INTERPOLATION OF CHARACTERISTIC CLASSES OF SINGULAR HYPERSURFACES

PAOLO ALUFFI AND JEAN-PAUL BRASSELET

ABSTRACT. We show that the Chern-Schwartz-MacPherson class of a hypersurface X in a nonsingular variety M 'interpolates' between two other notions of characteristic classes for singular varieties, provided that the singular locus of X is smooth and that certain numerical invariants of X are constant along this locus. This allows us to define a lift of the Chern-Schwartz-MacPherson class of such 'nice' hypersurfaces to intersection homology. As another application, the interpolation result leads to an explicit formula for the Chern-Schwartz-MacPherson class of X in terms of its polar classes.

Dedicated to Prof. Tatsuo Suwa on the occasion of his 60th birthday

1. INTRODUCTION AND STATEMENT OF THE RESULT

There are several different notions of 'characteristic classes' of possibly singular varieties, generalizing the notion of (homology) Chern classes of the tangent bundle of nonsingular ones; the relationship between some of these classes has been the object of recent work. In this note we prove a formula relating the Chern-Mather class, the Chern-Schwartz-MacPherson class, and the class of the virtual tangent bundle of a hypersurface in a nonsingular variety, under the assumption that the singular locus of X is smooth and that certain numerical invariants of X are constant along this locus. As an application, we obtain an explicit formula for Chern-Schwartz-MacPherson's class of X in terms of its polar classes (generalizing a result in [6]), under the same hypothesis on its singularity, and assuming that X is quasi-projective.

A more immediate, but perhaps more striking, application is to the problem of lifting Chern-Schwartz-MacPherson's classes of a hypersurface to intersection homology. While the class of the virtual tangent bundle of X trivially has a natural lift to intersection homology, examples of Mark Goresky and Jean-Louis Verdier show that the problem of lifting Chern-Schwartz-MacPherson's classes is much subtler (see [7] for a discussion of these examples). A lift exists to intersection homology with rational coefficients in middle perversity as a consequence of [4]; but there is no known way to construct a 'canonical' such lift in general. For quasi-projective hypersurfaces X satisfying our hypothesis, the interpolation formula given below defines a lift of the Chern-Schwartz-MacPherson class of X in $IH_*(X)$ with rational coefficients, in middle perversity. This could be used to define *Chern numbers* $c_i\, c_{\dim X - i}$ for projective singular hypersurfaces satisfying the condition considered here. Computing such numbers explicitly would be very interesting; also it would be interesting to establish the exact dependence (if any) of these numbers or of our lift on the embedding of X.

Let X be a reduced hypersurface of a nonsingular complex algebraic variety M. We denote by

$$c_{\rm SM}(X) \quad , \quad c_{\rm Ma}(X) \quad , \quad c_{\rm F}(X)$$

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the three classes mentioned above; the first two are defined in [13], while the third is the class of the virtual tangent bundle of X:

$$c_{\mathrm{F}}(X) = c\left(\frac{TM}{\mathcal{O}(X)}\right) \cap [X] = c(TM) \cap s(X,M) \quad .$$

Here, $s(X,M)$ is the *Segre class* of X in M, cf. [10], Chapter 4. The class $c_{\mathrm{F}}(X)$ equals William Fulton's intrinsic class of X, which can be defined for *every* scheme embeddable in a nonsingular variety (cf. [10], §4.2.6), and is independent of the ambient variety M.

All these classes can be defined in a good homology theory on X; in this paper we work in the Chow group of X with rational coefficient, denoted $(A_*X)_{\mathbb{Q}}$ (except for the application to lifting to IH_*).

We denote by Y the singularity *subscheme* of X (locally defined by the partial derivatives of a generator of its ideal in M) and by Y' its support, that is, the singular *locus* of X. For $p \in Y'$ we consider two numerical invariants of X at p:

—the local Euler obstruction of X at p, $\mathrm{Eu}_X(p)$, and

—the Euler characteristic χ_p of the Milnor fiber of X at p.

Definition 1.1. A variety X is a *nice hypersurface* if it can be realized as a hypersurface in a nonsingular variety M, and further its singular locus Y' is nonsingular and irreducible, and the numbers $\mathrm{Eu}_X(p)$, χ_p are constant along Y'.

