DIFFERENTIAL FORMS WITH LOGARITHMIC POLES AND CHERN-SCHWARTZ-MACPHERSON CLASSES OF SINGULAR VARIETIES

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Abstract. We express the Chern-Schwartz-MacPherson class of a possibly singular variety in terms of the total Chern class of a bundle of differential forms with logarithmic poles. As an application, we obtain a formula for the Chern-Schwartz-MacPherson class of a hypersurface of a nonsingular variety, in terms of the Chern-Mather class of a suitable sheaf.

§1. Introduction and statement of the result

In relation with the question of the existence of a *canonical* lift of the Chern-Schwartz-MacPherson homology classes of a singular variety to intersection homology (with rational coefficients), Jean-Paul Brasselet has asked whether it is possible to compute these classes by means of differential forms. The main aim of this short note is to propose an answer to Brasselet's question. The result is stated in this §1, and proved in §2. An application of this result is given in §3, where we compute the Chern-Schwartz-MacPherson class of a *hypersurface* of a nonsingular variety in terms of the Chern-Mather class of a certain sheaf.

Let X be a (possibly singular) algebraic variety over an algebraically closed field of characteristic 0. There is a notion of characteristic class of X, agreeing with the total Chern class of the tangent bundle of X when X is nonsingular, and satisfying good functoriality properties. This class was introduced in homology by Robert MacPherson for complex varieties ([9]), and was shown to agree with the Alexander dual of the class introduced ten years earlier by Marie-Hélène Schwartz (see [13], [4]). Gary Kennedy extended the definition to varieties over arbitrary algebraically closed fields of characteristic 0 ([7]), after ideas of Claude Sabbah ([12]); in this context, which we will assume here, the class lives in the Chow group A_*X of X. We will denote by $c_{\mathrm{SM}}(X)$ the Chern-Schwartz-MacPherson class of X.

The functoriality properties of the class amount to the existence of a natural transformation c_* from the functor of constructible functions to Chow group (or homology), such that $c_{\mathrm{SM}}(X)$ is the image of the constant function 1_X by the homomorphism induced by c_*; abusing notations,

$$c_{\mathrm{SM}}(X) = c_*(1_X) \in A_*X \quad .$$

The push-forward of a constructible function by a proper map is defined by taking Euler characteristics of fibers.

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Now assume that X is embedded as a closed subvariety of a nonsingular variety M, by $i : X \to M$. We are interested in the image $i_* c_{\rm SM}(X)$ of the Chern-Schwartz-MacPherson class of X in A_*M. Also, recall that if X' is a (reduced) divisor with smooth components and normal crossings in a nonsingular variety $\widetilde{M}$ we have a sheaf $\Omega^1_{\widetilde{M}}(\log X')$ of *differential forms with logarithmic poles along* X'. This is a locally free sheaf, of rank equal to the dimension of $\widetilde{M}$. The main result of this note is the following:

Theorem 1. *Let $i : X \to M$ be as above. Let $\pi : \widetilde{M} \to M$ be a proper birational map, with $\widetilde{M}$ a nonsingular variety, such that $X' = (\pi^{-1}(X))_{\rm red}$ is a divisor with smooth components and normal crossings in $\widetilde{M}$, and $\pi_{|\widetilde{M}-X'}$ is an isomorphism. Then*

$$i_* c_{\rm SM}(X) = c(TM) \cap [M] - \pi_* \left(c(\Omega^1_{\widetilde{M}}(\log X')^\vee) \cap [\widetilde{M}] \right) \in A_*M \quad .$$

The proof of this result is given in the next section. We remark that embedded resolution of singularities in characteristic 0 guarantees that a variety $\widetilde{M}$ as specified in the statement of the theorem always exists.

