CHERN CLASSES OF GRAPH HYPERSURFACES AND DELETION-CONTRACTION RELATIONS

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Abstract. We study the behavior of the Chern classes of graph hypersurfaces under the operation of deletion-contraction of an edge of the corresponding graph. We obtain an explicit formula when the edge satisfies two technical conditions, and prove that both these conditions hold when the edge is multiple in the graph. This leads to recursions for the Chern classes of graph hypersurfaces for graphs obtained by adding parallel edges to a given (regular) edge.

Analogous results for the case of Grothendieck classes of graph hypersurfaces were obtained in previous work, and both Grothendieck classes and Chern classes were used to define 'algebro-geometric' Feynman rules. The results in this paper provide further evidence that the polynomial Feynman rule defined in terms of the Chern-Schwartz-MacPherson class of a graph hypersurface reflects closely the combinatorics of the corresponding graph.

The key to the proof of the main result is a more general formula for the Chern-Schwartz-MacPherson class of a transversal intersection (see §3), which may be of independent interest.

We also describe a more geometric approach, using the apparatus of 'Verdier specialization'.

1. Introduction

1.1. *Graph hypersurfaces* are hypersurfaces of projective space associated with the parametric formulation of Feynman integrals in scalar quantum field theories. The study of their geometry was prompted by certain conjectures concerning the appearance of multiple zeta values in the results of computation of Feynman amplitudes, and has been the object of intense investigation (see e.g., [BK97], [Ste98], [BB03], [BEK06], [Blo07], [Mar10], [BY11], [Dor], [BS], and many others). In this paper we study Chern classes of graph hypersurfaces, from the point of view of deletion-contraction and multiple-edge formulas.

1.2. 'Algebro-geometric Feynman rules' are invariants of graphs which only depend on the isomorphism class of the corresponding hypersurfaces, and have controlled behavior with respect to unions. (This definition of course captures only a very small portion of the quantum field theory Feynman rules; it would be very interesting to have examples of algebro-geometric Feynman rules mirroring more faithfully their physical counterparts.) Matilde Marcolli and the author note in [AM11a] that the classes of graph hypersurfaces in the Grothendieck ring of varieties satisfy this basic requirement; Grothendieck classes of graph hypersurfaces have been studied rather thoroughly, given their relevance to the conjectures mentioned above. In the same paper we produced a different example of algebro-geometric Feynman rules, with values in $\mathbb{Z}[t]$, based on the *Chern classes* of graph hypersurfaces. The definition of this polynomial invariant $C_\Gamma(t)$ will be recalled below; its main interest lies in the fact that it carries intersection-theoretic information about the singularities of graph hypersurfaces. For example, the (push-forward to projective space of the) Segre class of the singularity subscheme of a hypersurface may be recovered from the polynomial Feynman rules. The Milnor number of the hypersurface (in its natural generalization to arbitrary hypersurfaces as defined by Parusiński, [Par88]) is but one piece of the information carried by $C_\Gamma(t)$.

The fact that the invariant satisfies the basic requirements of algebro-geometric Feynman rules is proved in [AM11a], Theorem 3.6, and is substantially less straightforward than the corresponding fact for Grothendieck classes. Other properties of $C_\Gamma(t)$ are listed in [AM11a], Proposition 3.1.

Grothendieck classes of graph hypersurfaces also satisfy a 'deletion-contraction' relation: this fact has been pointed out by several authors, see e.g., [Ste98], [BEK06], [AM11b]. The purpose of this article is to examine deletion-contraction relations for Chern classes of graph hypersurfaces, in terms of the polynomial Feynman rules mentioned above. As in [AM11a], it is natural to expect the situation for Chern classes to be substantially subtler than for classes in the Grothendieck ring, and in fact our first guess concerning a double-edge formula for Chern classes, based on somewhat extensive evidence computed for small graphs, turns out to be *incorrect* as stated in Conjecture 6.1 in [AM11b]. Nevertheless, we will be able to show here that these invariants do satisfy the expected general structure underlying multiple-edge formulas examined in [AM11b].

1.3. The *graph hypersurface* associated with a graph Γ is the zero-set of the polynomial

$$\Psi_\Gamma = \sum_T \prod_{e\notin T} t_e$$

where T ranges over the maximal spanning forests of Γ. This is a homogeneous polynomial in variables t_e corresponding to the edges e of Γ, and its zero set may be viewed in $\mathbb{P}^{n-1}$ or $\mathbb{A}^n$, depending on the context.