In the results given below, we assume that X is a nice hypersurface, and we denote the constant values of $\mathrm{Eu}_X(p)$, χ_p by Eu, χ respectively. This condition is satisfied for example if the stratification $\{Y', X \setminus Y', M \setminus X\}$ of the ambient variety M is Whitney regular, or satisfies the weaker condition of *c-regularity* of Karim Bekka, [5][1]. The precise algebro-geometric requirement is that the normal cone of the singularity subscheme Y in M be irreducible, cf. Lemma 2.3.

Under this assumption, we will show that the Chern-Schwartz-MacPherson class of X is an 'interpolation' of the classes $c_{\mathrm{Ma}}(X)$ and $c_{\mathrm{F}}(X)$; we will now state this result precisely. As X is a hypersurface in M, it determines a line bundle $\mathcal{O}_M(X)$; we adopt a common abuse of notation and denote by X the first Chern class $c_1(\mathcal{O}_M(X))$ of this line bundle, and its restrictions to subschemes of M. Thus $s(X,M) = \frac{[X]}{1+X}$ in our notations, and $\frac{1}{1+\alpha X}$ is shorthand for $1 - \alpha\, c_1(\mathcal{O}(X)) + \alpha^2\, c_1(\mathcal{O}(X))^2 + \cdots$.

For all rational numbers α, we let

$$c_{(\alpha)}(X) = c_{\mathrm{F}}(X) + \frac{(1-\alpha)}{1+\alpha X}\left(c_{\mathrm{Ma}}(X) - c_{\mathrm{F}}(X)\right) \quad \in (A_*X)_{\mathbb{Q}} \quad .$$

Thus

$$c_{(0)}(X) = c_{\mathrm{Ma}}(X) \quad \text{and} \quad c_{(1)}(X) = c_{\mathrm{F}}(X)$$

trivially. Also, $c_{(\alpha)}(X)$ does *not* depend on the ambient manifold M in which X is realized as a hypersurface: indeed, the class $(c_{\mathrm{Ma}}(X) - c_{\mathrm{F}}(X))$ is supported on the singular locus Y' of X, and it can be shown that the action of $X = c_1(\mathcal{O}_M(X))$ on Y' is independent of M. In fact, the restriction of $\mathcal{O}_M(X)$ to Y does not depend on M.

Theorem 1.1. *If X is a nice hypersurface, then*

$$c_{(\rho)}(X) = c_{SM}(X) \quad \text{in } (A_*X)_{\mathbb{Q}},$$

where $c_{SM}(X)$ denotes the Chern-Schwartz-MacPherson class of X, and $\rho = \frac{1-Eu}{\chi-Eu}$.

[1]We are indebted to Jörg Schürmann for this remark; see for example [17], V.3 and VI.2.

We note that, under our assumption on the singularity of X, and by the very definition of $c_{\rm SM}(X)$ in [13], the class $c_{\rm SM}(X)$ is a simple linear combination of $c_{\rm Ma}(X)$ and of the total homology Chern class of the singular locus Y' of X. Our formula replaces this latter *local* ingredient with the *global* information of the class of the virtual tangent bundle of X. If the Chern classes of Y' are known, our formula can be used 'in reverse' to obtain information about the invariants Eu, χ, bypassing a local study of X near its singular locus. See §4, for a precise statement (Proposition 4.1) and an example of such a computation.

Also note that while the condition of being 'nice' is of course very strong in general, it is automatically satisfied in codimension equal to the codimension of the singular locus Y' of X, provided that the top-dimensional part of Y' is irreducible: indeed, the hypersurface obtained from X by removing the locus where the invariants jump is then trivially nice. The formula of the theorem gives then a relation between the classes up to that codimension, under the sole assumption that the singular locus is irreducible in top dimension; see Example 4.2 for an illustration of this fact.

