# PLANE CURVES WITH SMALL LINEAR ORBITS II

PAOLO ALUFFI
CAREL FABER

October 1999

ABSTRACT. The 'linear orbit' of a plane curve of degree $d$ is its orbit in $\mathbb{P}^{d(d+3)/2}$ under the natural action of PGL(3). We classify curves with positive dimensional stabilizer, and we compute the degree of the closure of the linear orbits of curves supported on unions of lines. Together with the results of [3], this encompasses the enumerative geometry of all plane curves with small linear orbit. This information will serve elsewhere as an ingredient in the computation of the degree of the orbit closure of an arbitrary plane curve.

## §0. INTRODUCTION

The 'linear orbit' of a plane curve is its orbit under the natural action of PGL(3) on $\mathbb{P}^N = \mathbb{P}^{d(d+3)/2}$, the space parameterizing plane curves of degree $d$. We say that a plane curve has 'small' linear orbit if its stabilizer has positive dimension, so that the dimension of the orbit is less than the expected dimension $8 = \dim \mathrm{PGL}(3)$. This paper is concerned with such curves: we classify them (§1), then compute the degree of the closure of the orbits of such curves as well as other enumerative information (§2). The enumerative computations in this paper regard curves supported on unions of lines; if all lines but at most one are concurrent, the configuration has small orbit. The enumerative geometry of all other curves with small orbit (that is, of curves with small orbit and containing some non-linear component) is substantially more involved, and is studied in [3].

Orbits of *smooth* plane curves are studied in [2]. The results of this paper, together with the results in [3], form an essential ingredient towards the computation of the degree of the orbit closure for an arbitrary plane curve [5]. For a slightly more expanded discussion of the general context, we refer the reader to the introduction of [3]. The enumerative computations in this paper rely on the strategy employed in [2] and [3]: we construct a smooth complete variety dominating the orbit closure, and perform the necessary intersection theory. The variety is obtained by suitably blowing up the $\mathbb{P}^8$ of $3 \times 3$ matrices, viewed as a compactification of PGL(3): the action on a given curve extends to a rational map on this $\mathbb{P}^8$, and the variety is

1991 *Mathematics Subject Classification.* Primary 14N10; Secondary 14L30.

Both authors gratefully acknowledge partial NSF support, under grants DMS-9500843 and DMS-9801257.

obtained by performing a sequence of blow-ups removing the indeterminacies of this map. We have taken the *degree* of the orbit closure of a curve as the main focus of our work, but it should be noted that explicit constructions of good varieties dominating an orbit closure in principle allow the computation of other invariants, such as the multiplicity of the orbit closure along its boundary, or various kinds of characteristic classes.

As the referee pointed out, the subject of plane curves with positive dimensional stabilizer is very classical, going back as it does to F. Klein and S. Lie [7]. Their classification of 'W-curves' includes transcendental curves as well, and essentially coincides with ours in the algebraic case. Our viewpoint is however somewhat different, as we need to compile information regarding the degree of the stabilizers of the curves, especially when these are reducible. We do not know of any work on the enumerative geometry of these curves previous to ours.

**Acknowledgments.** P.A. thanks Florida State University for a 'Developing Scholar Award', and C.F. thanks the Max-Planck-Institut für Mathematik, Bonn, for support during the final stages of the preparation of this paper.

## §1. Classification of curves with small orbits

In this section we classify the plane curves with small orbits, i.e., the plane curves with positive dimensional stabilizer. We work over an algebraically closed field of characteristic 0. See §2.4 for a few remarks about linear orbits in positive characteristic.

Let $C$ be a plane curve with $\dim \mathrm{Stab}(C) > 0$. (By $\mathrm{Stab}(C)$ we mean the subgroup of $\mathrm{PGL}(3)$ of transformations fixing $C$.) Denote by $G$ the connected component of the identity of $\mathrm{Stab}(C)$. It is clear that $G$ fixes the irreducible components of $C$ and does not depend on the multiplicities of these components.

Every curve that is not just a line contains 4 points that form a frame; hence $G$ acts faithfully on such curves. It follows that $C$ contains infinitely many points which are not fixed by $G$. By imposing the condition that such a point be fixed, we find a subgroup of $G$ of dimension one less; repeating this, we find subgroups of $G$ of every dimension between 0 and $\dim G$, so that in particular $G$ contains a 1-dimensional subgroup. It is well-known that a connected 1-dimensional linear algebraic group is isomorphic either to the additive group $\mathbb{G}_a$ or to the multiplicative group $\mathbb{G}_m$.

First we consider the case where $G$ contains a $\mathbb{G}_m$. It can be diagonalized: denoting the embedding of $\mathbb{G}_m$ into $G$ by $\gamma$, we can find homogeneous coordinates $(x : y : z)$ on $\mathbb{P}^2$ such that the effect of $\gamma(t)$ is given by $(x : y : z) \mapsto (x : t^a y : t^b z)$, with $0 \le 2a \le b$ and $a$ and $b$ coprime. It is easy to see that all irreducible curves that are fixed by $\gamma(\mathbb{G}_m)$ are given by the equations $x$, $y$, $z$ or $y^b + \lambda z^a x^{b-a}$ with $\lambda \neq 0$.

It follows that $C$ consists of irreducible components of the form above, with arbitrary multiplicities. We list here the types of curves $C$ so obtained, together with the dimension of the orbit $\mathcal{O}_C$.

(1) $C$ consists of a single line; $\dim \mathcal{O}_C = 2$ (this is the only case where $\mathrm{Stab}(C)$ doesn't act faithfully on $C$).
(2) $C$ consists of 2 (distinct) lines; $\dim \mathcal{O}_C = 4$.
(3) $C$ consists of 3 or more concurrent lines; $\dim \mathcal{O}_C = 5$. (We call this configuration a *star*.)

(4) $C$ is a triangle (consisting of 3 lines in general position); $\dim \mathcal{O}_C = 6$.
(5) $C$ consists of 3 or more concurrent lines, together with 1 other (non-concurrent) line; $\dim \mathcal{O}_C = 7$. (We call this configuration a *fan*.)
(6) $C$ consists of a single conic; $\dim \mathcal{O}_C = 5$.
(7) $C$ consists of a conic and a tangent line; $\dim \mathcal{O}_C = 6$.
(8) $C$ consists of a conic and 2 (distinct) tangent lines; $\dim \mathcal{O}_C = 7$.
(9) $C$ consists of a conic and a transversal line and may contain either one of the tangent lines at the 2 points of intersection or both of them; $\dim \mathcal{O}_C = 7$.
(10) $C$ consists of 2 or more bitangent conics (conics in the pencil $y^2 + \lambda xz$) and may contain the line $y$ through the two points of intersection as well as the lines $x$ and/or $z$, tangent lines to the conics at the points of intersection; again, $\dim \mathcal{O}_C = 7$.
(11) $C$ consists of 1 or more (irreducible) curves from the pencil $y^b + \lambda z^a x^{b-a}$, with $b \geq 3$, and may contain the lines $x$ and/or $y$ and/or $z$; $\dim \mathcal{O}_C = 7$.