Graph hypersurfaces are singular in all but the simplest cases, and in this article (as in [AM11a]) we employ the theory of *Chern-Schwartz-MacPherson* (CSM) classes. CSM classes are defined for arbitrary varieties, and agree with the ordinary (total homology) Chern class of the tangent bundle when evaluated on nonsingular varieties. The reader may refer to §2.2-3 of [AM09] for a quick summary of this theory, which has a long and well-documented history. CSM classes can be viewed as a generalization of the topological Euler characteristic: indeed, the degree of the CSM class of a variety *is* its Euler characteristic, and to some extent CSM classes maintain the same additive and multiplicative behavior of the Euler characteristic. In this respect they are similar in flavor to the Grothendieck class. They also offer a direct measure of 'how singular' a variety is, by comparison with other characteristic classes of singular varieties, see e.g. §2.2 in [AM09] and references therein.

CSM classes are in fact defined for *constructible functions* on a variety ([Mac74]), and what we call the CSM class of X is the class $c_{\mathrm{SM}}(1\!\!1_X)$ of the constant function $1\!\!1_X$.

As our objects of study are hypersurfaces of projective space, we view CSM classes as elements of the Chow group of projective space, i.e., as polynomials in the hyperplane class. The polynomial Feynman rules mentioned above are closely related to the CSM class of the *complement* of a graph hypersurface $X_\Gamma \subseteq \mathbb{P}^{n-1}$: if a graph Γ with n edges is not a forest, then the polynomial Feynman rules $C_\Gamma(t)$ are determined by the relation

$$c_{\mathrm{SM}}(1\!\!1_{\mathbb{P}^{n-1}\smallsetminus X_\Gamma}) = \big(H^n C_\Gamma(1/H)\big) \cap [\mathbb{P}^{n-1}]\,,$$

where H denotes the hyperplane class; see Prop. 3.7 in [AM11a].

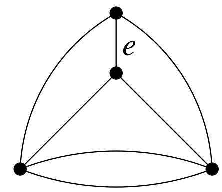
1.4. In graph theory, *deletion-contraction* formulas express invariants for a graph Γ directly in terms of invariants for the 'deletion' graph $\Gamma \smallsetminus e$ obtained by removing an edge e, and the 'contraction' graph Γ/e obtained by contracting the same edge. *Tutte-Grothendieck* invariants are the most general invariants with controlled behavior with respect to deletion-contraction; an impressive list of important graph invariants are of this kind, ranging from chromatic polynomials to partition functions for Potts models. In fact, these invariants may be viewed as 'Feynman rules' in a sense closely related to the one adopted in [AM11a], see Prop. 2.2 in [AM11b].

In [AM11b] we show that the invariant arising from the Grothendieck class satisfies a weak form of deletion-contraction (which involves 'non-combinatorial' terms); and that enough of this structure is preserved to trigger combinatorial 'multiple-edge' formulas. More precisely, let $\mathbb{U}(\Gamma) = [\mathbb{A}^n - \hat{X}_\Gamma]$ denote the Grothendieck class of the complement of the affine graph hypersurface, and denote by Γ_{2e} the graph obtained by doubling the edge e in Γ. If e is neither a bridge nor a looping edge, then ([AM11b], Proposition 5.2)

$$(*) \qquad \mathbb{U}(\Gamma_{2e}) = (\mathbb{T}-1)\,\mathbb{U}(\Gamma) + \mathbb{T}\,\mathbb{U}(\Gamma\smallsetminus e) + (\mathbb{T}+1)\,\mathbb{U}(\Gamma/e)\,,$$

where $\mathbb{T}$ is the class of $\mathbb{A}^1 \smallsetminus \mathbb{A}^0$. Note that (*) holds without further requirements on e. (Simpler formulas hold in case e is a bridge or a looping edge.)

1.5. As we will show in this paper, the situation concerning the polynomial invariant $C_\Gamma(t)$ recalled above is somewhat different. As in the case of the Grothendieck class, this invariant does not satisfy on the nose a deletion-contraction relation (this was already observed in [AM11b], Prop. 3.2). Unlike in the case of the invariant $\mathbb{U}(\Gamma)$, however, even a weaker non-combinatorial form of deletion-contraction only holds under special hypotheses on the edge e. The main result of this article is the determination of conditions on a pair (Γ, e) such that a sufficiently strong deletion-contraction relation (and corresponding consequences, such as multiple-edge formulas) holds for the edge e of Γ. These conditions are presented in §2; somewhat surprisingly, they appear to hold for 'many' graphs. One of them can be formulated as follows. Assume that e is neither a bridge nor a looping edge of Γ. We may consider the polynomial $\Psi_{\Gamma\smallsetminus e}$ for the deletion $\Gamma \smallsetminus e$. The condition is then that Ψ_Γ *belongs to the Jacobian ideal of* $\Psi_{\Gamma\smallsetminus e}$. The smallest counterexample to this requirement appears to be the graph



with respect to the vertical edge. We find it surprising that this condition is satisfied as often as it is.