As mentioned earlier, the interpolation formula can be used to lift Chern-Schwartz-MacPherson's classes of a quasi-projective nice hypersurface X to $IH_*(X)$ (with rational coefficients, in middle perversity). This relies precisely on our trading the 'local' information of $c(TY')\cap[Y']$ (which is hard to transfer into $IH_*(X)$) for the 'global' information of $c_{\rm F}(X)$. In order to define the lift, we just note that a lift of $c_{\rm Ma}(X)$ for quasi-projective X is defined in [9]; $c_{\rm F}(X)$ lifts as it is the class of the virtual tangent bundle of X; and the other ingredients in the interpolation formula also involve elements in cohomology, so they trivially lift to $IH_*(X)$. Hence the formula defines an element of $IH_*(X)$ for nice singular hypersurfaces, which lifts $c_{\rm SM}(X)$ by the main theorem. We note that as the lift of $c_{\rm Ma}(X)$ of [9] potentially depends on the realization of X as a quasi-projective variety, our lift of $c_{\rm SM}(X)$ may also depend on this choice. It would be interesting to establish whether it is in fact uniquely determined by X itself.

Theorem 1.1 is proved in §2; the main tools are formulas from [2], manipulations of Segre classes, and a key result of Adam Parusiński and Piotr Pragacz ([14]). In §3 we consider a situation in which $c_{\rm Ma}(X)$ can be expressed very concretely, that is, when X is given explicitly as a subvariety of a projective space $\mathbb{P}^n$. In this case, a result of Ragni Piene can be used to express the class in terms of a suitable combination $[P]$ of the *polar classes* of X:

$$[P]=-\sum_{k\geq 0}[P_k]^\vee\otimes_{\mathbb{P}^n}\mathcal{O}(1)\quad ,$$

where $[P_k]$ denotes the class of the k-th polar locus of X, and we use the notations introduced in [1]. We recall precise definitions and Piene's result in §3. We then have:

Corollary 1.2. *If $X\subset\mathbb{P}^n$, and X is a nice hypersurface (in some variety M), let*

$$\rho=\frac{1-Eu}{\chi-Eu}\quad,\quad \sigma=1-\rho=\frac{\chi-1}{\chi-Eu}\quad .$$

Then the Chern-Schwartz-MacPherson class of X is given by

$$c_{SM}(X)=c(TM)\cap\frac{\rho\,[X]}{1+\rho X}+c(T\mathbb{P}^n)\cap\frac{\sigma\,[P]}{1+\rho X}\in(A_*X)_{\mathbb{Q}}\quad .$$

As in Theorem 1.1, X is used here to denote $c_1(N_XM)$. Note that we are not requiring X to be a hypersurface *in* $\mathbb{P}^n$*;* all we need is that X can be abstractly realized as a nice hypersurface in *some* variety M, and that X is itself quasi-projective.

If $Y = Y'$ and the (nice) hypersurface X has multiplicity 2 along Y, then $(\chi, \mathrm{Eu}) = (2,0)$ if the codimension of Y in X is even, and $(0,2)$ if it is odd (this follows from Lemma 2.3 in §2). In both cases $\rho = \sigma = \frac{1}{2}$; if further $M = \mathbb{P}^n$, then the formula in the corollary specializes to the case considered in [6]; this was our starting point in this work.

Other expressions for Chern-Schwartz-MacPherson's class of a singular variety X in the context of the study of polar varieties are known, notably those given in §6 of [12] (without any restriction on X!). The work of Ragni Piene, Lê Dũng Tráng, and Bernard Teissier ([12], [16]) has exposed the close relationship between characteristic classes of singular varieties and their polar varieties. Our motivation in §3 is however somewhat different than in these references—we have aimed specifically at identifying the contribution of polar varieties to 'correction terms' between different notions of characteristic classes. The term $c(TM) \cap \frac{\rho[X]}{1+\rho X}$ is the analog of Fulton's intrinsic class for a 'virtual' hypersurface ρX. The other term, $c(T\mathbb{P}^n) \cap \frac{\sigma[P]}{1+\rho X}$, could then be interpreted as a *Milnor class* (in the sense of [6]) for such a virtual hypersurface. Thus, Corollary 1.2 brings evidence to the possibility that Milnor classes admit simple expressions in terms of polar classes. Positive results in this direction could lead to a good treatment of Milnor classes for more general varieties, which would be highly desirable.

This paper is dedicated to Prof. Tatsuo Suwa, with best wishes for his 60th birthday. We would also like to thank him for pointing out to us that alternative proofs of some of the results presented here may be obtained by applying results of [8] (for example, Corollary 5.13 in [8] can be used to provide a different proof of our Proposition 4.1).