Next we consider the case where $G$ contains a $\mathbb{G}_a$. The $\mathbb{G}_a$-orbit of a point which is not a fixed point is an affine curve. Upon taking the projective closure of this curve, we find a fixed point of the $\mathbb{G}_a$-action as well as a fixed (tangent) line through that point. Hence in suitable coordinates $(x : y : z)$ the image of the embedding $\gamma : \mathbb{G}_a \to G$ consists of lower triangular matrices; it is easy to see that the diagonal entries are equal to 1.

$$\gamma(t) = \begin{pmatrix} 1 & 0 & 0 \\ a(t) & 1 & 0 \\ b(t) & c(t) & 1 \end{pmatrix}$$

Here $a(t)$, $b(t)$ and $c(t)$ are polynomials. Necessary and sufficient conditions for $\gamma$ to be a homomorphism of algebraic groups are that $a$ and $c$ are additive polynomials ($a(s+t) = a(s)+a(t)$, $c(s+t) = c(s)+c(t)$) and that $b$ satisfies $b(s)+b(t)+a(s)c(t) = b(s+t)$.

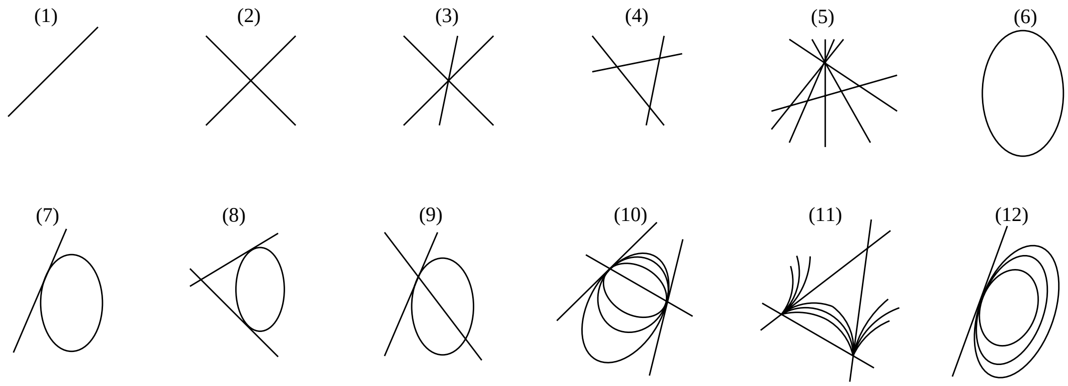
Since the characteristic is 0, both $a$ and $c$ are necessarily linear, forcing $b$ to be quadratic; the general solution can be written in the form

$$\gamma(t) = \begin{pmatrix} 1 & 0 & 0 \\ at & 1 & 0 \\ bt + \frac{1}{2}act^2 & ct & 1 \end{pmatrix} .$$

Since $\gamma$ is an embedding, the constants $a$, $b$ and $c$ cannot all vanish. When $a$ and $c$ are both non-zero, the only fixed point is $(0 : 0 : 1)$. When $c = 0$, the locus of fixed points is the line $x = 0$; when $a = 0$, the locus of fixed points is the line $bx + cy = 0$. It follows that every irreducible component of $C$ has degree 1 or 2; if it doesn't consist entirely of fixed points, it has a unique fixed point. If $C$ consists only of lines, these necessarily form a star. If $C$ contains a unique conic, it may also contain the tangent line at one point; again, we find no new configurations this way. If however $C$ contains two or more conics, every conic intersects every other conic in exactly one point (with multiplicity 4); this point, $(0 : 0 : 1)$, lies on all the conics. The curve $C$ may also contain the (tangent) line $x = 0$. (The conics are elements of the pencil through $cy^2 + 2x(by - az)$ and $x^2$.) We find thus one new configuration:

(12) $C$ contains 2 or more conics from a pencil through a conic and a double tangent line; it may also contain that tangent line. In this case, $\dim \mathcal{O}_C = 7$ (the identity component of $\mathrm{Stab}(C)$ is $\mathbb{G}_a$).

The following picture represents schematically the curves described above.




We conclude this section with a description of the stabilizers $\mathrm{Stab}(C)$ of the various types of plane curves $C$ with small orbit; the degree of the stabilizer is one ingredient in the computation of the degree of the orbit closure of the corresponding curve.

Choose coordinates $(x : y : z)$ on $\mathbb{P}^2$, $(p_0 : p_1 : p_2 : p_3 : p_4 : p_5 : p_6 : p_7 : p_8)$ on $\mathbb{P}^8$, and $(a : b : c : d)$ on $\mathbb{P}^3$. The list below corresponds to the list above.

(1) The stabilizer is (the intersection of PGL(3) and) a linear subspace of dimension 6. For $C$ with equation $x^d$ it is given by $p_1 = p_2 = 0$.

(2) The group $G$, the connected component of the identity of $\mathrm{Stab}(C)$, is a linear subspace of dimension 4. For $C$ with equation $x^k y^{d-k}$ it is given by $p_1 = p_2 = p_3 = p_5 = 0$. The index of $G$ in $\mathrm{Stab}(C)$ is 2 when $2k = d$ and 1 otherwise.

(3) The group $G$ is a linear subspace of dimension 3. For $C$ with equation $f(x, y)$ (with 3 or more distinct linear factors) it is given by $p_0 = p_4$ and $p_1 = p_2 = p_3 = p_5 = 0$. The quotient $\mathrm{Stab}(C)/G$ is isomorphic to the stabilizer in PGL(2) of the $d$-tuple with equation $f(x, y)$, via the map sending a $3 \times 3$ matrix $(p_0 : p_1 : 0 : p_3 : p_4 : 0 : p_6 : p_7 : p_8)$ in $\mathrm{Stab}(C)$ to the $2 \times 2$ matrix $(p_0 : p_1 : p_3 : p_4)$ in the stabilizer of the $d$-tuple.

(4) The group $G$ is a linear subspace of dimension 2. For $C$ with equation $x^a y^b z^c$ it is given by $p_1 = p_2 = p_3 = p_5 = p_6 = p_7 = 0$. The quotient $\mathrm{Stab}(C)/G$ is isomorphic to the symmetry group of the triple $(a, b, c)$.

(5) The group $G$ is a line in PGL(3). For $C$ with equation $f(x, y)z^k$ it is given by $p_0 = p_4$ and $p_1 = p_2 = p_3 = p_5 = p_6 = p_7 = 0$. As in (3), the quotient $\mathrm{Stab}(C)/G$ is isomorphic to the stabilizer in PGL(2) of the $(d - k)$-tuple with equation $f(x, y)$.