The second condition is more technical. See §2 for a discussion of both conditions.

Once a (weak) deletion-contraction formula is available, one should expect combinatorial multiple-edge formulas to hold. And indeed, we will prove the following analogue of (*) for the Chern class Feynman rules:

Theorem 1.1. *If the conditions on* (Γ, e) *mentioned above are satisfied, then*

$$C_{\Gamma_{2e}}(t) = (2t-1)\, C_\Gamma(t) - t(t-1)\, C_{\Gamma\smallsetminus e}(t) + C_{\Gamma/e}(t)\,.$$

This is the formula that was proposed in [AM11b], Conjecture 6.1, on the basis of many examples computed explicitly in [Str11]. However, the additional conditions on (Γ, e) had not been identified at the time; the formula proposed in [AM11b] for the class of a triangle with doubled edges is incorrect, and will be corrected here in Example 5.6.

1.6. As mentioned above, we do not have a sharp combinatorial characterization on (Γ, e) ensuring that the technical hypotheses needed for Theorem 1.1 are satisfied. However, there is one important case in which we are able to prove that these conditions are indeed satisfied: the conditions hold if e is itself a multiple edge, i.e., if the endpoints of e are adjacent in $\Gamma \smallsetminus e$. Thus, Theorem 1.1 implies that the polynomial Feynman rules satisfy essentially the same recursive structure for multiple-edge formulas that is studied in general in [AM11b], §6. If e is neither a bridge nor a looping edge of Γ, then

$$C_{\Gamma^{(m+3)}}(t) = (3t-1)\, C_{\Gamma^{(m+2)}}(t) - (3t^2 - 2t)\, C_{\Gamma^{(m+1)}}(t) + (t^3 - t^2)\, C_{\Gamma^{(m)}}(t)\,.$$

In a sense, this recursion is 'nicer' than the corresponding one for Grothendieck classes (see the comments following Lemma 5.3).

1.7. This paper is organized as follows. In §2 we state precisely the technical conditions mentioned above, in the case of graph hypersurfaces, providing a few simple examples to illustrate them. This is also done in the hope that others may identify sharp combinatorial versions of these conditions. We prove (Lemma 2.3) that the conditions hold for (Γ, e) if e has parallel edges in Γ, and describe one class of examples in which the conditions do not (both) hold. In §3 we discuss a formula for the CSM class of a transversal intersection, needed for the application to graph hypersurfaces presented here; this section can be read independently of the rest of the paper. In §4 we apply these formulas to the case of graph hypersurfaces, obtaining the deletion-contraction relation (Theorem 4.7). In §5 we apply this relation to obtain multiple-edge formulas as mentioned above (Theorem 5.2, Lemma 5.3). In §6 we describe a different and more 'geometric' (but in practice less applicable) approach to the main deletion-contraction formula, using *Verdier's specialization.*

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2. Two technical conditions

We work over an algebraically closed field k of characteristic zero.

2.1. As in §1, Γ denotes a finite graph, with n edges; we allow looping edges as well as parallel edges. We associate with each edge e a variable t_e, and we consider the *graph polynomial*

$$\Psi_\Gamma = \sum_T \prod_{e \notin T} t_e \,,$$

where T ranges over the maximal spanning forests of Γ. (Note: According to this definition, the polynomial for a graph is the product of the polynomials for its connected components.)

We denote by X_Γ the projective hypersurface defined by $\Psi_\Gamma = 0$. We present in this section two conditions for a pair (Γ, e), where e is an edge of Γ, encoding some geometric features of X_Γ. Finding more transparent, combinatorial versions of these conditions is an interesting problem.

We will say that an edge e of Γ is *regular* if e is neither a bridge nor a looping edge, and further $\Gamma \smallsetminus e$ is not a forest. We will essentially always assume that e is regular on Γ; non-regular edges are easy to treat separately (see e.g., §4.7).

If e is a regular edge of Γ, then

$$\Psi_\Gamma = t_e \Psi_{\Gamma \smallsetminus e} + \Psi_{\Gamma/e}\,;$$

this is well-known, and easily checked. As $\Psi_{\Gamma \smallsetminus e}$ is not a forest, $\deg \Psi_{\Gamma \smallsetminus e} > 0$; in this case a point p of X_Γ is determined by setting all variables except t_e to 0. We denote by $\widetilde{X}_\Gamma$ the blow-up of X_Γ at p, and by E the exceptional divisor of this blow-up. The variety $\widetilde{X}_\Gamma$ may be realized as a hypersurface in the blow-up of $\mathbb{P}^{n-1}$ at p. Denoting by D the exceptional divisor of this latter blow-up, E is the intersection $D \cap \widetilde{X}_\Gamma$. Heuristically, the conditions we will present below amount to requiring this intersection to be sufficiently transversal.