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§2. Proof of Theorem 1

We first note (cf. for example [15], §3) that there is an exact sequence of sheaves on $\widetilde{M}$:

$$0 \to \Omega^1_{\widetilde{M}} \to \Omega^1_{\widetilde{M}}(\log X') \to \oplus \mathcal{O}_{X_i} \to 0$$

where X_i, $1 = 1, \dots, r$ are the components of X', and the map $\Omega^1_{\widetilde{M}}(\log X') \to \oplus \mathcal{O}_{X_i}$ is defined by taking residues. Therefore

$$c(\Omega^1_{\widetilde{M}}(\log X')) = c(\Omega^1_{\widetilde{M}}) \cdot \prod c(\mathcal{O}_{X_i}) = \frac{c(\Omega^1_{\widetilde{M}})}{(1-X_1)\cdots(1-X_r)}$$

$$\text{and hence} \qquad c(\Omega^1_{\widetilde{M}}(\log X')^\vee) = \frac{c(T\widetilde{M})}{(1+X_1)\cdots(1+X_r)} \quad .$$

Next, denote by j the inclusion $X' \subset \widetilde{M}$; then we claim that

$$j_* c_{\rm SM}(X') = c(T\widetilde{M}) \left(1 - \frac{1}{(1+X_1)\cdots(1+X_r)} \right) \cap [\widetilde{M}] \quad .$$

To see this, one may argue by induction on the number r of components of the divisor with normal crossings X': for $r = 1$,

$$c(T\widetilde{M}) \left(1 - \frac{1}{(1+X_1)} \right) \cap [\widetilde{M}] = j_* \frac{c(T\widetilde{M})}{(1+X_1)} \cap [X_1] = j_* c(TX_1) \cap [X_1] = j_* c_{\rm SM}(X_1);$$

and the equality for general r follows since both sides satisfy 'inclusion-exclusion'.

Combining the two ingredients shows that

$$c(\Omega^1_{\widetilde{M}}(\log X')^\vee) \cap [\widetilde{M}] = c(T\widetilde{M}) \cap [\widetilde{M}] - j_* c_{\rm SM}(X') = c_*(1_{\widetilde{M}-X'}) \quad .$$

Now applying the functoriality of MacPherson's classes yields the statement of the theorem:

$$\begin{aligned} \pi_* \left(c(\Omega^1_{\widetilde{M}}(\log X')^\vee) \cap [\widetilde{M}] \right) &= \pi_* c_*(1_{\widetilde{M}-X'}) = c_* \pi_*(1_{\widetilde{M}-X'}) = c_*(1_{M-X}) \\ &= c(TM) \cap [M] - i_* c_{\rm SM}(X) \end{aligned}$$

as needed. □

§3. Variations on the theme

If X is a *hypersurface* in M, the theorem in §1 can be used to obtain an alternative description of the Chern-Schwartz-MacPherson class of X. For this, denote by $\mathcal{L}$ the line bundle $\mathcal{O}(X)$ on M. A section s of $\mathcal{L}$ defining X determines a section $\mathcal{O} \xrightarrow{S} \mathcal{P}^1_M\mathcal{L}$ of the bundle of principal parts of $\mathcal{L}$ (a very accessible reference for bundles of principal parts is Appendix A in [11]). Denote by $\overline{\Omega}_X$ the cokernel of this section after tensoring by the dual line bundle $\mathcal{L}^\vee$, so that

$$(*) \qquad 0 \longrightarrow \mathcal{L}^\vee \longrightarrow \mathcal{L}^\vee \otimes \mathcal{P}^1_M\mathcal{L} \longrightarrow \overline{\Omega}_X \longrightarrow 0$$

is an exact sequence of sheaves on M; $\overline{\Omega}_X$ is a coherent sheaf on M, of rank equal to the dimension of M.

Remark. Note that X is nonsingular precisely when (*) is an exact sequence of *vector bundles,* and $\overline{\Omega}_X$ is locally free in that case; and

$$c(\overline{\Omega}_X) \cap [X] = c(T^*X) \cap [X] \quad \text{if } X \text{ is nonsingular, since}$$
$$c(\overline{\Omega}_X) = \frac{c(T^*M \otimes \mathcal{L} \otimes \mathcal{L}^\vee)c(\mathcal{L} \otimes \mathcal{L}^\vee)}{c(\mathcal{L}^\vee)} = \frac{c(T^*M)}{1-X} \quad .$$

As a consequence (*if X is nonsingular*)

$$\begin{aligned} i_*c(TX) \cap [X] &= c(TM) \cap \frac{[X]}{1+X} = c(TM)\left(1 - \frac{1}{1+X}\right) \cap [M] \\ &= c(TM) \cap [M] - (c(\overline{\Omega}_X) \cap [M])^\vee \end{aligned}$$

where, for a class $\alpha \in A_kM$, $\alpha^\vee$ denotes $(-1)^{\dim M - k}\alpha$. Theorem 2 will generalize this formula to the case in which X is singular.