(6) The stabilizer of a conic $x^2 + yz$ is the image of the embedding of PGL(2) into PGL(3) given by $(a : b : c : d) \mapsto (ad + bc : bd : -ac : 2cd : d^2 : -c^2 : -2ab : -b^2 : a^2)$. Its closure has degree 8 since it is a projection of the second Veronese of $\mathbb{P}^3$ from the $\mathbb{P}^9$ of quadrics in $(a : b : c : d)$, from the determinant quadric $ad - bc$.

(7) The stabilizer of a conic with tangent line $y(x^2 + yz)$ is the image of the embedding $(a : b : 0 : d) \mapsto (ad : bd : 0 : 0 : d^2 : 0 : -2ab : -b^2 : a^2)$. Its closure has degree 4 since it is the second Veronese of $\mathbb{P}^2$.

(8) For a conic with two tangent lines $y^k z^l (x^2 + yz)^m$, the group $G$ consists of the diagonal matrices with entries $(ad, d^2, a^2)$. Its closure has degree 2. The quotient $\mathrm{Stab}(C)/G$ is isomorphic to the symmetry group of the pair $(k, l)$.

(9), (10), (11) The stabilizers in these cases are discussed in [3], Lemma 3.1.
(12) These curves and their stabilizers are discussed in [3], §4.1.

## 2. Predegree polynomials for configurations of lines

§**2.1 Predegree polynomials.** We move now to the enumerative geometry of these curves. Our main task is to compute the *degree* of the orbit closure of a given curve with small orbit. We have chosen this question because of its clear enumerative interpretation, and because the tool needed to solve it (that is, a nonsingular variety dominating the orbit closure) could in principle allow us to compute several other invariants of the orbit closure. If the curve $C$ is reduced and its orbit has dimension $j$, then the degree of the orbit closure of $C$ is the number of translates of $C$ which contain $j$ points in general position in the plane. In very special cases (for example, when $C$ is a triangle) this number can be computed by naive combinatorial considerations. In general this is not possible.

The computation of the degree of the orbit closure of a curve with small orbit and containing non-linear components (that is, items (6) through (12) in the classification of §1) requires a lengthy study, paralleling the embedded resolution of such curves. As this study is rather involved, we have devoted to it a separate paper ([3]). Here we deal with curves consisting of configurations of (possibly multiple) lines. We have seen in §1 that, among these, the ones with small orbit are the configurations in which all but at most one of the lines are concurrent (types (1) through (5)). We call a set of concurrent lines a *star*, and the union of a set of concurrent lines with a general line a *fan*.

The machinery necessary to study the enumerative geometry of stars and fans yields in fact analogous results for *arbitrary* unions of lines (whose orbits are not necessarily small); we will state the more general result in §2.3, and derive from it formulas for stars and fans.

As pointed out already in [3], a more refined degree question is natural in our framework. In view of the applications of the results we obtain here and in [3], it is useful to compute the degree of the orbit closure as well as of the subsets of the orbit closure determined by imposing linear conditions on the transformation applied to the given curve.

To state this question precisely, let $C$ be a plane curve, with equation $F(x:y:z)=0$. The orbit of $C$ is the image of the action map $\mathrm{PGL}(3) \longrightarrow \mathbb{P}^N$ sending $\varphi$ to the curve with equation $F(\varphi(x):\varphi(y):\varphi(z))=0$. Embedding PGL(3) in the $\mathbb{P}^8$ of $3\times 3$ matrices, the action extends to a rational map

$$\mathbb{P}^8 \overset{c}{\dashrightarrow} \mathbb{P}^N \quad ,$$

and the orbit closure of $C$ is the closure of the image of this rational map. 'Imposing general linear conditions' on $\varphi \in \mathrm{PGL}(3)$ amounts to restricting this map $c$ to general linear subspaces of $\mathbb{P}^8$.

We will compute here the degree of the closure of the image of general subspaces of $\mathbb{P}^8$ of all dimensions, for $C =$ an arbitrary union of (possibly multiple) lines. For $0 \le j \le 8$, consider a general $\mathbb{P}^j$ in $\mathbb{P}^8$, and let $f_j$, $d_j$ denote respectively the number of points in the general fiber of $c_{|\mathbb{P}^j}$ and the intersection number of $c(\mathbb{P}^j)$ and a codimension-$j$ linear subspace of $\mathbb{P}^N$. The products $f_j \cdot d_j$ are the *predegrees* of $C$. We assemble these numbers into a generating function

$$\sum_{j\geq 0}(f_j\cdot d_j)\,\frac{t^j}{j!}\quad ,$$

which we call the 'adjusted predegree polynomial' of $C$.

*Remarks.* (1) If the orbit of $C$ has dimension $j$, then
— $d_k=0$ for $k>j$, and $d_j$ is the degree of the orbit closure;
— $f_j$ is the degree of the stabilizer $\mathrm{Stab}(C)$ of $C$, and $f_k=1$ for $k<j$;
— since $C$ consists of lines, the closure of every connected component of the stabilizer of $C$ is a linear subspace of $\mathbb{P}^8$; hence $f_j$ is the number of connected components of the stabilizer of $C$. For example, $f_j=1$ if the lines in the configuration appear with distinct multiplicities.

(2) $C$ has small orbit if and only if its adjusted predegree polynomial has degree $<8$. By (1), the degree of the orbit closure of $C$ can be recovered from the leading coefficient of the adjusted predegree polynomial of $C$ by dividing by the number of components of the stabilizer of $C$.

(3) The denominators are introduced in the definition of the adjusted predegree polynomial because this uncovers a certain amount of structure which would otherwise be lost (and is completely invisible in the individual predegrees). This structure amounts to a multiplicativity of the adjusted predegree polynomial with respect to union of transversal lines, and allows a convenient shortcut in the computation of the predegrees of stars and fans, cf. Lemma 2.6 and Corollary 2.11.

Computing adjusted predegree polynomials (rather than degrees alone) allows us to deal at once with all configurations of lines, regardless of the dimension of their stabilizer. Adjusted predegree polynomials for all other small orbits are computed in [3], §4.2.

From a geometric point of view, our approach to the problem will be to construct (in §2.2) a nonsingular variety $\widetilde{V}$ resolving the indeterminacies of the rational map $c$:

$$\begin{tikzcd} & \widetilde{V} \arrow[dl] \arrow[dr, "\tilde{c}"] & \\ \mathbb{P}^8 \arrow[rr, dashed, "c"] & & \mathbb{P}^N \end{tikzcd}$$

Once this is accomplished, we can let $\widetilde{H}$, $\widetilde{W}$ respectively be the pull-backs of the hyperplane class $H$ in $\mathbb{P}^8$ and $W$ in $\mathbb{P}^N$; then tracing the definitions shows that the adjusted predegree polynomial equals

$$\sum_{j\geq 0}\left(\int \widetilde{H}^{8-j}\widetilde{W}^j\right)\frac{t^j}{j!}\quad .$$

The construction of §2.2 and intersection theory allow us in §2.3 to compute the individual intersection degrees $\int \widetilde{H}^{8-j}\widetilde{W}^j$ of classes in $\widetilde{V}$.

