$$

We do not have a conceptual explanation for this surprisingly simple expression (although we again feel that the action of PGL(3) must be responsible for it).

Finally, the knowledge of $c_{\mathrm{SM}}(D)$ allows us to compute the (push-forward of the) Segre class of the singularity subscheme JD of D:

Proposition 13.

$$s(JD,\mathbb{P}^9)=69[\mathbb{P}^7]-1086[\mathbb{P}^6]+12093[\mathbb{P}^5]-108660[\mathbb{P}^4]\\+750015[\mathbb{P}^3]-2369910[\mathbb{P}^2]-40989270[\mathbb{P}^1]+1143250160[\mathbb{P}^0]\quad.$$

Proof. This is obtained by using the result of [A3], relating Chern-Schwartz-MacPherson's class, obtained above, to the class of the virtual tangent bundle of D:

$$c_F(D)=(1+h)^{10}\frac{[D]}{(1+12h)}=12[\mathbb{P}^8]-24[\mathbb{P}^7]+828[\mathbb{P}^6]-8496[\mathbb{P}^5]+104472[\mathbb{P}^4]\\-1250640[\mathbb{P}^3]+15010200[\mathbb{P}^2]-180120960[\mathbb{P}^1]+2161452060[\mathbb{P}^0]$$

as a class in $\mathbb{P}^9$. Theorem I.5 in [A3] gives (with notation as in [A3])

$$c_{\mathrm{SM}}(D)=c_F(D)+c(\mathcal{O}(12))^8\cap(\mu_{\mathcal{O}(12)}(JD)^\vee\otimes_{\mathbb{P}^9}\mathcal{O}(12))$$

from which one can then obtain

$$\mu_{\mathcal{O}(12)}(JD)=69[\mathbb{P}^7]+5676[\mathbb{P}^6]+200226[\mathbb{P}^5]+3926268[\mathbb{P}^4]\\+46220754[\mathbb{P}^3]+326651280[\mathbb{P}^2]+1283190093[\mathbb{P}^1]+2161452050[\mathbb{P}^0]$$

As $\mu_{\mathcal{O}(12)}(JD)=c(T^*\mathbb{P}^9\otimes\mathcal{O}(12))\cap s(JD,\mathbb{P}^9)$, this information is enough to obtain $s(JD,\mathbb{P}^9)$, with the stated result. □

For example, using [A-C] and Theorem 3 in §2 we can now compute the multiplicity m_3 and Euler obstruction e_3 at the discriminant D of plane cubics in the discriminant hypersurface of $\mathcal{O}(12)$ on $\mathbb{P}^9$:

Corollary 14.

$$m_3=17559733166$$
$$e_3=2161452050$$

In fact it is not hard to give a compact formula for the Euler obstruction e_d at the discriminant of degree-d plane curves of the corresponding discriminant ($d\geq 2$):

$$e_d=\frac{1}{3(d-1)^2}\left((3(d-1)^2-1)^{\binom{d+2}{2}}-(-1)^{\binom{d+2}{2}}\right)$$

(we will leave this as an exercise for the entertainment of the interested reader); we know of no such formula for the multiplicity m_d.

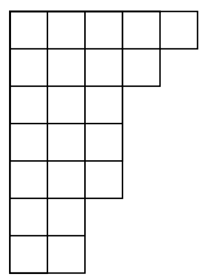
§5. Discriminant of $\mathcal{O}(d)$ on $\mathbb{P}^1$

In this section we compute Schwartz-MacPherson's class of strata of the discriminant of $\mathcal{O}(d)$ on $\mathbb{P}^1$. We picture $\mathbb{P}^d = H^0(\mathbb{P}^1, \mathcal{O}(d))$ as the space of unordered d-tuples of points in $\mathbb{P}^1$, and the discriminant consists then of *nonreduced* d-tuples. We consider the stratification of the discriminant by the loci in $\mathbb{P}^d$ parametrizing d-tuples of a given combinatorial type. More precisely, for every partition $m = \{m_1, \dots, m_s\}$ of d

$$d = m_1 + \cdots + m_s$$

into nonincreasing summands $m_1 \geq m_2 \geq \cdots \geq m_s$, we compute the image in $A_*\mathbb{P}^d$ (via MacPherson's transformation) of the characteristic function of the locus of d-tuples $m_1 p_1 + \cdots + m_s p_s$ with distinct p_i's.