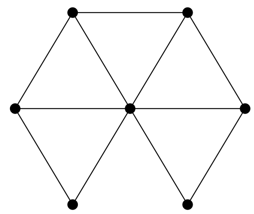
2.2. Assume e is regular. Both $X_{\Gamma \smallsetminus e}$ and $X_{\Gamma/e}$ are hypersurfaces of a projective space $\mathbb{P}^{n-2}$ with homogeneous coordinates corresponding to the edges of Γ other than e. The first condition on the pair (Γ, e) may be expressed as a relation between them:

$$\text{(Condition I)} \qquad \Psi_{\Gamma/e} \in (\partial \Psi_{\Gamma \smallsetminus e})\,.$$

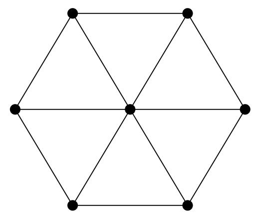
Here, $(\partial \Psi_{\Gamma \smallsetminus e})$ denotes the ideal of partial derivatives of $\Psi_{\Gamma \smallsetminus e}$, defining the *singularity subscheme* $\partial X_{\Gamma \smallsetminus e}$ of $X_{\Gamma \smallsetminus e}$. The condition essentially (that is, up to saturating $(\partial \Psi_{\Gamma \smallsetminus e})$) amounts to requiring $\partial X_{\Gamma \smallsetminus e}$ to be a subscheme of $X_{\Gamma/e}$.

It is of course easy to verify whether this condition holds on any given graph, by employing a symbolic manipulation package such as Macaulay2 ([GS]). The following examples illustrate a few cases, showing in particular that the condition depends on global features of the graph.

Example 2.1. For the graph

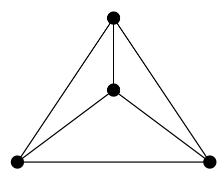


condition I is satisfied with respect to all edges. For the wheel

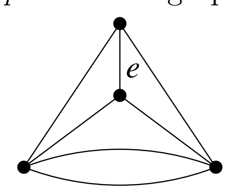


condition I is satisfied with respect to the spokes, and it is not satisfied with respect to the rim edges.

Condition I is satisfied with respect to all edges for the graph



and with respect to all edges *except* e for the graph



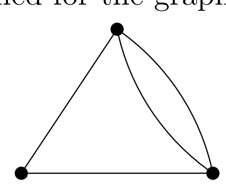
This is the 'smallest' example not satisfying condition I. ⌟

2.3. The second condition we will consider is more technical than condition I. Let e be a regular edge on Γ, and consider the blow-up introduced in §2.1. For any point q of $E \cap \partial\widetilde{X}_\Gamma$, let I be the ideal of $\partial\widetilde{X}$ at q, and denote by u an equation for E at q. The second condition we must consider on the pair (Γ, e) is

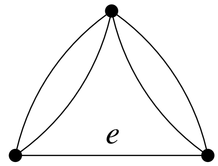
(Condition II) For all $q \in E \cap \partial\widetilde{X}_\Gamma$, u is a non-zero-divisor modulo I^j for $j \gg 0$.

Again, checking this condition on any given graph is possible with a tool such as Macaulay2, although computing power will limit the size of graphs that can be analyzed in practice. (Note that, by Artin-Rees, only finitely many j need be checked.)

Example 2.2. Condition II is verified for the graph



with respect to all edges, while it does *not* hold for



with respect to e. (Both assertions may be verified with Macaulay2; also, see §2.6 for a generalization.) ⌟

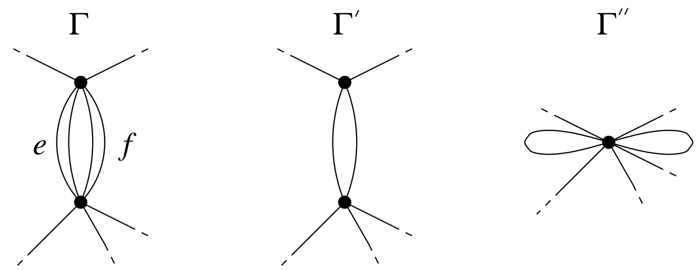
2.4. We would be interested in purely combinatorial interpretations in terms of Γ and e of the conditions presented in §2.2 and §2.3; it is not even clear to us that such sharp characterizations exist. However, we can provide one combinatorial situation in which both conditions are satisfied, and this situation is at the root of the application to multiple-edge formulas in §5.