If X is singular, the sheaf $\overline{\Omega}_X$ is not locally free. Now, for an arbitrary coherent sheaf $\mathcal{F}$ there is a notion of *Chern-Mather* class, which we denote $c_{\mathrm{Ma}}(\mathcal{F})$, agreeing with the (homology) Chern class for locally free sheaves. This notion stems from work of Marie-Hélène Schwartz [14], and is discussed in detail in Michał Kwieciński's thesis ([8]). It can be viewed as the result of performing for arbitrary coherent sheaves the operation described for the cotangent sheaf Ω^1_X of X in Example 4.2.9. (a) of [6]. In particular, $c_{\mathrm{Ma}}(\Omega^1_X)^\vee$ recovers the ordinary Chern-Mather class of X, defined in [9].

Theorem 1 allows us to extend the formula given in the remark to the case when X is a singular hypersurface, by using this notion of Chern-Mather class. The precise statement is the following:

Theorem 2. *Let X be a hypersurface in a nonsingular variety M, and let i denote the inclusion $X \hookrightarrow M$. Then, with notations as above:*

$$i_*c_{\mathrm{SM}}(X) = c(TM) \cap [M] - c_{\mathrm{Ma}}(\overline{\Omega}_X)^\vee$$

Proof. By Theorem 1, and with the notations used there, we only need to prove that

$$c_{\mathrm{Ma}}(\overline{\Omega}_X) = \pi_*\left(c(\Omega^1_{\widetilde{M}}(\log X')) \cap [\widetilde{M}]\right)$$

where $\pi : \widetilde{M} \longrightarrow M$ is a proper birational map such that $\pi^{-1}X$ is a divisor with smooth (possibly multiple) components and normal crossings, and $X' = (\pi^{-1}X)_{\mathrm{red}}$. We may in fact assume that, further, the singularity subscheme Y of X (locally defined by the partials of a section defining X) pulls back in $\widetilde{M}$ to a Cartier divisor $Y' = \pi^{-1}Y$.

Observe that Y is precisely the zero-scheme of the section $S : \mathcal{O} \to \mathcal{P}^1_M\mathcal{L}$ induced by the section of $\mathcal{L}$ defining X; therefore, S induces an embedding *of vector bundles* $0 \to \mathcal{O}(Y') \to \pi^*\mathcal{P}^1_M\mathcal{L}$. It follows that we have a surjection of sheaves of the same rank:

$$\pi^*\overline{\Omega}_X \to \frac{\mathcal{L}^\vee \otimes \pi^*\mathcal{P}^1_M\mathcal{L}}{\mathcal{L}^\vee \otimes \mathcal{O}(Y')} \to 0 \quad ,$$

and as the target is locally free we have

$$c_{\mathrm{Ma}}(\overline{\Omega}_X) = \pi_* c\left(\frac{\mathcal{L}^\vee \otimes \pi^*\mathcal{P}^1_M\mathcal{L}}{\mathcal{L}^\vee \otimes \mathcal{O}(Y')} \cap [\widetilde{M}]\right) \quad .$$

Our main tool now is a morphism of locally-free sheaves on $\widetilde{M}$:

$$\mathcal{L}^\vee \otimes \pi^*\mathcal{P}^1_M\mathcal{L} \to \Omega^1_{\widetilde{M}}(\log X') \quad .$$

To define this morphism on (local) sections, assume U is an open subset of $\widetilde{M}$ such that X' has equation $u_1 \cdots u_r = 0$ for local parameters $u_1, \cdots, u_n$ in U; then $\pi^{-1}X$ has ideal $(u_1^{m_1} \cdots u_n^{m_n})$ for suitable integers $m_i \geq 0$, with $m_i = 0$ for $i > r$. Sections of $\Omega_{\widetilde{M}}(\log X')$ over U can be written

$$\alpha_1 \frac{du_1}{u_1} + \cdots + \alpha_r \frac{du_r}{u_r} + \alpha_{r+1} du_{r+1} + \cdots + \alpha_n du_n \quad ,$$

and we can describe sections of $\mathcal{L}^\vee \otimes \pi^*\mathcal{P}^1_M\mathcal{L}$ over U by $(f; f_1 du_1 + \cdots + f_n du_n)$.