The main ingredients in the answer are the elementary symmetric functions $s_j(m) = s_j(m_1, \dots, m_s)$, and the order $\operatorname{aut} m$ of m, by which we mean the number of row-shufflings preserving the corresponding Young diagram. For example, for $m = \{5, 4, 3, 3, 3, 2, 2\}$:



we have $\operatorname{aut} m = 1!\,1!\,3!\,2! = 12$.

With these notations, the answer is:

Proposition 15. *Schwartz-MacPherson's class of the locus of d-tuples $m_1 p_1 + \cdots + m_s p_s$ with distinct p_i is*

$$\frac{1}{\operatorname{aut} m} \sum_{k=0}^{s} \binom{s-3}{k} (-1)^k k!\,(s-k)!\, s_{s-k}(m)\, [\mathbb{P}^{s-k}]$$

(We use the convention $\binom{-a}{b} = (-1)^b \binom{a+b-1}{b}$.)

Proposition 15 assigns a class to each Young diagram with s rows. This class has no components in dimension < 3 for $s \geq 3$; for $s = 3$ it consists of one component:

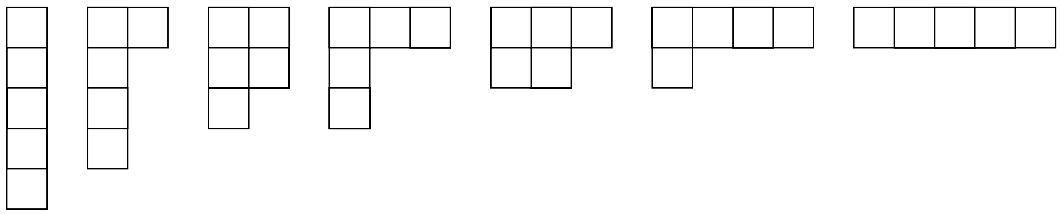
$$\frac{6}{\operatorname{aut} m} m_1 m_2 m_3\, [\mathbb{P}^3] \quad ;$$

for $s \leq 2$ it equals

$$\begin{cases} 2m_1 m_2\, [\mathbb{P}^2] + (m_1 + m_2)\, [\mathbb{P}^1] + 2\, [\mathbb{P}^0] & m_1 \neq m_2 \\ m^2\, [\mathbb{P}^2] + m\, [\mathbb{P}^1] + [\mathbb{P}^0] & m_1 = m_2 = m \end{cases}$$

The good properties of Schwartz-MacPherson's classes endow the classes of Proposition 15 of many interesting features. For example, the sum of the classes over all Young diagrams with d boxes must equal the class of the tangent bundle of $\mathbb{P}^d$, since the corresponding characteristic functions add up to the constant $1_{\mathbb{P}^d}$ on $\mathbb{P}^d$.

Example. For $d = 5$ we have the diagrams



which by Proposition 15 correspond respectively to

$$[\mathbb{P}^5] - 2[\mathbb{P}^4] + [\mathbb{P}^3] \quad , \quad 8[\mathbb{P}^4] - 7[\mathbb{P}^3] \quad , \quad 12[\mathbb{P}^3] \quad , \quad 9[\mathbb{P}^3]$$
$$12[\mathbb{P}^2] + 5[\mathbb{P}^1] + 2[\mathbb{P}^0] \quad , \quad 8[\mathbb{P}^2] + 5[\mathbb{P}^1] + 2[\mathbb{P}^0] \quad , \quad 5[\mathbb{P}^1] + 2[\mathbb{P}^0]$$

Adding up these classes gives

$$[\mathbb{P}^5] + 6[\mathbb{P}^4] + 15[\mathbb{P}^3] + 20[\mathbb{P}^2] + 15[\mathbb{P}^1] + 6[\mathbb{P}^0] = (1+h)^6 \cdot [\mathbb{P}^5] = c(T\mathbb{P}^5) \cap [\mathbb{P}^5]$$