§**2.2 Blow-ups.** Let $C$ be a degree-$d$ plane curve consisting of a union of lines: that is, the ideal of $C$ is generated by a product

$$F(x,y,z)=\prod L_i(x,y,z)^{r_i}\quad ,\quad \sum r_i=d$$

where the $L_i$ are distinct linear homogeneous polynomials. In this section we resolve the indeterminacies of the corresponding rational map $c : \mathbb{P}^8 \dashrightarrow \mathbb{P}^N$, $N = d(d+3)/2$. This construction does not use any information regarding whether the orbit of $C$ is small or not.

First, observe that the base locus of $c$ reflects the combinatorics of the configuration of lines. More precisely, $\varphi \in \mathbb{P}^8$ is in the base locus if and only if $F(\varphi(x, y, z))$ is identically 0, that is, if the image of $\varphi$ is contained in one of the lines of the configuration. The condition $\operatorname{im}\varphi \subset$ line specifies a certain $\mathbb{P}^5$ in the $\mathbb{P}^8$ of matrices. Hence, the base locus consists of a union of $\mathbb{P}^5$'s, each corresponding to a line of the configuration; the intersection of two $\mathbb{P}^5$'s corresponding to two lines $\ell$, $m$ is the $\mathbb{P}^2$ of matrices whose image is the point $\ell \cap m$.

Summarizing, the line configuration determines a set of disjoint $\mathbb{P}^2$'s in $\mathbb{P}^8$, one for each point of intersection of two lines in the configuration, and a set of $\mathbb{P}^5$'s in $\mathbb{P}^8$, one for each line in the configuration. This is the base locus of $c$.

**Proposition 2.1.** *Let $\widetilde{V} \xrightarrow{\pi} \mathbb{P}^8$ be the variety obtained by first blowing up $\mathbb{P}^8$ along the distinguished $\mathbb{P}^2$'s, and then along the proper transforms of the distinguished $\mathbb{P}^5$'s. Then $\widetilde{V}$ resolves the indeterminacies of $c$; that is, there is a commutative diagram*

$$\begin{tikzcd} & \widetilde{V} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{c}"] & \\ \mathbb{P}^8 \arrow[rr, dashed, "c"] & & \mathbb{P}^N \end{tikzcd}$$

*with $\tilde{c}$ a regular map.*

The rest of this section is devoted to the proof of this proposition.

Given $C$ as above, every point $(x_0 : y_0 : z_0)$ in the plane determines a hypersurface in $\mathbb{P}^8$, with ideal generated by $F(\varphi(x_0, y_0, z_0))$ : concretely, this hypersurface consists of the matrices $\varphi$ such that the translate of $C$ by $\varphi$ contains the point $(x_0, y_0, z_0)$ (or is undefined). We call such hypersurfaces (and their proper transforms) 'point-conditions'. Note that the rational map $c$, and its lift to varieties dominating $\mathbb{P}^8$, is precisely the map determined by the linear system generated by the point-conditions. What we have to show is that the (proper transforms of the) point-conditions in $\widetilde{V}$ generate a base-point-free linear system.

Let $V'$ be the variety obtained after the first set of blow-ups.

**Claim 2.2.** *(i) The proper transforms of the distinguished $\mathbb{P}^5$'s in $V'$ are disjoint.*

*(ii) The multiplicity of a point-condition along the $\mathbb{P}^2$ corresponding to the intersection of lines of the configuration equals the sum of the multiplicities of those lines.*

*(iii) The intersection of the point-conditions $W'$ in $V'$ consists of the proper transforms of the distinguished $\mathbb{P}^5$'s.*

*Proof.* (i) is clear, while (ii) and (iii) require a computation. Statement (iii) is vacuously true in the complement of the exceptional divisors of the first set of blow-ups, so we only need to check it along a component of the exceptional divisor. Consider then a point of intersection of two lines of the configuration; without loss of generality we may assume this is the point $(1 : 0 : 0)$. Write the equation of $C$ as

$$\prod_i (\beta_i y + \gamma_i z)^{r_i} \cdot \prod_j (\alpha_j x + \beta_j y + \gamma_j z)^{r_j} = 0$$

with $\alpha_j \neq 0$. We can study the blow-up over the affine piece $(1 : p_1 : p_2 : p_3 : p_4 : p_5 : p_6 : p_7 : p_8)$; in these coordinates, the point-condition corresponding to $(x_0 : y_0 : z_0)$ has equation

$$\prod_i (\beta_i(p_3x_0 + p_4y_0 + p_5z_0) + \gamma_i(p_6x_0 + p_7y_0 + p_8z_0))^{r_i}$$
$$\cdot \prod_j (\alpha_j(x_0 + p_1y_0 + p_2z_0) + \beta_j(p_3x_0 + p_4y_0 + p_5z_0) + \gamma_j(p_6x_0 + p_7y_0 + p_8z_0))^{r_j}$$

and $(1 : 0 : 0)$ corresponds to the distinguished 2-dimensional locus $p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = 0$. In a representative chart of $V'$, we can choose coordinates $q_1, \dots, q_8$ so that the blow-up map is given by

$$(q_1, \dots, q_8) \mapsto (q_1, q_2, q_3, q_3q_4, \dots, q_3q_8) \quad .$$

Therefore the point-condition pulls back to

$$\prod_i q_3^{r_i} (\beta_i(x_0 + q_4y_0 + q_5z_0) + \gamma_i(q_6x_0 + q_7y_0 + q_8z_0))^{r_i}$$
$$\cdot \prod_j (\alpha_j(x_0 + q_1y_0 + q_2z_0) + \beta_j q_3(x_0 + q_4y_0 + q_5z_0) + \gamma_j q_3(q_6x_0 + q_7y_0 + q_8z_0))^{r_j}$$

so it contains the exceptional divisor $q_3 = 0$ with multiplicity $\sum r_i$, proving (ii). Also, its proper transform has intersection with the exceptional divisor

$$(*) \quad \prod_i (\beta_i(x_0 + q_4y_0 + q_5z_0) + \gamma_i(q_6x_0 + q_7y_0 + q_8z_0))^{r_i} \cdot (x_0 + q_1y_0 + q_2z_0)^{\sum r_j} \quad .$$

A point $(q_1, q_2, 0, q_4, \dots, q_8)$ is in the intersection of all point-conditions in $V'$ if (*) is identically zero for all $(x_0, y_0, z_0)$. From (*) we see that the set of such points is given by the equations

$$\begin{cases} \beta_i + \gamma_i q_6 = 0 \\ \beta_i q_4 + \gamma_i q_7 = 0 \\ \beta_i q_5 + \gamma_i q_8 = 0 \end{cases}$$

It is easy to check that these are precisely the equations of the proper transforms of the distinguished $\mathbb{P}^5$'s corresponding to those lines in the configuration which contain $(1 : 0 : 0)$, and this verifies (iii). $\square$

**Claim 2.3.** *(i) The point-conditions in $V'$ contain the proper transform of a distinguished $\mathbb{P}^5$ with multiplicity equal to the multiplicity of the corresponding line in the configuration.*

*(ii) The intersection of all point-conditions in $\widetilde{V}$ is empty.*

*Proof.* $\widetilde{V}$ is obtained from $V'$ by blowing up the proper transforms of the distinguished $\mathbb{P}^5$'s. The statement can be checked by another straightforward coordinate computation, which we leave to the reader. $\square$

This claim implies Proposition 2.1, as $\tilde{c}$ is the map corresponding to the linear system generated by the point-conditions, and this linear system has empty base locus by the claim.