We define a map $(\mathcal{L}^\vee \otimes \pi^*\mathcal{P}^1_M\mathcal{L})(U) \to \Omega^1_{\widetilde{M}}(\log X')(U)$ by

$$(f; f_1 du_1 + \cdots + f_n du_n) \mapsto \sum (f_i u_i - m_i f)\frac{du_i}{u_i} \quad .$$

In order to see that this local description patches up to a global morphism of sheaves, observe that it is induced by a morphism defined at the level of meromorphic sections between

$$\mathcal{L}^\vee \overset{L}{\otimes} \mathcal{P}^1_M\mathcal{L} \quad \text{and} \quad \mathcal{P}^1_M\mathcal{L} \overset{R}{\otimes} \mathcal{L}^\vee \cong \mathcal{P}^1_M\mathcal{O}_M \quad :$$

here the first tensor is computed (as above) using the usual $\mathcal{O}_M$-module structure of $\mathcal{P}^1_M\mathcal{L}$; the second is obtained according to the *other* $\mathcal{O}_M$-module structure, cf. [11], §A.5. The isomorphism with $\mathcal{P}^1_M\mathcal{O}_M$ is [5], 16.7.2.1. One defines a morphism between meromorphic sections of the two tensors in the most natural way that involves the section s of $\mathcal{L}$ defining X, that is:

$$u \overset{L}{\otimes} g \mapsto (su) g \overset{R}{\otimes} \frac{1}{s} \quad .$$

Pulling back to $\widetilde{M}$, one checks that this morphism is given on a trivializing open set by the local description given above, and in particular that the image of a holomorphic section is a section of $\Omega^1_{\widetilde{M}}(\log X')$. Also, it is easy to check that the subbundle $\mathcal{L}^\vee \otimes \mathcal{O}(Y')$ of $\mathcal{L}^\vee \otimes \pi^*\mathcal{P}^1_M\mathcal{L}$ is in the kernel of this morphism. So we obtain a morphism of vector bundles

$$\frac{\mathcal{L}^\vee \otimes \pi^*\mathcal{P}^1_M\mathcal{L}}{\mathcal{L}^\vee \otimes \mathcal{O}(Y')} \to \Omega^1_{\widetilde{M}}(\log X') \quad .$$

This morphism has maximal rank ($=\dim M$) off Y'. The difference

$$\left(c\left(\frac{\mathcal{L}^\vee\otimes\pi^*\mathcal{P}^1_M\mathcal{L}}{\mathcal{L}^\vee\otimes\mathcal{O}(Y')}\right)-c(\Omega^1_{\widetilde{M}}(\log X'))\right)\cap[\widetilde{M}]$$

can be evaluated by means of the *graph construction*—see for example [6], Example 18.1.6. The details of the construction needed here are similar to those given in [9], p. 429. Applying the graph construction shows that the push-forward of the difference to M by π_* vanishes, so

$$c_{\mathrm{Ma}}(\overline{\Omega}_X)=\pi_*\left(c\left(\frac{\mathcal{L}^\vee\otimes\pi^*\mathcal{P}^1_M\mathcal{L}}{\mathcal{L}^\vee\otimes\mathcal{O}(Y')}\right)\cap[\widetilde{M}]\right)=\pi_*\left(c(\Omega^1_{\widetilde{M}}(\log X'))\cap[\widetilde{M}]\right)$$

as needed. □

Alternative proofs of the formula given in Theorem 2 can be derived from recent results on Chern-Schwartz-MacPherson classes of hypersurfaces. In fact, Theorem 2 is equivalent to a weak (that is, after push-forward to the ambient variety) version of the main result in [1], of which it provides a considerably more streamlined proof. More specifically, the reader should have no difficulties obtaining the (weak form of the) formula in Theorem I.3 in [1] from the statement of Theorem 2. A different proof of the same formula in [1] can also be found in §3 of [10].