Other characteristic classes of the discriminant can be obtained either directly, or by pasting together a suitable combination of the classes obtained in Proposition 15. For example, according to Theorem 5 Mather's Chern class must be

$$\left((2d-2)(1+h)^d - d\,h\,(1+h)^{d-1}\right) \cdot [\mathbb{P}^{d-1}] \quad .$$

This class must be the image via MacPherson's map of the Euler obstruction; for a d-tuple with partition $m = m_1 \geq \cdots \geq m_s$ as above, this is easily computed to be

$$\sum (m_i - 1) \quad ,$$

that is, the number of boxes in the diagram obtained by removing the rightmost box from each row of the Young diagram of m. Adding up Schwartz-MacPherson's classes (as computed in Proposition 15) weighted according to the Euler obstruction must give Mather's class. For example, for $d = 5$ we have the above Young diagrams, with local Euler obstructions respectively

$$0 \quad , \quad 1 \quad , \quad 2 \quad , \quad 2 \quad , \quad 3 \quad , \quad 3 \quad , \quad 4$$

and adding up the corresponding weighted MacPherson's classes (also listed above) gives

$$8\,[\mathbb{P}^4] + 35\,[\mathbb{P}^3] + 60\,[\mathbb{P}^2] + 50\,[\mathbb{P}^1] + 20\,[\mathbb{P}^0]$$

agreeing with the direct computation. Can Proposition 15 (or Lemma 16 below) be proved 'geometrically' (by providing enough direct computations of characteristic classes)?

Proof of Proposition 15. The proposition follows from the analogous result for *ordered* s-tuples of points on $\mathbb{P}^1$. Consider the space

$$\underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{s}$$

and the open subset parametrizing s-tuples of distinct points. Let h_i denote the pull-back of the hyperplane from the i-th factor (and note $h_i^2 = 0$ for all i).

Lemma 16. *The image in $A_*(\mathbb{P}^1\times\cdots\times\mathbb{P}^1)$ of the characteristic function of the set set of ordered s-tuples of distinct points is*

$$(1-h_1-\cdots-h_s)^{s-3}\cdot[\mathbb{P}^1\times\cdots\times\mathbb{P}^1]$$

Granting the Lemma for a moment, the proposition follows: for a partition m as above, consider the map

$$\begin{aligned}\mathbb{P}^1\times\cdots\times\mathbb{P}^1 &\xrightarrow{\rho}\mathbb{P}^d\\ (p_1,\dots,p_s)&\mapsto m_1p_1+\cdots+m_sp_s\end{aligned}$$

surjecting onto the closure of the corresponding stratum. It is readily understood that $\operatorname{aut} m$ equals the degree of ρ; this is the exact number of preimages over $m_1p_1+\cdots+m_sp_s$ with distinct p_i's, so Schwartz-MacPherson's class of the stratum is given by

$$(*)\quad \frac{1}{\operatorname{aut} m}\rho_*(\text{MacPherson's class of the set of } d\text{-tuples of distinct points})$$
$$=\frac{1}{\operatorname{aut} m}\rho_*((1-h_1-\cdots-h_s)^{s-3}\cdot[\mathbb{P}^1\times\cdots\times\mathbb{P}^1])$$

by the Lemma. Now denote by h the hyperplane in $\mathbb{P}^d$, and note that

$$\rho^*h=m_1\,h_1+\cdots+m_s\,h_s\quad .$$

By the projection formula, and using that $h_i^2=0$,

$$\begin{aligned}&h^{s-k}\rho_*((h_1+\cdots+h_s)^k\cdot[\mathbb{P}^1\times\cdots\times\mathbb{P}^1])\\ &\quad=\rho_*((m_1h_1+\cdots+m_sh_s)^{s-k}(h_1+\cdots+h_s)^k\cdot[\mathbb{P}^1\times\cdots\times\mathbb{P}^1])\\ &\quad=\rho_*((s-k)!s_{s-k}(m_1,\dots,m_s)k!\,h_1\cdots h_s\,\cdot[\mathbb{P}^1\times\cdots\times\mathbb{P}^1])\\ &\quad=(s-k)!\,k!\,s_{s-k}(m)\,[\mathrm{pt}]\end{aligned}$$