*Remark 2.4.* The orbit-closure of a star is dominated via $\widetilde{c}$ by the proper transform $\widetilde{\mathbb{P}}^5$ of a general $\mathbb{P}^5 \subset \mathbb{P}^8$; such a $\mathbb{P}^5$ is for example the closure of the set of matrices of rank 2 whose image is a fixed line not containing the center of the star. Indeed, this locus avoids the distinguished $\mathbb{P}^2$ corresponding to the center of the star, and intersects transversally the other centers of the blow-ups considered in this section. This implies that the class of $\widetilde{\mathbb{P}}^5$ in $\widetilde{V}$ is $\widetilde{H}^3$, and hence

$$\frac{1}{j!}\int_{\widetilde{\mathbb{P}}^5} \widetilde{H}^{5-j}\widetilde{W}^j = \frac{1}{j!}\int_{\widetilde{V}} \widetilde{H}^{8-j}\widetilde{W}^j \qquad (j = 0, \dots, 5)$$

is the coefficient of $t^j$ in the adjusted predegree polynomial for the star, for such a $\mathbb{P}^5$. This fact is used in [5].

§**2.3 Predegree computations.** For an arbitrary configuration of lines $C$, the construction of §2.2 provides us with a nonsingular variety $\widetilde{V}$ and a regular map

$$\tilde{c} : \widetilde{V} \longrightarrow \mathbb{P}^N$$

whose image is the orbit closure of $C$. Intersection theory allows us to compute the degree of the image of this map, and more generally to compute the 'adjusted predegree polynomial' of $C$:

$$P(t) = \sum_{j \geq 0} \left( \int \widetilde{H}^{8-j}\widetilde{W}^j \right) \frac{t^j}{j!} \quad .$$

For stars and fans, the result of this computation can be stated quite succinctly:

**Theorem 2.5.** *(i) The adjusted predegree polynomial of a star of lines $L_i$, appearing with multiplicity $r_i$, is*

$$\left\{ \prod_i \left( 1 + r_i t + \frac{r_i^2 t^2}{2} \right) \right\}_5 \quad ,$$

*where $\{\}_5$ denotes the truncation to $t^5$. In particular, a star with three or more lines has predegree*

$$30(e_2e_3 - e_1e_4 - e_5) \quad ,$$

*where $e_j$ denotes the $j$-th elementary symmetric function in the multiplicities $r_i$ of the lines in the star.*

*(ii) Let $C'$ be a star of lines with multiplicities $r_i$, and let $C$ be a fan obtained as the union of $C'$ with a transversal line with multiplicity $r$. Then the adjusted predegree polynomial of $C$ is*

$$\left( 1 + rt + \frac{r^2t^2}{2} \right) \left\{ \prod_i \left( 1 + r_i t + \frac{r_i^2 t^2}{2} \right) \right\}_5 \quad .$$

*In particular, if the star has three or more lines, then the predegree of the fan is*

$$630r^2(e_2e_3 - e_1e_4 - e_5) \quad ,$$

*where $e_j$ denotes the $j$-th elementary symmetric function in the multiplicities $r_i$ of the lines in the star.*

For example, according to (i) the adjusted predegree polynomial for a star consisting of three reduced lines is

$$\left\{\left(1+t+\frac{t^2}{2}\right)^3\right\}_5 = 1+3t+\frac{9t^2}{2}+4t^3+\frac{9t^4}{4}+\frac{3t^5}{4} \quad ,$$

yielding a predegree of $\frac{3}{4}\cdot 5! = 90$. According to our enumerative interpretation this must be the number of ordered triples of concurrent lines containing 5 general points; and indeed this is $3\binom{5}{2}\binom{3}{2} = 90$. By the theorem, the predegree of a star of $d \geq 3$ simple lines is

$$(d-2)\,(d-1)\,d\,(d^2+3d-3) \quad ;$$

for $d \geq 4$ this cannot be checked by simple combinatorics, as a star of four or more lines has moduli. The predegree of the 4-dimensional orbit closure of a pair of lines is 6 (twice its degree: this is the divisor of singular conics); and the predegree for a single line is of course 1.

As another example that can be checked combinatorially, consider the orbit closure of a *triangle* (item (4) in the classification in §1). A triangle is a fan consisting of a star of two lines union a general line, so according to (ii) its adjusted predegree polynomial must be

$$1+3t+\frac{9t^2}{2}+4t^3+\frac{9t^4}{4}+\frac{3t^5}{4}+\frac{t^6}{8} \quad .$$

In particular, the predegree of the 6-dimensional orbit closure of a triangle must be $\frac{1}{8}\cdot 6! = 90$. The orbit closure of a triangle is however the closure of the set of *all* triangles, so this number should agree with the number of ordered triples of general lines in the plane, containing 6 general points, that is $\binom{6}{2}\binom{4}{2}\binom{2}{2} = 90$, as expected.

*Proof of Theorem 2.5.* The statements given above are particular cases of the computation for arbitrary configurations of lines (Theorem 2.8). A particularly economical way to package all the information relevant to stars and fans is the following:

**Lemma 2.6.** *Let $C'$ be a configuration of lines with adjusted predegree polynomial $P(t)$, and let $C$ be the configuration obtained by adding to $C'$ a line with multiplicity $r$ that intersects $C'$ transversally. Then the adjusted predegree polynomial of $C$ is the truncation to $t^8$ of*

$$\left(1+rt+\frac{r^2t^2}{2}\right)P(t) \quad .$$

Assuming this statement, Theorem 2.5 can be proved as follows. First, observe that the adjusted predegree polynomial for a single $r$-multiple line is

$$1+rt+\frac{r^2t^2}{2}$$

(in this case, the orbit is simply an $r$-tuple embedding of $\mathbb{P}^2$ in $\mathbb{P}^N$); by the lemma, it follows that the adjusted predegree polynomial for a configuration of general distinct lines with multiplicities $r_1, r_2, \dots$ is the truncation to $t^8$ of

$$\prod_i \left(1+r_it+\frac{r_i^2t^2}{2}\right).$$

Now, the key remarks are that

(1) a star has stabilizer of dimension $\geq 3$, so its adjusted predegree polynomial has degree $\leq 5$; and
(2) the combinatorics of the intersections of the lines in a configuration only affects the coefficients of degree $\geq 6$ in the adjusted predegree polynomial.