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2. Segre classes and characteristic classes

Let X be a reduced hypersurface of an arbitrary nonsingular variety M; we work over $\mathbb{C}$ for convenience, but most of what we say can easily be extended to arbitrary algebraically closed fields of characteristic 0.

We denote by Y the singularity subscheme of X, and for the moment we make no further assumptions on X or Y. The following formulas for the Mather and Schwartz-MacPherson classes (denoted respectively $c_{\mathrm{Ma}}(X)$, $c_{\mathrm{SM}}(X)$) are given in [2] (Lemma I.2, Theorem I.3):

Lemma 2.1. *Let $\pi : \widetilde{M} \to M$ be the blow-up of M along Y; let $\mathcal{Y}$ be the exceptional divisor in this blow-up, and let $\mathcal{X}$, $\widetilde{X}$ respectively be the total and strict transforms of X in $\widetilde{M}$. Then*

$$c_{Ma}(X) = c(TM) \cap \pi_* \left(\frac{[\widetilde{X}]}{1 + \mathcal{X} - \mathcal{Y}} \right)$$

$$c_{SM}(X) = c(TM) \cap \pi_* \left(\frac{[\mathcal{X} - \mathcal{Y}]}{1 + \mathcal{X} - \mathcal{Y}} \right) \quad .$$

These formulas can be conveniently rewritten without reference to classes in $\widetilde{M}$, by adopting the notations introduced in [1]: for $A = \sum a_p$, a_p a class of dimension p in a subscheme of M, and $\mathcal{L}$ a line bundle, we let $A \otimes \mathcal{L}$ be the class $\sum_p c(\mathcal{L})^{p - \dim M} \cap a_p$

and $A^\vee$ be $\sum(-1)^{p-\dim M} a_p$. Note that these notations depend on the ambient variety M; this will be understood in the following. With these notations:

Proposition 2.2. *Denote by $\mathcal{L}$ the restriction of the line bundle $\mathcal{O}(X)$ to Y. Then*

$$c_{Ma}(X) = c(TM) \cap \left(\frac{[X]}{1+X} + s(Y,X)^\vee \otimes \mathcal{L} \right)$$

$$c_{SM}(X) = c(TM) \cap \left(\frac{[X]}{1+X} + (c(\mathcal{L}) \cap s(Y,M))^\vee \otimes \mathcal{L} \right)$$

Proof. The statement follows at once from the formulas given in Lemma 2.1, with standard manipulations involving the notations recalled above. For the first formula, write

$$\frac{[\widetilde{X}]}{1+\mathcal{X}-\mathcal{Y}} = \frac{[\widetilde{X}]}{1-\mathcal{Y}} \otimes \mathcal{L} = \left([\widetilde{X}] + \frac{\mathcal{Y}\cdot[\widetilde{X}]}{1-\mathcal{Y}} \right) \otimes \mathcal{L} = \frac{[\widetilde{X}]}{1+\mathcal{X}} + \left(\frac{\mathcal{Y}\cdot[\widetilde{X}]}{1+\mathcal{Y}} \right)^\vee \otimes \mathcal{L} \quad :$$

pushing this expression forward by π gives

$$\frac{[X]}{1+X} + s(Y,X)^\vee \otimes \mathcal{L} \quad ,$$

yielding the first formula in the statement. The second formula is proven similarly; it is in fact Theorem I.4 in [2]. □

Proposition 2.2 highlights the difference between the two notions of Chern-Mather and Chern-Schwartz-MacPherson classes of a hypersurface X in a nonsingular variety M: the distinction lies in the difference between

$$s(Y,X) \quad \text{and} \quad c(\mathcal{O}(X)) \cap s(Y,M)$$

and in this sense it is precisely captured by the singularity subscheme Y of the hypersurface. Relating the two characteristic classes directly amounts then to comparing $s(Y,X)$ and $s(Y,M)$ directly. Unfortunately, very few such comparison results are known in general (cf. [10], Example 4.2.8 for a counterexample to the naive guess for such a comparison). At present, the strong assumption posed in §1 to state the main theorem of this paper is necessary precisely because it allows us to perform this comparison.

First we gather more information from the blow-up $\widetilde{M}$; here we make crucial use of a result from [14].