Lemma 2.3. *Let Γ be a graph, and let e be a regular edge that has parallel edges in Γ. Then both conditions I and II are verified for (Γ, e).*

Proof. Let f be an edge parallel to e. We first assume that f is not a bridge in $\Gamma \smallsetminus e$. Then

$$\Psi_\Gamma = t_e\Psi_{\Gamma\smallsetminus e} + \Psi_{\Gamma/e} = t_e(t_f\Psi_{\Gamma'} + \Psi_{\Gamma''}) + t_f\Psi_{\Gamma''}$$

where $\Gamma' = (\Gamma \smallsetminus e) \smallsetminus f$ and $\Gamma'' = (\Gamma/e)/f$.



Condition I. Among the partials of $\Psi_{\Gamma\smallsetminus e}$ is $\Psi_{\Gamma'} = \frac{\partial\Psi_{\Gamma\smallsetminus e}}{\partial t_f}$. Since $\Psi_{\Gamma\smallsetminus e}$ is homogeneous (and we are in characteristic zero), it is in the ideal of partials $(\partial\Psi_{\Gamma\smallsetminus e})$. It follows that so is $\Psi_{\Gamma''}$, and as a consequence $\Psi_{\Gamma/e} = t_f\Psi_{\Gamma''} \in (\partial\Psi_{\Gamma\smallsetminus e})$ as needed.

Condition II. Since the condition only depends on the part of $\widetilde{X}_\Gamma$ over p, it is unaffected by analytic changes of coordinates at p. First, we can center an affine chart at p by setting $t_e = 1$, and the equation of X_Γ in this chart is

$$t_f\Psi_{\Gamma'} + (1 + t_f)\Psi_{\Gamma''} = 0\,.$$

Next, we can set $t_f = \frac{\tau}{1-\tau}$, i.e., $\tau = \frac{t_f}{1+t_f}$; this does not affect the geometry of X_Γ near p (where $t_f = 0$). In coordinates τ, t_{e_i}, the equation for X_Γ is

$$\tau\Psi_{\Gamma'} + \Psi_{\Gamma''} = 0\,.$$

This equation is *homogeneous,* so X_Γ is a cone with vertex at p in these coordinates. It is then clear that condition II holds: the equations of $\widetilde{X}_\Gamma$ in the standard charts do not depend on the variable u defining the exceptional divisor, so at each $q \in E \cap \partial\widetilde{X}_\Gamma$

the ideal I has a set of generators independent of u, and it follows that u is a non-zero-divisor modulo I^j for all j.

This concludes the proof in the case in which f is not a bridge in $\Gamma \smallsetminus e$. If f is a bridge in $\Gamma \smallsetminus e$, then $\Psi_{\Gamma'} = \Psi_{\Gamma''}$, and one verifies easily that

$$\Psi_\Gamma = (t_e + t_f)\Psi_{\Gamma'}\,.$$

Since $\Psi_{\Gamma\smallsetminus e} = \Psi_{\Gamma'}$ in this case, it is immediate that $\Psi_\Gamma \in (\partial\Psi_{\Gamma\smallsetminus e})$. Further, X_Γ has equation

$$\Psi_{\Gamma'} = 0$$

in the affine chart $t_e = 1$, and near p. Again this is a cone with vertex at p, so condition II holds by the same argument used above. □

2.5. We now discuss more in detail the geometric meaning of the two conditions presented above. This will also clarify the sense in which the two conditions may be interpreted as transversality statements.

Claim 2.4. *If Condition I is satisfied, then* $\partial E = E \cap \partial\widetilde{X}_\Gamma$.

Proof. Recall that $\widetilde{X}$ denotes the blow-up of X at the point p obtained by setting t_e to 1 and all other coordinates $t_1, \dots, t_{n-1}$ to 0 (where n is the number of edges of Γ, and e is assumed to be a regular edge). Working in the affine chart $\mathbb{A}^{n-1}$ centered at p, X has equation

$$\Psi_{\Gamma\smallsetminus e} + \Psi_{\Gamma/e} = 0\,,$$

where the summands are homogeneous polynomials of degree $d-1$ and d respectively, with $d = b_1(\Gamma)$. We can cover the blow-up of $\mathbb{A}^{n-1}$ at p with standard coordinate patches; in one of them we have coordinates $(u_1, \dots, u_{n-2}, u)$ so that the blow-up map is given by

$$\begin{cases} t_1 & = u\,u_1 \\ & \dots \\ t_{n-2} & = u\,u_{n-2} \\ t_{n-1} & = u \end{cases}.$$

The equation of $\widetilde{X}$ in this chart is then

$$\Psi_{\Gamma\smallsetminus e}(u_1, \dots, u_{n-2}, 1) + u\,\Psi_{\Gamma/e}(u_1, \dots, u_{n-2}, 1) = 0\,,$$

and $u = 0$ is the equation of the exceptional divisor in this chart; thus, E has ideal

$$(u, \Psi_{\Gamma\smallsetminus e}(u_1, \dots, u_{n-2}, 1))$$

in this chart. Note that the exceptional divisor D of the blow-up of $\mathbb{A}^{n-1}$ is a projective space $\mathbb{P}^{n-2}$. The above computation (together to the same in the other patches) shows that $E \cong X_{\Gamma\smallsetminus e}$, a hypersurface in $D \cong \mathbb{P}^{n-2}$.