The first observation should be clear; the second follows from the construction in §2.2. More explicitly, the coefficients of degree $\leq 5$ in the adjusted predegree polynomial are intersection degrees

$$\frac{1}{j!}\int \widetilde{H}^{8-j}\widetilde{W}^j$$

in $\widetilde{V}$, with $j \leq 5$; that is, they involve the intersection of 3 or more proper transforms $\widetilde{H}$ of hyperplanes from $\mathbb{P}^8$. However, points of intersection of the lines in the configuration correspond to 2-dimensional centers of blow-up in the construction of $\widetilde{V}$; the intersection of three or more general hyperplanes avoids these centers, so the numbers $\int \widetilde{H}^{8-j}\widetilde{W}^j$ are unaffected by how the lines meet, for $j \leq 5$.

These observations imply part (i) of Theorem 2.5: by (1), the coefficient of $t^j$ of the adjusted predegree polynomial of a star must be 0 for $j \geq 6$; by (2), the adjusted predegree polynomial for a star must agree with the adjusted predegree polynomial of a general configuration up to the term of degree 5.

The second part of the theorem follows then immediately from the first, by applying Lemma 2.6 again. This proves Theorem 2.5 (modulo Lemma 2.6). □

Lemma 2.6 seems of independent interest. It follows immediately from a more general multiplicativity formula (Cor. 2.11) for transversal configurations, proved below. This in turn will follow from the computation of the adjusted predegree polynomial for *arbitrary* configurations of lines (Theorem 2.8), which will be obtained by applying intersection theory to the construction given in §2.2. As in [2] and [3], this can be done by applying Theorem II from [1]; in the form that we need here, this amounts to the following formula.

**Lemma 2.7.** *Let $B \overset{i}{\hookrightarrow} V$ be nonsingular varieties; $X$, $Y$ hypersurfaces in $V$; $\widetilde{X}$, $\widetilde{Y}$ their proper transforms in the blow-up of $V$ along $B$. Then*

$$\int_{\widetilde{V}} \widetilde{X}^{\dim V-j}\cdot\widetilde{Y}^j = \int_V X^{\dim V-j}\cdot Y^j - \int_B \frac{(m_{B,X}[B]+i^*[X])^{\dim V-j}(m_{B,Y}[B]+i^*[Y])^j}{c(N_BV)}$$

*where $m_{B,X}$, $m_{B,Y}$ denote the multiplicities of $X$, $Y$ along $B$.* □

This formula and the 'Bézout' product in $\mathbb{P}^8$ will yield the following result.

Assume $C$ is an arbitrary degree-$d$ configuration of lines $L_i$, appearing with multiplicities $r_i$ (so $d=\sum r_i$). Line $L_i$ intersects the rest of the configuration in a tuple of points $p_j$; for each of these points $p_j \in L_i$ we let

$$\rho_{\alpha,ij} = \Big(\sum_{L_k\ni p_j, k\neq i} r_k\Big)^\alpha - \Big(\sum_{L_k\ni p_j, k\neq i} r_k^\alpha\Big) \quad .$$

Note that all $\rho_{\alpha,ij} = 0$ if $L_i$ intersects the rest of the configuration transversally; that is, these functions measure how far from transversal the intersection is. Next, we let

$$\rho_{\alpha,i} = \sum_{p_j \in L_i} \rho_{\alpha,ij} \quad ,$$

and define the following three functions:

$$S_{6,i} = -\rho_{5,i} r_i - 5\rho_{4,i} r_i^2 + 10\rho_{3,i} r_i^3 + 5\rho_{2,i} r_i^4$$
$$S_{7,i} = 6\rho_{6,i} r_i + 29\rho_{5,i} r_i^2 - 50\rho_{4,i} r_i^3 - 20\rho_{3,i} r_i^4 - \rho_{2,i} r_i^5$$
$$S_{8,i} = -21\rho_{7,i} r_i - 99\rho_{6,i} r_i^2 + 155\rho_{5,i} r_i^3 + 55\rho_{4,i} r_i^4 + \rho_{3,i} r_i^5 - \rho_{2,i} r_i^6$$

Again, note that these expressions vanish if $L_i$ meets the rest of the configuration transversally.

**Theorem 2.8.** *With the above notations, the adjusted predegree polynomial of a configurations of lines $L_i$ is*

$$\prod_i \left(1 + r_i t + \frac{r_i^2 t^2}{2}\right) + \sum_i \left(1 + (d - r_i)t + \frac{(d - r_i)^2 t^2}{2}\right) \left(\frac{S_{6,i} t^6}{6!} + \frac{S_{7,i} t^7}{7!} + \frac{S_{8,i} t^8}{8!}\right)$$

*(truncated to the $t^8$ term).*

*Proof.* This can be viewed as an excess intersection problem in $\mathbb{P}^8$. Lemma 2.7 is used to evaluate the contribution to the Bézout numbers due to the excess components, supported on the base locus of $c$. The discussion in §2.2 identifies a contribution from the first set of blow-ups (due to the intersection points of the configuration), and a contribution from the second set of blow-ups (due to the lines of the configuration). Here is the raw form that the answer takes by applying Lemma 2.7:

**Lemma 2.9.** *Denote by $L_i$ the lines of the configuration $C$, and by $p_j$ the intersection points. Let $r_i$ be the multiplicity of $L_i$ in $C$, and let $m_j$ be the multiplicity of $p_j$ in $C$. Let $d = \sum r_i$ be the degree of $C$. Then:*

*(1) the contribution due to the first set of blow-ups is*

$$\sum_{p_j} \left(\frac{m_j^6 t^6}{6!} + \frac{(7dm_j^6 - 6m_j^7)t^7}{7!} + \frac{(28d^2 m_j^6 - 48dm_j^7 + 21m_j^8)t^8}{8!}\right)$$

*(2) the contribution due to the second set of blow-ups is*

$$\sum_{L_i} r_i^3 \left(\frac{t^3}{3!} + (4d - 3r_i)\frac{t^4}{4!} + (10d^2 - 15dr_i + 6r_i^2)\frac{t^5}{5!} + (20d^3 - 45d^2 r_i \right.$$
$$+ 36dr_i^2 - 10r_i^3)\frac{t^6}{6!} + (35d^4 - 105d^3 r_i + 126d^2 r_i^2 - 70dr_i^3 + 15r_i^4)\frac{t^7}{7!}$$
$$\left. + (56d^5 - 210d^4 r_i + 336d^3 r_i^2 - 280d^2 r_i^3 + 120dr_i^4 - 21r_i^5)\frac{t^8}{8!}\right)$$