hence

$$\rho_*((h_1+\cdots+h_s)^k\cdot[\mathbb{P}^1\times\cdots\times\mathbb{P}^1])=(s-k)!\,k!\,s_{s-k}(m)\,[\mathbb{P}^{s-k}]\quad .$$

The formula in the proposition follows from applying this to the obvious expansion of $(*)$. $\square$

Our proof of the Lemma is purely combinatorial:

Proof of Lemma 16. The product $\mathbb{P}^1\times\cdots\times\mathbb{P}^1$ is the disjoint union of the set under scrutiny and the 'diagonals' parametrizing s-tuples $\{p_1,\dots,p_s\}$ for which certain subsets of the p_i's coincide. There is one such diagonal for each partition

$$\mathcal{P}=\{A_1,\dots,A_k\}$$

of the set $\{1,\dots,s\}$, and the set of s-tuples of distinct points corresponds to the longest partition $\{\{1\},\dots,\{s\}\}$. Denoting by $c_{\mathrm{SM}}(\mathcal{P})$ Schwartz-MacPherson's class of the characteristic function of the subset corresponding to $\mathcal{P}$, we observe that as $1_{\mathbb{P}^1\times\cdots\times\mathbb{P}^1}$ is the sum of all such characteristic functions, we must have

$$\sum_{\mathcal{P}} c_{\mathrm{SM}}(\mathcal{P}) = c_{\mathrm{SM}}(\mathbb{P}^1\times\cdots\times\mathbb{P}^1) = \prod_{i=1}^{s}(1+h_i)^2\cdot[\mathbb{P}^1\times\cdots\times\mathbb{P}^1]$$

Now the key observation is that each diagonal is itself a set of n-tuples of distinct points, so its c_{SM} is recursively known. Hence, the last display gives a recursive relation of which we claim the formula given in the lemma is the solution.

More precisely, denote by $[\mathcal{P}]$ the class of the diagonal corresponding to $\mathcal{P} = \{A_1,\dots,A_k\}$. Each A_i lists a set of factors agreeing in the diagonal, so the (closure of the) set corresponding to $\mathcal{P}$ is the image of the map

$$\underbrace{\mathbb{P}^1\times\cdots\times\mathbb{P}^1}_{k} \to \underbrace{\mathbb{P}^1\times\cdots\times\mathbb{P}^1}_{s}$$

sending the i-th factor on the left to the small diagonal in ${\mathbb{P}^1}^{A_i}$. Observe that any h_{a_i} for $a_i\in A_i$ restricts to the hyperplane in the i-th factor, so according to the formula proposed in the Lemma

$$c_{\mathrm{SM}}(\mathcal{P}) = C(\mathcal{P})\cdot[\mathcal{P}]$$

where

$$C(\mathcal{P}) := (1-h^{(1)}-\cdots-h^{(k)})^{k-3}$$

and where $h^{(i)} = h_{a_i}$ for any $a_i\in A_i$. Also, a moment's thought shows that the class of the small diagonal in ${\mathbb{P}^1}^{A_i}$ is

$$\frac{1}{(|A_i|-1)!}\,(\sum_{a\in A_i} h_a)^{|A_i|-1}\cdot[{\mathbb{P}^1}^{A_i}]$$

where $|A_i|$ denotes the cardinality of A_i. Indeed (keeping in mind $h_i^2=0$) this equals the 'penultimate' elementary symmetric function in the h_a's, which is the class dotting to 1 against each h_a. The class $[\mathcal{P}]$ is therefore the product of all such functions as A_i ranges in $\mathcal{P}$:

$$[\mathcal{P}] = S(\mathcal{P})\cdot[\mathbb{P}^1\times\cdots\times\mathbb{P}^1]$$

where

$$S(\mathcal{P}) := \prod_{A_i\in\mathcal{P}}\frac{1}{(|A_i|-1)!}\,(\sum_{a\in A_i} h_a)^{|A_i|-1} \quad .$$

Summarizing, in order to prove the Lemma we have to show that

$$\sum_{\mathcal{P}} S(\mathcal{P})\,C(\mathcal{P}) = \prod_{i=1}^{s}(1-h_i)^2$$

where the summation ranges over all partitions $\mathcal{P}$ of $\{1,\dots,s\}$.