This computation also shows that $\partial E = \partial X_{\Gamma\smallsetminus e}$ has ideal $(\partial\Psi_{\Gamma\smallsetminus e})$. In the representative patch chosen above, this is

$$(\dagger) \qquad \left(u,\ \Psi_{\Gamma\smallsetminus e},\ \frac{\partial\Psi_{\Gamma\smallsetminus e}}{\partial u_i}\right)_{i=1,\dots,n-2}.$$

On the other hand, in the same patch, $\partial\widetilde{X}_\Gamma$ has ideal

$$\left(\Psi_{\Gamma\smallsetminus e}+u\,\Psi_{\Gamma/e},\ \Psi_{\Gamma/e},\ \frac{\partial\Psi_{\Gamma\smallsetminus e}}{\partial u_i}+u\,\frac{\partial\Psi_{\Gamma/e}}{\partial u_i}\right)_{i=1,\dots,n-2}$$

and hence $E\cap\partial\widetilde{X}_\Gamma$ has ideal

$$(\ddagger)\qquad \left(u,\ \Psi_{\Gamma\smallsetminus e},\ \Psi_{\Gamma/e},\ \frac{\partial\Psi_{\Gamma\smallsetminus e}}{\partial u_i}\right)_{i=1,\dots,n-2}.$$

Comparing (†) and (‡) (and the analogous ideals in all patches covering the blow-up), we see that the ideals agree if $\Psi_{\Gamma/e}\in(\partial\Psi_{\Gamma\smallsetminus e})$, that is, if Condition I holds. This verifies Claim 2.4. □

Remark 2.5. The picture we have in mind is that of the nonsingular exceptional divisor $D\cong\mathbb{P}^{n-2}$ intersecting $\widetilde{X}$ along E. According to Claim 2.4, Condition I implies that the intersection $E=D\cap\widetilde{X}_\Gamma$ is only singular along the intersection of D with the singularity subscheme of $\widetilde{X}_\Gamma$, as would be expected if D met $\widetilde{X}_\Gamma$ transversally. ⌟

In order to interpret Condition II, we have to introduce the blow-up $\mu:\widehat{X}_\Gamma\to\widetilde{X}_\Gamma$ of $\widetilde{X}_\Gamma$ along its singularity subscheme $\partial\widetilde{X}_\Gamma$. In this blow-up we may consider two subschemes: the proper transform $\widehat{E}$ (isomorphic to the blow-up of E along $E\cap\partial\widetilde{X}$) and the inverse image $\mu^{-1}(E)$.

Claim 2.6. *If Condition II holds, then $\widehat{E}=\mu^{-1}(E)$.*

Proof. Indeed, assume Condition II holds. Then letting $\mathcal{I}$, $\mathcal{J}$ denote respectively the ideal sheaves of $\partial\widetilde{X}_\Gamma$ and E in $\widetilde{X}_\Gamma$, Condition II implies $\mathcal{J}\cap\mathcal{I}^j=\mathcal{J}\cdot\mathcal{I}^j$ for $j\gg 0$, and hence the natural morphism

$$\frac{\mathcal{I}^j}{\mathcal{J}\cdot\mathcal{I}^j}\to\frac{\mathcal{I}^j+\mathcal{J}}{\mathcal{J}}$$

is an isomorphism for $j\gg 0$. It follows that the natural inclusion

$$\mathrm{Proj}_{\mathscr{O}_{\widetilde{X}}}\left(\bigoplus_j\frac{\mathcal{I}^j+\mathcal{J}}{\mathcal{J}}\right)=B\ell_{E\cap\partial\widetilde{X}}E\subseteq\mu^{-1}(E)=\mathrm{Proj}_{\mathscr{O}_{\widetilde{X}}}\left(\bigoplus_j\frac{\mathcal{I}^j}{\mathcal{J}\cdot\mathcal{I}^j}\right)$$

is an equality, verifying Claim 2.6. □

Remark 2.7. By the same token, the proper transform of the divisor D equals its inverse image in the blow-up along $\partial\widetilde{X}_\Gamma$. This is the behavior expected if the (nonsingular) hypersurface D meets the center of a blow-up transversally, so Condition II, like Condition I, appears to express a measure of transversality of the intersection of D with $\widetilde{X}$. ⌟

Remark 2.8. The two conditions differ: for example, Condition II does not hold for the second graph displayed in Example 2.2 with respect to edge e, while Condition I does hold in this case.