*plus*

$$-\sum_{L_i} \sum_{p_j \in L_i} r_i^3 \left((20m_j^3 - 45m_j^2 r_i + 36m_j r_i^2 - 10r_i^3)\frac{t^6}{6!} + (140dm_j^3 - 105m_j^4 - 315dm_j^2 r_i \right.$$
$$+ 210m_j^3 r_i + 252dm_j r_i^2 - 126m_j^2 r_i^2 - 70dr_i^3 + 15r_i^4)\frac{t^7}{7!} + (560d^2 m_j^3 - 840dm_j^4$$
$$+ 336m_j^5 - 1260d^2 m_j^2 r_i + 1680dm_j^3 r_i - 630m_j^4 r_i + 1008d^2 m_j r_i^2 - 1008dm_j^2 r_i^2$$
$$\left. + 336m_j^3 r_i^2 - 280d^2 r_i^3 + 120dr_i^4 - 21r_i^5)\frac{t^8}{8!}\right)$$

*Proof of Lemma 2.9.* Consider one point $p$ of intersection of lines in the configuration; as we have seen in §2.2, $p$ contributes a $\mathbb{P}^2$ to the first set of blow-ups. Denote by $k$ the hyperplane class in this $\mathbb{P}^2$. Then the contribution of $p$ to the first correction term given by applying the formula in Lemma 2.7 is

$$\sum_\alpha \left( \int_{\mathbb{P}^2} \frac{k^{8-\alpha}(m+dk)^\alpha}{(1+k)^6} \right) \frac{t^\alpha}{\alpha!}$$

where $m$ is the multiplicity of $p$ in $C$. Indeed, a general hyperplane in $\mathbb{P}^8$ does not contain $B = \mathbb{P}^2$ (so its multiplicity along $B$ is 0), while we have checked in Claim 2.2 that point-conditions contain $B$ with the stated multiplicity; $(1+k)^6$ is the total Chern class of the normal bundle of $\mathbb{P}^2$ in $\mathbb{P}^8$. Evaluating the expression and summing over the points $p_j$ gives (1).

Next, consider a line $L$ in the configuration, appearing with multiplicity $r$. In $V'$ we find the proper transform $\widetilde{\mathbb{P}}^5$ of the distinguished $\mathbb{P}^5$ corresponding to $L$. Let $\ell$ denote the hyperplane class in $\mathbb{P}^5$, and its pull-back to $\widetilde{\mathbb{P}}^5$; also, let $e_1, \dots, e_s$ denote the pull-back to $\widetilde{\mathbb{P}}^5$ of the exceptional divisors from the first set of blow-ups, corresponding to the intersection of $L$ with the rest of the configuration. Note that the other exceptional divisors do not meet this $\widetilde{\mathbb{P}}^5$. The following claim can be checked by using standard intersection theory:

**Claim 2.10.** *(i)* $c(N_{\widetilde{\mathbb{P}}^5} V') = (1 + \ell - e_1 - \cdots - e_s)^3$.

*(ii) Let $m_j$ denote the multiplicity of $C$ at the $j$-th point of intersection of $L$ with the rest of the configuration; then the pull-back of the class $W'$ of the point-conditions in $V'$ to $\widetilde{\mathbb{P}}^5$ is $d\ell - m_1 e_1 - \cdots - m_s e_s$.*

*(iii) For each $j$,*

$$e_j^0 \mapsto 1\,,\ e_j^1 \mapsto 0\,,\ e_j^2 \mapsto 0\,,\ e_j^3 \mapsto \ell^3\,,\ e_j^4 \mapsto 3\ell^4\,,\ e_j^5 \mapsto 6\ell^5$$

*via the push-forward from $\widetilde{\mathbb{P}}^5$ to $\mathbb{P}^5$, while $e_i e_j = 0$ for $i \neq j$.* □

Using this information and Claim 2.3(ii), Lemma 2.7 evaluates the contribution of $L$ to the correction term for the second blow-up. The individual intersection products are corrected by

$$\int_{\widetilde{\mathbb{P}}^5} \frac{\ell^{8-j}(r + d\ell - m_1 e_1 - \cdots - m_s e_s)^j}{(1 + \ell - e_1 - \cdots - e_s)^3}$$

This degree can be computed by pushing forward to $\mathbb{P}^5$ and using Claim 2.10(iii). Summing over the lines and the intersection points on each line gives (2). This finishes the proof of Lemma 2.9. □

The statement of Theorem 2.8 simply packages the information assembled in Lemma 2.9 in a more legible format. This finishes the proof of Theorem 2.8. □

For example, if all lines meet transversally, then the adjusted predegree polynomial of $C$ is the truncation to $t^8$ of

$$\prod_i \left(1 + r_i t + \frac{r_i^2 t^2}{2}\right) \quad :$$

indeed, as observed above, in this case the terms $S_{k,i}$ all vanish.

In particular, the predegree of a configuration of four or more simple lines in general position is $2520(e_4^2 - 2e_2e_6 + 2e_8)$, where $e_j$ denotes the $j$-th elementary symmetric function in the multiplicities $r_i$ of the lines in the configuration. For example, a configuration of four general (simple) lines has predegree 2520. This is in agreement with the naive combinatorial count: the number of (ordered) 4-tuples of lines through 8 general points is $\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 2520$. For five or more lines, this combinatorial argument cannot be applied since the configuration has moduli. If $d \geq 4$ lines have multiplicity one and intersect transversally, then the predegree of the configuration is

$$2520\left(\binom{d}{4}^2 - 2\binom{d}{2}\binom{d}{6} + 2\binom{d}{8}\right)$$
$$= (d-3)\,(d-2)\,(d-1)\,d\,(d^4 + 6d^3 - 31d^2 - 36d + 105) \quad .$$

This is 0 for $d < 4$, since the configuration then has small orbit.

Lemma 2.6 amounts to a 'multiplicativity of adjusted predegree polynomials' when one of the lines meets the rest of the configuration transversally. In fact such a multiplicativity holds more generally whenever two configurations $C'$, $C''$ are *transversal* in the sense that every line of each configuration meets the other configuration transversally:

**Corollary 2.11.** *Let $C$ be the union of two configurations $C'$ and $C''$ meeting transversally, and let $P_{C'}(t)$, $P_{C''}(t)$ respectively be the adjusted predegree polynomials of the two configurations. Then the adjusted predegree polynomial of $C$ is*

$$\{P_{C'}(t) \cdot P_{C''}(t)\}_8 \quad .$$

*Proof.* Denote by $d, r_i, S$, $d', r'_j, S'$, and $d'', r''_k, S''$ the data for $C$, $C'$, $C''$ used in Theorem 2.8. First, we observe that

$$\left\{\prod_j \left(1 + r'_j t + \frac{{r'_j}^2 t^2}{2}\right)\right\}_2 = \left(1 + d't + \frac{{d'}^2 t^2}{2}\right) \quad ,$$

and similarly for $C''$. Taking the product of the expressions for $P_{C'}(t)$, $P_{C''}(t)$ given by the theorem and truncating to $t^8$ agrees then with the truncation of