The reader is addressed to [1], [10], and [3], for recent work on the Chern-Schwartz-MacPherson class of a hypersurface (and, in [3], the more general case of a complete intersection). These references deal primarily with measuring the difference between the Chern-Schwartz-MacPherson class and other 'canonical' classes such as Fulton's class and Fulton-Johnson's class (cf. [6], Example 4.2.6). The sheaf $\overline{\Omega}_X$ seems particularly suited to study such differences: by Theorem 2, its Chern-Mather class relates to the Chern-Schwartz-MacPherson class of X; while its ordinary Chern class recovers Fulton's class $c_F(X)$ (that is, the class of the virtual tangent bundle of X):

$$c(\overline{\Omega}_X)\cap[X]=\frac{c(\mathcal{L}^\vee\otimes\mathcal{P}^1_M\mathcal{L})}{c(\mathcal{L}^\vee)}\cap[X]=\frac{c(T^*M)}{c(\mathcal{L}^\vee)}\cap[X]=c_F(X)^\vee \quad .$$

As a final remark we also note that as, according to Theorem 2, $c(TM)\cap[M]-c_{\mathrm{Ma}}(\overline{\Omega}_X)^\vee$ computes the Chern-Schwartz-MacPherson class of X, it is not hard to see that

$$c(TM)\cap[M]-c(\mathcal{O}(X))\cap c_{\mathrm{Ma}}(\overline{\Omega}_X)^\vee$$

computes (up to sign and pushing forward to the ambient variety M) the *weighted Chern-Mather class* of the singularity scheme Y of X (cf. [2]). The details are left to the interested reader.

References

[1] P. Aluffi, *Chern classes for singular hypersurfaces*, Trans. Amer. Math. Soc. **351** (1999), 3989–4026.

[2] P. Aluffi, *Weighted Chern-Mather classes and Milnor classes of hypersurfaces*, Singularities—Sapporo 1998, Adv. Stud. Pure Math., 29, 2000, pp. 1–20.

[3] J.-P. Brasselet, D. Lehmann, J. Seade, T. Suwa, *Milnor classes of local complete intersections*, Preprint (1998).

[4] J.–P. Brasselet, M.–H. Schwartz, *Sur les classes de Chern d'un ensemble analytique complexe*, Astérisque **82–83** (1981), 93–147.

[5] A. Grothendieck, J.A. Dieudonné, *Éléments de Géométrie Algébrique*, Publ. Math. IHES **32** (1967).

[6] W. Fulton, *Intersection Theory*, Springer Verlag, 1984.

[7] G. Kennedy, *MacPherson's Chern classes of singular algebraic varieties*, Comm. in Alg. **18(9)** (1990), 2821–2839.

[8] M. Kwieciński, *Sur le transformé de Nash et la construction du graphe de MacPherson*, Thèse, Doctorat de l'Université de Provence (Aix-Marseille I) (1994).

[9] R. MacPherson, *Chern classes for singular algebraic varieties*, Annals of Math. **100** (1974), 423–432.

[10] A. Parusiński, P. Pragacz, *Characteristic classes of hypersurfaces and characteristic cycles*, J. Algebraic Geom. **10** (2001), 63–79.
[11] D. Perkinson, *Curves in Grassmannians*, Trans. Amer. Math. Soc. **347** (1995), 3179–3246.
[12] C. Sabbah, *Quelques remarques sur la géométrie des espaces conormaux*, Asterisque **130** (1985), 161–192.
[13] M.–H. Schwartz, *Classes caractéristiques définies par une stratification d'une variété analytique complexe*, C. R. Acad. Sci. Paris **260** (1965), 3262–3264; 3535–3537.
[14] M.–H. Schwartz, *Classes et caractères de Chern-Mather des espaces linéaires*, C. R. Acad. Sci. Paris **295** (1982), 399–402.
[15] R. Silvotti, *On a conjecture of Varchenko*, Invent. Math. **126** (1996), 235–248.

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