Lemma 2.3. *Under the hypotheses of the theorem, $\mathcal{Y}$ is irreducible. Denoting by $\mathcal{Y}'$ its support, and writing*

$$\mathcal{Y} = m\mathcal{Y}' \quad , \quad \mathcal{X} = \widetilde{X} + n\mathcal{Y}'$$

as cycles, then we have

$$m = (-1)^{\dim X - \dim Y}(\chi - 1) \quad , \quad n = (-1)^{\dim X - \dim Y}(\chi - Eu) \quad .$$

with χ and Eu defined as in §1.

Proof. Let $\mathcal{Y} = \sum m_i \mathcal{Y}_i$ be the irreducible decomposition of $\mathcal{Y}$; by [11], each $\mathcal{Y}_i$ can be identified with the conormal variety of its support. In particular, there is exactly one component $\mathcal{Y}'$ over the support Y' of Y. Following [14] we let $\mu = (-1)^{\dim X}(\chi - 1)$, and remark that under our assumption this is a multiple of the characteristic function 1_Y of Y', hence of the local Euler obstruction $\mathrm{Eu}_{Y'}$ since Y' is nonsingular by hypothesis. By Theorem 2.3(iii) of [14], the cycle $\sum m_i \mathcal{Y}_i$ must equal a constant times $\mathcal{Y}'$; it follows that $\mathcal{Y}'$ is the only irreducible component of $\mathcal{Y}$, and further that

$$\mu = m(-1)^{\dim Y} 1_Y \quad .$$

The first assertion in the statement follows, as well as the formula for m. The formula for n can be obtained similarly from Theorem 2.3 in [14], or from the fact that

$$1_X = \mathrm{Eu}_X(p) + (n-m)(-1)^{\dim X - \dim Y} \mathrm{Eu}_Y(p)$$

for all $p \in Y$ (as proved in [3], §2). $\square$

Note that, for nice hypersurfaces, necessarily $\chi \neq 1$ and $\chi \neq \mathrm{Eu}$ (this follows from Lemma 2.3). The next lemma relates $s(Y,X)$ and $s(Y,M)$ in the special case of nice hypersurfaces.

Lemma 2.4. *With assumptions and notations as above,*

$$s(Y,X) = \left(\frac{\chi - Eu}{\chi - 1} + X\right) \cdot s(Y,M)$$

in $(A_*Y)_{\mathbb{Q}}$.

Proof. Since Y has codimension at least 2 in M, note that

$$\pi_* \frac{\mathcal{Y} \cdot [\mathcal{Y}]}{1+\mathcal{Y}} = \pi_* \frac{[\mathcal{Y}]}{1+\mathcal{Y}} = s(Y,M) \quad .$$

Therefore with notations as in Lemma 2.3

$$\begin{aligned} s(Y,X) = \pi_* \frac{\mathcal{Y} \cdot [\widetilde{X}]}{1+\mathcal{Y}} &= \pi_* \frac{\mathcal{Y} \cdot ([\mathcal{X}] - n[\mathcal{Y}'])}{1+\mathcal{Y}} = X \cdot \pi_* \frac{[\mathcal{Y}]}{1+\mathcal{Y}} - \frac{n}{m} \pi_* \frac{\mathcal{Y} \cdot [\mathcal{Y}]}{1+\mathcal{Y}} \\ &= X \cdot s(Y,M) + \frac{n}{m} s(Y,M) = \left(\frac{n}{m} + X\right) \cdot s(Y,M) \end{aligned}$$

$\square$

After these preliminary considerations, we are ready to prove the main theorem. Mimicking the relation between $s(X,M)$ and $c_{\mathrm{F}}(X)$, we write $s_{\mathrm{Ma}}(X,M)$ for $c(TM)^{-1} \cap c_{\mathrm{Ma}}(X)$; and, as above, $\sigma = \frac{\chi - 1}{\chi - \mathrm{Eu}}$ and $\rho = 1 - \sigma$.