Of course this is shown by induction on s. For $s=1$ we have the single partitions $\mathcal{P}=\{\{1\}\}$, for which

$$S(\mathcal{P})=1 \quad , \quad C(\mathcal{P})=(1-h_1)^{1-3}=1+2h_1=(1+h_1)^2$$

(once more, since $h_i^2=0$), as needed.

To establish the induction step, observe that removing h_s from a partition of $\{1,\dots,s\}$ determines uniquely a partition of $\{1,\dots,s-1\}$; that is, a partition

$$\mathcal{P}=\{A_1,\dots,A_k\}$$

of $\{1,\dots,s-1\}$ spawns the $(k+1)$ partitions of $\{1,\dots,s\}$:

$$\begin{aligned}
&\mathcal{P}'_0=\{A_1,\dots,A_k,\{h_s\}\}\\
&\mathcal{P}'_i=\{A_1,\dots,A'_i,\dots,A_k\} \qquad i=1,\dots,k
\end{aligned}$$

where $A'_i=A_i\cup\{h_s\}$. Now observe that $S(\mathcal{P}'_0)=S(\mathcal{P})$ (since the penultimate elementary symmetric function of $\{h_s\}$ is 1), while

$$\begin{aligned}
C(\mathcal{P}'_0)&=(1-h^{(1)}-\dots-h^{(k)}-h_s)^{(k+1)-3}\\
&=(1-h^{(1)}-\dots-h^{(k)})^{k-2}-(k-2)(1-h^{(1)}-\dots-h^{(k)})^{k-3}\,h_s\\
&=(1-h^{(1)}-\dots-h^{(k)}-(k-2)\,h_s)\,C(\mathcal{P}) \quad .
\end{aligned}$$

On the other hand, clearly $C(\mathcal{P}'_i)=C(\mathcal{P})$, and the contributions to $S(\mathcal{P}'_i)$ due to the A_j, $j\neq i$, equal the corresponding contributions to $S(\mathcal{P})$; for $j=i$, write

$$A_i=\{h_{a_1},\dots,h_{a_r}\} \quad ;$$

then the contribution of A'_i to $S(\mathcal{P}'_i)$ equals

$$\begin{aligned}
&\frac{1}{r!}(h_{a_1}+\dots+h_{a_r}+h_s)^r\\
&\qquad=\frac{1}{r!}(h_{a_1}+\dots+h_{a_r})^r+\frac{1}{(r-1)!}(h_{a_1}+\dots+h_{a_r})^{r-1}\,h_s\\
&\qquad=\left(\frac{1}{r}(h_{a_1}+\dots+h_{a_r})+h_s\right)\,\frac{1}{(r-1)!}(h_{a_1}+\dots+h_{a_r})^{r-1}
\end{aligned}$$

and, further, each h_a ($a\in A_i$) hits $C(\mathcal{P}')$ as $h^{(i)}$; so this shows

$$S(\mathcal{P}'_i)=(h^{(i)}+h_s)\,S(\mathcal{P})$$

(after restriction to the diagonal). Putting everything together,

$$\begin{aligned}
\sum_{i\geq 0}S(\mathcal{P}'_i)C(\mathcal{P}'_i)&=(1-h^{(1)}-\dots-h^{(k)}-(k-2)\,h_s+\sum_i(h^{(i)}+h_s))\,S(\mathcal{P})C(\mathcal{P})\\
&=(1+2h_s)\,S(\mathcal{P})C(\mathcal{P})\\
&=(1+h_s)^2\,S(\mathcal{P})C(\mathcal{P})
\end{aligned}$$