However, we do not know of examples of graphs for which Condition II holds and Condition I does not. It is conceivable that such examples exist (cf. §3.3). ⌟

2.6. Finally, we discuss one case in which conditions I and II do not both hold. If a graph is obtained by taking the union of two graphs Γ', Γ'', joined at a vertex, then its graph polynomial equals the product $\Psi_{\Gamma'} \cdot \Psi_{\Gamma''}$. If neither Γ' nor Γ'' is a forest, we say that the graph is 'disjoinable'; note that the graph hypersurface of a disjoinable graph is singular in codimension 1 (We do not know whether the converse holds.)

Claim 2.9. *Let Γ be a graph such that X_Γ is nonsingular in codimension 1. Let e be an edge such that $\Gamma \smallsetminus e$ is disjoinable. Then at least one of conditions I and II fails for (Γ, e).*

For example, the second graph drawn in Example 2.2 is of this type, with respect to the bottom edge. (The first is not, since while removing the bottom edge does produce the join of two graphs, one of these is a tree.)

Proof. If condition I does not hold, we are done; so we may assume that condition I holds, and we will show that condition II does not hold in this case.

We use notation as in the discussion following the statement of Claim 2.4. After setting t_e to 1, we have

$$\Psi_\Gamma = \Psi_{\Gamma'} \cdot \Psi_{\Gamma''} + \Psi_{\Gamma/e}$$

by assumption, where $\Psi_{\Gamma'}$ and $\Psi_{\Gamma''}$ are graph polynomials of degree ≥ 1. In the chart of $\widetilde{\mathbb{A}}^{n-1}$ with coordinates $(u_1, \dots, u_{n-2}, u)$, the equation of $\widetilde{X}$ is

$$\Psi_{\Gamma'}(u_1, \dots, u_{n-2}, 1) \cdot \Psi_{\Gamma''}(u_1, \dots, u_{n-2}, 1) + u\, \Psi_{\Gamma/e}(u_1, \dots, u_{n-2}, 1) = 0\,,$$

and E has ideal

$$(u, \Psi_{\Gamma'}(u_1, \dots, u_{n-2}, 1) \cdot \Psi_{\Gamma''}(u_1, \dots, u_{n-2}, 1))\,.$$

In particular, the singularity subscheme ∂E of E contains the locus Z with ideal $(u, \Psi_{\Gamma'}(u_1, \dots, u_{n-2}, 1), \Psi_{\Gamma''}(u_1, \dots, u_{n-2}, 1))$. As we are assuming that condition I holds, we have that $\partial E = E \cap \partial\widetilde{X}$ (cf. Claim 2.4); in particular, $Z \subseteq \partial\widetilde{X}$. On the other hand, note that Z has codimension 1 in E, hence codimension 2 in $\widetilde{X}$; since X is nonsingular in codimension 1 and $Z \subseteq E$, Z must consist of a collection of components of $\partial\widetilde{X}$. But then E contains components of $\partial\widetilde{X}$, and it follows that the condition in Claim 2.6 is not verified. Hence (Γ, e) does not satisfy condition II. $\square$

3. CSM classes of transversal intersections

3.1. In this section we discuss a formula expressing the Chern-Schwartz-MacPherson class of the intersection of a variety X with a hypersurface D, in terms of $c_{\mathrm{SM}}(X)$ and of the class of D. This section can be read independently of the rest of the paper.

Our template is the transversal intersection of nonsingular varieties. Let V be a nonsingular variety, and let D, X be nonsingular subvarieties of V. Assume that D is a hypersurface, and that D and X meet transversally. In this case $D \cap X$ is a nonsingular hypersurface of X, and $\mathscr{O}_X(D \cap X) = \mathscr{O}_V(D)|_X$, hence (harmlessly abusing notation)

$$c(T(D \cap X)) \cap [D \cap X] = \frac{c(TX)}{c(N_D V)} \cap [D \cap X] = \frac{D}{1 + D} \cap c(TX) \cap [X]\,,$$

i.e.,

$$c_{\mathrm{SM}}(D\cap X)=\frac{D}{1+D}\cap c_{\mathrm{SM}}(X)\,.$$

(This equality holds in A_*X.) We are interested in generalizing this formula to the case in which X is possibly singular.

3.2. The key question is, of course, what 'transversal' should mean in the singular case. The conditions presented in §2 are precisely concocted to make this requirement precise.

We assume V is a nonsingular variety, D and X are reduced hypersurfaces of V, and D is nonsingular. We denote by ∂X the singularity subscheme of X, defined by the ideal of partial derivatives of a local equation. We denote by $\rho:\widetilde{V}\to V$ the blow-up along ∂X. Further, $\widetilde{D}$ denotes the proper transform of D in $\widetilde{V}$.