Proof of Theorem 1.1. In view of the first formula in Proposition 2.2,

$$s_{\mathrm{Ma}}(X,M) = \frac{[X]}{1+X} + s(Y,X)^\vee \otimes \mathcal{L} \quad .$$

By Lemma 2.4, $s(Y,X) = \frac{1}{\sigma}(1+\sigma X) \cap s(Y,M)$, hence

$$s(Y,M) = \sigma \frac{1}{1+\sigma X} \cap s(Y,X) \quad .$$

Therefore

$$\begin{aligned} (c(\mathcal{L}) \cap s(Y,M))^\vee \otimes \mathcal{L} &= \sigma \left(\frac{1-X}{1-\sigma X} \cap s(Y,X)^\vee\right) \otimes \mathcal{L} \\ &= (1-\rho) \frac{1}{1+\rho X} \cap (s(Y,X)^\vee \otimes \mathcal{L}) \end{aligned}$$

(using Proposition 1 in [1]). That is,

$$(c(\mathcal{L}) \cap s(Y,M))^\vee \otimes \mathcal{L} = \frac{(1-\rho)}{1+\rho X} \left(s_{\mathrm{Ma}}(X,M) - \frac{[X]}{(1+X)}\right) \quad ,$$

and the formula given in the theorem now follows from the second formula of Proposition 2.2 and from $c_{\mathrm{F}}(X) = c(TM) \cap \frac{[X]}{1+X}$. $\square$

3. Relation with polar classes

We now turn to the situation in which X is embedded in a projective space $\mathbb{P}^n$, and to polar classes; all we need to do is rewrite a result of Ragni Piene into our language. If X is a (closed) subvariety of dimension r in $\mathbb{P}^n$, the k-th *polar locus* P_k of X with respect to a general linear subspace $L_k = \mathbb{P}^{k-2+n-r} \subset \mathbb{P}^n$ is the closure of the locus

$$\{x \in X_{\text{smooth}} \mid \dim(T_xX \cap L_k) \geq k-1\} \quad .$$

The class $[P_k] \in A_{\dim X - k}$ is independent of the (general) choice of the subspace L_k (cf. [15], Proposition 1.2). Note that $[P_0] = [X]$, since the condition defining P_0 is vacuous. We define the 'total polar class' of X by

$$[P] = (-1)^{n-r} \sum_{k\geq 0} [P_k]^\vee \otimes_{\mathbb{P}^n} \mathcal{O}(1) \quad ;$$

again we are using the notations of [1], for convenience.

The main observation here is that the class $[P]$ is closely related to the class $c_{\text{Ma}}(X)$. The precise relation is given in part (a) in the following theorem, due to Ragni Piene; we include part (b) for completeness, and stress that (b) holds for *arbitrary* hypersurfaces.

Theorem 3.1. *(a) (R. Piene) For any subvariety X of $\mathbb{P}^n$ as above:*

$$c_{Ma}(X) = c(T\mathbb{P}^n) \cap [P] \quad .$$

(b) If, further, X is a hypersurface in a nonsingular variety M, with singularity subscheme Y, and $\mathcal{L}$ denotes $\mathcal{O}_M(X)_{|X}$, then

$$s(Y, X) = [X] + \frac{c(N_X^*\mathbb{P}^n \otimes \mathcal{L})}{c(\mathcal{L})^{n-r-1}} \cap ([P]^\vee \otimes_M \mathcal{L}) \quad \in A_*X \quad .$$

Proof. (a) This is the translation in our notations of the second formula in Ragni Piene's Théorème 3 in [16]:

$$c_{\text{Ma}}(X) = \sum_{k\geq 0} \sum_{i=0}^{k} (-1)^{k-i} \binom{r+1-k+i}{i} H^i \cdot [P_{k-i}] \quad ,$$

where H is the hyperplane class.

(b) From part (a) and Proposition 2, we have

$$c(T\mathbb{P}^n) \cap [P] = c(TM) \cap \left(\frac{[X]}{1+X} + s(Y,X)^\vee \otimes \mathcal{L} \right) \quad ;$$

therefore, using Propositions 1 and 2 from [1]:

$$\begin{aligned}
\frac{[X]}{1+X} + s(Y,X)^\vee \otimes \mathcal{L} &= \frac{c(T\mathbb{P}^n)}{c(TM)} \cap [P] = \frac{c(N_X\mathbb{P}^n)}{c(\mathcal{L})} \cap [P] \\
s(Y,X)^\vee \otimes \mathcal{L} &= \frac{1}{c(\mathcal{L})} \left(c(N_X\mathbb{P}^n) \cap [P] - [X] \right) \\
s(Y,X) \otimes \mathcal{L}^\vee &= \frac{1}{c(\mathcal{L}^\vee)} \left(c(N_X^*\mathbb{P}^n) \cap [P]^\vee + [X] \right) \\
s(Y,X) &= c(\mathcal{L}) \frac{c(N_X^*\mathbb{P}^n \otimes \mathcal{L})}{c(\mathcal{L})^{n-r}} \cap ([P]^\vee \otimes \mathcal{L}) + [X] \quad ,
\end{aligned}$$

with the stated result. □

In the particular case in which X is a hypersurface *of* $\mathbb{P}^n$, the formula in part (b) reduces to

$$s(Y,X) = [P]^\vee \otimes \mathcal{L} + [X] \quad ;$$

as the reader may check, this is equivalent to Piene's *Plücker formulae* (cf. Theorem 2.3 in [15]).

Corollary 1.2 follows from part (a), Theorem 1.1, and straightforward manipulations.

4. Remarks and examples

Regarding the computability of the key coefficient ρ needed in order to apply Theorem 1.1, the following observation may be useful.

Proposition 4.1. *With notations and assumptions as in § 2,*

$$(1+X)\,(c_{Ma}(X) - c_F(X)) = ((Eu-\chi) + (Eu-1)X)\cdot(c(TY')\cap[Y']) \quad .$$

The point is that if $c_{\mathrm{Ma}}(X)$, $c_{\mathrm{F}}(X)$, and $c(TY')$ are known, and $X\cdot\dim Y' \neq 0$ (for example, $\dim Y' > 0$ if $M = \mathbb{P}^n$), then this formula determines $(\mathrm{Eu}-1)$ and $(\mathrm{Eu}-\chi)$; ρ is the quotient of these two numbers.

Proof. Since Y' is assumed to be nonsingular, the weighted Chern-Mather class of Y (cf. [3] and Lemma 2.3) is given by

$$c_{\mathrm{wMa}}(Y) = mc(TY')\cap[Y'] = (-1)^{\dim X - \dim Y}(\chi - 1)c(TM)\cap s(Y',M) \quad ;$$

on the other hand, by Proposition 1.3 in [3]

$$\begin{aligned} c_{\mathrm{wMa}}(Y) &= (-1)^{\dim Y}(c(T^*M\otimes\mathcal{L})\cap s(Y,M))_{\vee\mathcal{L}} \\ &= (-1)^{\dim M - \dim Y} c(TM)\cap(s(Y,M)^\vee\otimes\mathcal{L}) \quad . \end{aligned}$$

Therefore

$$s(Y,M)^\vee\otimes\mathcal{L} = (1-\chi)c(N_{Y'}M)^{-1}\cap[Y'] \quad .$$

Now arguing as in the proof of Lemma 2.4:

$$\begin{aligned} s(Y,X)^\vee\otimes\mathcal{L} &= \frac{\mathrm{Eu}-\chi}{1-\chi}\,\frac{1+\frac{\mathrm{Eu}-1}{\mathrm{Eu}-\chi}X}{1+X}\,(s(Y,M)^\vee\otimes\mathcal{L}) \\ &= \frac{(\mathrm{Eu}-\chi)+(\mathrm{Eu}-1)X}{1+X}\,c(N_{Y'}M)^{-1}\cap[Y'] \end{aligned}$$

and the statement follows from the expression for $s_{\mathrm{Ma}}(X,M)$ obtained in the proof of Theorem 1.1. $\square$

Example 4.1. *The tangent developable surface of the twisted cubic in* $\mathbb{P}^3$*.* For a concrete example, let X be the surface obtained as the union of all tangent lines to a fixed twisted cubic curve in $\mathbb{P}^3$. It is a standard but pleasant exercise to check that

—X is a surface of degree 4;

—its singular locus Y' is the twisted cubic;

—the polar classes of X are: $[P_0] = X$; $[P_1] = 3[\mathbb{P}^1]$; $[P_2] = 0$.