Adding over all partitions, we see

$$\sum_{\mathcal{P}' \text{ partition of } \{1,\dots,s\}} S(\mathcal{P}')\, C(\mathcal{P}') = \left(\sum_{\mathcal{P} \text{ partition of } \{1,\dots,s-1\}} S(\mathcal{P})\, C(\mathcal{P}) \right) (1+h_s)^2$$

and we are done, since by induction $\sum_{\mathcal{P}} S(\mathcal{P})\, C(\mathcal{P}) = \prod_{i=1}^{s-1}(1+h_i)^2$. $\square$

As an important special case of the proposition, the complement of the discriminant corresponds to the partition $1^d = \{1, 1, \dots, 1\}$, so we have $s = d$,

$$\text{aut } 1^d = d! \quad , \quad s_{s-k}(1^d) = \binom{d}{s-k}$$

and the class simplifies to

$$\sum_{k=0}^{d} \binom{d-3}{k} (-1)^k\, [\mathbb{P}^{d-k}] = \sum_{k \geq 0} \binom{d-3}{k} (-1)^k\, h^k [\mathbb{P}^d] = (1-h)^{d-3} [\mathbb{P}^d]$$

where h is the hyperplane in $[\mathbb{P}^d]$. We feel that there should be an explanation of the shape of this class in terms of the obvious $\mathrm{PGL}(2)$ action.

As an immediate consequence of this formula:

COROLLARY 17. *Schwartz-MacPherson's class of the discriminant of $\mathcal{O}(d)$ on $\mathbb{P}^1$ is*

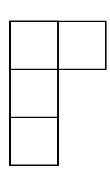
$$\left((1+h)^{d+1} - (1-h)^{d-3}\right) \cdot [\mathbb{P}^d] = \sum_{j \geq 1} \left(\binom{d+1}{j} - (-1)^j \binom{d-3}{j} \right) [\mathbb{P}^{n-j}]$$

We do not know of a similarly compact formula for Schwartz-MacPherson's class of *the closure* of an arbitrary stratum of the discriminant; the case of Corollary 17, that is the whole discriminant, corresponds to the closure of the stratum corresponding to the partition $\{2, 1, \dots, 1\}$.

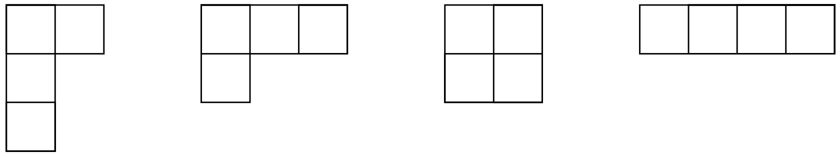
A formula can be given for the class obtained by weighing strata according roughly to the row automorphisms of the corresponding Young diagrams. We say that a Young diagram $\mathcal{P}'$ is a 'degeneration' of a diagram $\mathcal{P}$, and write $\mathcal{P}' \leq \mathcal{P}$, if $\mathcal{P}'$ can be obtained from $\mathcal{P}$ by collapsing several rows into one; that is, if the stratum corresponding to $\mathcal{P}'$ is in the closure of the stratum corresponding to $\mathcal{P}$. For diagrams $\mathcal{P}$, $\mathcal{P}'$ we define a weight

$$w(\mathcal{P}, \mathcal{P}')$$

as follows: $w(\mathcal{P}, \mathcal{P}')$ equals the number of row automorphisms of $\mathcal{P}'$ (that is, $\text{aut}\, m'$ for the corresponding partition m') times the number of ways $\mathcal{P}'$ can be obtained as a degeneration of $\mathcal{P}$. For example, the diagram $\mathcal{P}$:



has degenerations $\mathcal{P}'$



with weights respectively

$$2 \quad , \quad 2 \quad , \quad 2 \quad , \quad 1$$

Also, denote by $c_{\mathrm{SM}}(\mathcal{P})$ Schwartz-MacPherson's class of the stratum corresponding to $\mathcal{P}$, as computed in Proposition 15. Schwartz-MacPherson's class of the closure of the locus of d-tuples with combinatorial type $\mathcal{P}$ would be

$$\sum_{\mathcal{P}'\le\mathcal{P}} c_{\mathrm{SM}}(\mathcal{P}') \quad ;$$

we do not have a simple expression for this in general. However:

PROPOSITION 18. *Assume $\mathcal{P}$ corresponds to the partition m. Then*

$$\sum_{\mathcal{P}'\le\mathcal{P}} w(\mathcal{P},\mathcal{P}')\, c_{\mathrm{SM}}(\mathcal{P}') = \sum_{k=0}^{s} 2^{s-k} k!\, s_k(m_1,\dots,m_s)\,[\mathbb{P}^k]$$

PROOF. As in the proof of Proposition 15, consider the natural map

$$\mathbb{P}^1\times\cdots\times\mathbb{P}^1\to\mathbb{P}^d$$

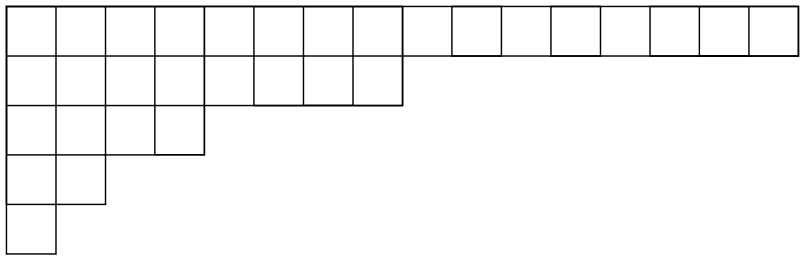
sending $(p_1,\dots,p_s)$ to $m_1p_1+\cdots+m_sp_s$. The weight $w(\mathcal{P},\mathcal{P}')$ computes the number of preimages of a given d-tuple, so the properties of Schwartz-MacPherson's class imply that

$$\sum_{\mathcal{P}'\le\mathcal{P}} w(\mathcal{P},\mathcal{P}')\, c_{\mathrm{SM}}(\mathcal{P}')$$

equals the push-forward of the total Chern class of the tangent bundle to $\mathbb{P}^1\times\cdots\times\mathbb{P}^1$. This is easily evaluated and gives the right-hand-side. $\square$

The degree of the class given in Proposition 18 computes the 'weighted' Euler characteristic; this equals 2^s for the closure of a stratum of dimension s. The actual Euler characteristic of the closure of a stratum, that is, the degree of its Schwartz-MacPherson's class, is more elusive. However, by the remarks following the statement of Proposition 15 we see that there is no contribution to the Euler characteristic from strata of dimension ≥ 3. It follows in fact that the Euler characteristic of the closure of the stratum corresponding to a diagram $\mathcal{P}$ equals *twice the number of degenerations of $\mathcal{P}$ with ≤ 2 rows, minus 1 if $\mathcal{P}$ degenerates to $\{\frac{d}{2},\frac{d}{2}\}$*. For example, $\{2,1,1\}$ degenerates to $\{3,1\}$, $\{2,2\}$, and $\{4\}$, hence the Euler characteristic of the closure of the corresponding stratum is 5.

Proposition 18 computes the actual Schwartz-MacPherson's class of the closure of the stratum when all substrata are counted with weight 1. This is the case when all sums of subsets of $\{m_1, \dots, m_s\}$ yield different numbers; for example, Schwartz-MacPherson's class of the closure of the stratum corresponding to



is computed by the right-hand side of Proposition 18:

$$\sum_{k=0}^{5} 2^{s-k} k!\, s_k(16, 8, 4, 2, 1)\, [\mathbb{P}^k]$$
$$= 122880[\mathbb{P}^5] + 95232[\mathbb{P}^4] + 29760[\mathbb{P}^3] + 4960[\mathbb{P}^2] + 496[\mathbb{P}^1] + 32[\mathbb{P}^0] \quad .$$