$$\prod_j \left(1 + r'_j t + \frac{{r'_j}^2 t^2}{2}\right) \prod_k \left(1 + r''_k t + \frac{{r''_k}^2 t^2}{2}\right)$$
$$+ \sum_j \left(1 + (d'' + d' - r'_j)t + \frac{(d'' + d' - r'_j)^2 t^2}{2}\right)\left(\frac{S'_{6,j}t^6}{6!} + \frac{S'_{7,j}t^7}{7!} + \frac{S'_{8,j}t^8}{8!}\right)$$
$$+ \sum_k \left(1 + (d' + d'' - r''_k)t + \frac{(d' + d'' - r''_k)^2 t^2}{2}\right)\left(\frac{S''_{6,k}t^6}{6!} + \frac{S''_{7,k}t^7}{7!} + \frac{S''_{8,k}t^8}{8!}\right)$$

that is, with the truncation of

$$\prod_i \left(1 + r_i t + \frac{r_i^2 t^2}{2}\right) + \sum_j \left(1 + (d - r'_j)t + \frac{(d - r'_j)^2 t^2}{2}\right)\left(\frac{S'_{6,j}t^6}{6!} + \frac{S'_{7,j}t^7}{7!} + \frac{S'_{8,j}t^8}{8!}\right)$$
$$+ \sum_k \left(1 + (d - r''_k)t + \frac{(d - r''_k)^2 t^2}{2}\right)\left(\frac{S''_{6,k}t^6}{6!} + \frac{S''_{7,k}t^7}{7!} + \frac{S''_{8,k}t^8}{8!}\right) \quad .$$

Now the crucial observation is that transversal intersections do not affect the $S$-functions. Since every line $\ell'$ of $C'$ intersects the whole configuration $C''$ transversally, the $S$-function for $\ell'$ (viewed as a line of $C$) agrees with its $S'$-function. Similarly, $S$-functions for lines of $C''$ must agree with the corresponding $S''$-functions. Therefore, the expression given above agrees with the truncation of

$$\prod_i \left(1 + r_i t + \frac{r_i^2 t^2}{2}\right) + \sum_i \left(1 + (d - r_i)t + \frac{(d - r_i)^2 t^2}{2}\right) \left(\frac{S_{6,i} t^6}{6!} + \frac{S_{7,i} t^7}{7!} + \frac{S_{8,i} t^8}{8!}\right) \quad ,$$

that is, with the adjusted predegree polynomial for $C$. $\square$

Lemma 2.6 follows immediately from this corollary by taking $C''$ to consist of a single line. The corollary goes however a little further: for example, it shows that multiplicativity holds for unions of stars, so long as each line of the configuration belongs to only one star.

§**2.4 Concluding remarks.** The 'multiplicativity lemma' and other qualitative considerations hold in fact in all dimensions. The result is that the orbit closure of an $r$-fold hyperplane in $\mathbb{P}^n$ has adjusted predegree polynomial

$$P_n(t) = \sum_{k=0}^{n} \frac{r^k t^k}{k!} \quad ,$$

(that is, '$\lim_{n\to\infty} P_n(t) = \exp(rt)$') and that the adjusted predegree polynomial of a union of transversal hyperplanes is the (truncated) product of such terms. For $n = 1$, this recovers [4], Proposition 1.3; for $n > 2$, this observation is apparently new. For example, it follows that the (pre)degree of the orbit closure of an arrangement of $d \geq 5$ (reduced) planes in general position in $\mathbb{P}^3$ is

$$\begin{aligned}(d-4)(d-3)(d-2)(d-1)d\,\big(&d^{10} + 10\,d^9 + 65\,d^8 - 1015\,d^7 + 63\,d^6 \\ &-10885\,d^5 + 190560\,d^4 - 658885\,d^3 + 1358936\,d^2 - 3034850\,d + 3503500\big) \quad ;\end{aligned}$$

as should be expected, this number is 0 for $d \leq 4$ since the orbit is small in that case. Of course, for $d = 5$ this agrees with the naive combinatorial count $\binom{15}{3}\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 168168000$.

Another issue of some interest is the study of linear orbits of curves in positive characteristic. The classification of small orbits is more difficult in this case, since there are many more additive polynomials. As an example, the (non-diagonalized) multiplicative automorphism group of the cuspidal cubic $x^2z + y^3 + y^2x$ in characteristic 0 reduces to the additive group in characteristic 3. Another phenomenon appearing in positive characteristic is the presence of curves whose general point is an inflection point; this may affect the dimension of the stabilizer of a curve. For example, let $p > 0$ be the characteristic of the ground field, and $n = p^r$ for $r \geq 1$; then on the curve $x^n = yz^{n-1}$ every nonsingular point is an $n$-flex. (In fact the curve is *strange,* since all tangent lines contain the point $(1 : 0 : 0)$.) The point $(0 : 0 : 1)$, which in other characteristics is distinguished by being an $n$-flex, becomes indistinguishable from any other nonsingular point. In practice this enlarges the stabilizer of the curve, making its orbit 'even smaller'. The intersection theoretic computations of [3] can be carried out for these curves, yielding their adjusted predegree polynomial:

$$1 + nt + \frac{n^2t^2}{2} + \frac{n^3t^3}{3!} + \frac{n^4t^4}{4!} + \frac{n^3(n^2-3)t^5}{5!} + \frac{3n^3(n-1)(n-2)t^6}{6!} \quad .$$

An additional subtlety is that the stabilizer for this curve is *nonreduced;* taking this into account, the degree of the orbit closure of the curve is then $3(n-1)(n-2) = 6\binom{n-1}{2}$. For example, the degree of the orbit closure of the 'strange cuspidal curve' $x^3 = yz^2$ in characteristic 3 equals 6, in agreement with a computation in [6].

## References


[1] P. Aluffi, *The enumerative geometry of plane cubics I: smooth cubics*, Trans. of the AMS **317** (1990), 501–539.

[2] P. Aluffi, C. Faber, *Linear orbits of smooth plane curves*, Jour. of Alg. Geom **2** (1993), 155–184.

[3] P. Aluffi, C. Faber, *Plane curves with small linear orbits I*, Annales de l'Institut Fourier (to appear), `math.AG/9805020`.

[4] P. Aluffi, C. Faber, *Linear orbits of d-tuples of points in* $\mathbb{P}^1$, J. reine angew. Math. **445** (1993), 205-220.

[5] P. Aluffi, C. Faber, *Linear orbits of arbitrary plane curves*, preprint (1999).

[6] A. Hefez, I. Vainsencher, *Varieties of strange plane curves*, Comm. in Algebra **19(1)** (1991), 333-345.

[7] F. Klein, S. Lie, *Ueber diejenigen ebenen Curven, welche durch ein geschlossenes System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen*, Math. Ann. **4** (1871), 50-84.



Mathematics Department, Florida State University, Tallahassee FL 32306, U.S.A.
*E-mail address*: `aluffi@math.fsu.edu`

Dept. of Mathematics, Oklahoma State University, Stillwater OK 74078, U.S.A.
*E-mail address*: `cffaber@math.okstate.edu`

Department of Mathematics, KTH, 100 44 Stockholm, Sweden
*E-mail address*: `carel@math.kth.se`