The numerical invariants of X are clearly constant along the twisted cubic: indeed, X is invariant under an action of PGL(2) that is transitive along the singular locus. Hence X is a nice hypersurface of $M = \mathbb{P}^3$.

Note that in this example Eu equals the multiplicity of X along Y' (for example by the fundamental formula in [12], §5); however, this multiplicity is not available without a local study of X. Also, we do not know of any direct way to compute χ that does not involve a deeper local study of X.

The global information listed above suffices however to determine both Eu and χ in this example, by means of Proposition 4.1. Indeed, we have

$$[P] = \frac{[X]}{1+H} - \frac{3[\mathbb{P}^1]}{(1+H)^2} = [X] - 7[\mathbb{P}^1] + 10[\mathbb{P}^2]$$

(where H denotes the hyperplane class), and $c(T\mathbb{P}^3) \cap [P] = c_{\mathrm{Ma}}(X, M)$ by Theorem 3.1 (a). Hence

$$(1+X)\left(c_{\mathrm{Ma}}(X) - c_{\mathrm{F}}(X)\right) = c(T\mathbb{P}^3) \cap ((1+X)[P] - [X]) = 9[\mathbb{P}^1] + 18[\mathbb{P}^0] \quad .$$

Applying Proposition 4.1 gives then

$$9[\mathbb{P}^1] + 18[\mathbb{P}^0] = ((\mathrm{Eu} - \chi) + (\mathrm{Eu} - 1)X) \cdot (3[\mathbb{P}^1] + 2[\mathbb{P}^0]) \quad ,$$

from which $\mathrm{Eu} - \chi = 3$ and $7\,\mathrm{Eu} - \chi = 15$. Therefore

$$\mathrm{Eu} = 2 \quad , \quad \chi = -1 \quad .$$

Hence $\rho = \frac{1}{3}$ here, and by Corollary 1.2

$$c_{\mathrm{SM}}(X) = c(T\mathbb{P}^3) \cap \frac{\frac{1}{3}[X] + \frac{2}{3}[P]}{1 + \frac{1}{3}X} = [X] + 6[\mathbb{P}^1] + 4[\mathbb{P}^0] \quad .$$

We end with perhaps the simplest example in which our formula does *not* apply.

Example 4.2. Consider a reduced plane curve of degree $d \geq 3$ with exactly one node, and let X be the cone in $\mathbb{P}^3$ over this curve. The singular locus Y' of X is then a line L, but the singularity scheme Y is 'fatter' at the vertex of the cone. The invariants considered here detect this feature of Y: it is not hard to check that $(\chi, \mathrm{Eu}) = (0, 2)$ at all points of L but the vertex, while $(\chi, \mathrm{Eu}) = (d(d-1)(d-2), 2 + 2d - d^2)$ at the vertex; in particular, these numbers are not constant along Y', so X is not 'nice'.

For this example we have $[P_0] = [X]$, $[P_1] = d^2 - d - 2$ lines through the vertex, and $[P_2] = 0$. The push-forward to $[\mathbb{P}^3]$ of the class $c_\alpha(X)$ defined in §1 is

$$d[\mathbb{P}^2] + \left(2 + 4d - d^2 - 2\alpha\right)[\mathbb{P}^1] + \left(4 + 5d - 2d^2 + (-4 - d - 2d^2 + d^3)\alpha + 2d\alpha^2\right)[\mathbb{P}^0].$$

It is immediate to check that this expression does *not* equal the push forward of $c_{\mathrm{SM}}(X)$:

$$d[\mathbb{P}^2] + (1 + 4d - d^2)[\mathbb{P}^1] + (2 + 3d - d^2)[\mathbb{P}^0]$$

for any value of α. Note however that the value $\alpha = \frac{1}{2}$ corresponding to $(\chi, \mathrm{Eu}) = (0, 2)$ at the general point on L *does* yield the correct term in codimension 1: indeed, the invariants jump on a locus of codimension 2 in X, so (as observed in the introduction) the formula in Theorem 1.1 is correct for all terms of lower codimension in X.

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Mathematics Department, Florida State University, Tallahassee FL 32306, U.S.A.
E-mail address: `aluffimath.fsu.edu`

IML – CNRS, Case 907, Luminy, 13288 Marseille Cedex 9, France
E-mail address: `jpbiml.univ-mrs.fr`