As a final remark, observe that the explicit form of Schwartz-MacPherson's class of the discriminant given in Corollary 17 allows us to obtain the Segre class of its singularity subscheme. As often is the case, the μ-class (in the sense of [A1]) looks a little nicer: denoting as usual by h the hyperplane class in $\mathbb{P}^d$, the μ-class equals (again by applying [A3], I.5)

$$\left(\frac{(1 + (2d-3)h)^{d+1}}{(1 + (2d-2)h)^2} - (1 + (2d-2)h)^2 (1 + (2d-1)h)^{d-3} + (1 + (2d-2)h)^{d-1} (2d-2)h \right) \cdot [\mathbb{P}^d]$$

For example, the degree of this class (that is, Parusiǹski's number of the discriminant) turns out to be

$$(2d-2)^d + (-1)^d (d+1) \qquad (d \ge 3)$$

This is the local Euler obstruction of the discriminant, seen as a point in the discriminant of the linear system it determines on $\mathbb{P}^d$, cf. Corollary 4. This information is equivalent to the statement that the Euler characteristic of the discriminant of $\mathcal{O}(d)$ on $\mathbb{P}^1$ is $(d+1)$, that is, it equals the Euler characteristic of $\mathbb{P}^d$; this also follows from observing that the complement of the discriminant is a union of three-dimensional PGL(2) orbits, hence its Euler characteristic must vanish. This simple argument however does not suffice to compute more sophisticated information, such as the *multiplicity* of the discriminant in its discriminant. This can be obtained from the above expression for the μ class, using [A-C] (cf. [A1], §2.2):

$$\frac{(2d-3)^{d+1} + (-1)^d}{2d-2} + d(2d-2)^d - (2d-2)^3 (2d-1)^{d-3} \qquad (d \ge 3) \quad .$$

For example ($d = 3$), the discriminant of triples of points on $\mathbb{P}^1$, a quartic surface singular along a twisted cubic, is a point of multiplicity 148 on the discriminant of $\mathcal{O}(4)$ on $\mathbb{P}^3$.

References

[A1] P. Aluffi, *Singular schemes of hypersurfaces*, Duke Math. J. **80** (1995), 325–351.

[A2] P. Aluffi, *MacPherson's and Fulton's Chern Classes of Hypersurfaces*, I.M.R.N. (1994), 455–465.

[A3] P. Aluffi, *Chern classes for singular hypersurfaces*, Trans. Amer. Math. Soc. **351** (1999), 3989–4026.

[A4] P. Aluffi, *Some characteristic numbers for nodal and cuspidal plane curves of any degree*, Manu. Math. **72** (1991), 425–444.

[A-C] P. Aluffi, F. Cukierman, *Multiplicities of discriminants*, Manu. Math. **78** (1993), 245–258.

[D-L] I. Dolgachev, A. Libgober, *On the fundamental group of the complement to a discriminant variety*, Lecture Notes in Math. **862**, Springer, 1–25.

[DF-I] P. Di Francesco, C. Itzykson, *Quantum intersection rings*, The moduli space of curves, (Dijkgraaf, Faber, van der Geer), Birkhäuser, 1995, pp. 81–148.

[E] L. Ernström, *Topological Radon transforms and the local Euler obstruction*, Duke Math. J. **76** (1994), 1–21.

[MP] R. MacPherson, *Chern classes for singular algebraic varieties*, Annals of Math. **100** (1974), 423–432.

[P1] A. Parusiǹski, *A generalization of the Milnor Number*, Math. Ann. **281** (1988), 247–254.

[P2] A. Parusiǹski, *Multiplicity of the dual variety*, Bull. of the London Math. Soc. **23** (1991), 429–436.

[R] J. Roberts, *A stratification of the dual variety*, Preprint, Univ. of Minnesota (1976).

[V1] I. Vainsencher, *The degrees of certain strata of the dual variety*, Compositio Math. **38** (1979), 241–252.

[V2] I. Vainsencher, *Generic singularities of hyperplane sections of a surface in* $\mathbb{P}^4$, Conference in Singularities and Topology, S. Carlos 1996.