Theorem 3.1. *Assume that*

(1) $\partial(D\cap X)=D\cap\partial X$*;*
(2) $\widetilde{D}=\rho^{-1}D$*.*

Then

$$c_{SM}(D\cap X)=\frac{D}{1+D}\cap c_{SM}(X)$$

in A_*X*.*

In this proof we will make use of the following notation: for a class $\alpha=\oplus_i\alpha^i$ indexed by codimension in A_*V, and a line bundle $\mathscr{L}$, we let

$$\alpha^\vee=\sum_i(-1)^i\alpha^i\quad\text{and}\quad\alpha\otimes_V\mathscr{L}=\sum_i\frac{\alpha^i}{c(\mathscr{L})^i}\,.$$

Basic properties of this notation are listed in [Alu99], §1.4.

Proof. Our main tool is Theorem I.4 in [Alu99], which relates the Chern-Schwartz-MacPherson class of a hypersurface with the Segre class of its singularity subscheme. Viewing $D\cap X$ as a hypersurface in D, this result yields

$$c_{\mathrm{SM}}(D\cap X)=c(TD)\cap\left(s(D\cap X,D)+c(\mathscr{O}(X))^{-1}\cap(s(\partial(D\cap X),D)^\vee\otimes_D\mathscr{O}(X))\right)\;;$$

by the same token,

$$c_{\mathrm{SM}}(X)=c(TV)\cap\left(s(X,V)+c(\mathscr{O}(X))^{-1}\cap(s(\partial X,V)^\vee\otimes_V\mathscr{O}(X))\right)\;.$$

It follows that

$$\frac{D}{1+D}\cap c_{\mathrm{SM}}(X)=c(TD)\cap(s(D\cap X,D)+D\cdot c(\mathscr{O}(X))^{-1}\cap(s(\partial X,V)^\vee\otimes_V\mathscr{O}(X))\,.$$

Therefore, the formula stated in Theorem 3.1 would follow from the equality

$$D\cdot c(\mathscr{O}(X))^{-1}\cap(s(\partial X,V)^\vee\otimes_V\mathscr{O}(X))=c(\mathscr{O}(X))^{-1}\cap(s(\partial(D\cap X),D)^\vee\otimes_D\mathscr{O}(X))\,,$$

and hence from

$$(D\cdot s(\partial X,V))^\vee\otimes_D\mathscr{O}(X))=s(\partial(D\cap X),D)^\vee\otimes_D\mathscr{O}(X)\,,$$

and finally from

$$D\cdot s(\partial X,V)=s(\partial(D\cap X),D)$$

(in A_*X). This reduces the proof of Theorem 3.1 to the following claim:

Claim 3.2. *Under hypotheses (1) and (2) from the statement of Theorem 3.1,*

$$s(\partial(D\cap X), D) = D\cdot s(\partial X, V)$$

in $A_*(D\cap X)$.

To prove this, consider the blow-up of V along ∂X, with exceptional divisor F; the proper transform of D may be viewed as the blow-up of the latter along $D\cap\partial X$:

$$\begin{tikzcd}
& F \arrow[rr, hook] \arrow[dd, "\sigma"] & & \widetilde{V} \arrow[dd, "\rho"] \\
\widetilde{D}\cap F \arrow[ur, hook] \arrow[rr, hook] \arrow[dd, "\sigma'"'] & & \widetilde{D} \arrow[ur, hook] \arrow[dd] & \\
& \partial X \arrow[rr, hook] & & V \\
D\cap\partial X \arrow[ur, hook] \arrow[rr, hook] & & D \arrow[ur, hook] &
\end{tikzcd}$$

By the birational invariance of Segre classes,

$$s(\partial X, V) = \sigma_* s(F, \widetilde{V}) = \sigma_*\left(\frac{F}{1+F}\cap[\widetilde{V}]\right)$$
$$s(D\cap\partial X, D) = \sigma'_* s(\widetilde{D}\cap F, \widetilde{D}) = \sigma'_*\left(\frac{F}{1+F}\cap[\widetilde{D}]\right)\,.$$

By hypothesis (2), $\rho^{-1}D = \widetilde{D}$, hence $c_1(\sigma^*\mathscr{O}(D)|_{\partial X})$ is represented by $\widetilde{D}\cap F$. Therefore, by the projection formula ([Ful84], Proposition 2.3 (c))

$$\sigma'_*\left(\frac{F}{1+F}\cap[\widetilde{D}]\right) = \sigma'_*\left(\frac{[\widetilde{D}\cap F]}{1+F}\right) = \sigma'_*\left(\sigma^*D\cdot\frac{F}{1+F}\cap[\widetilde{V}]\right) = D\cdot s(\partial X, V)\,.$$

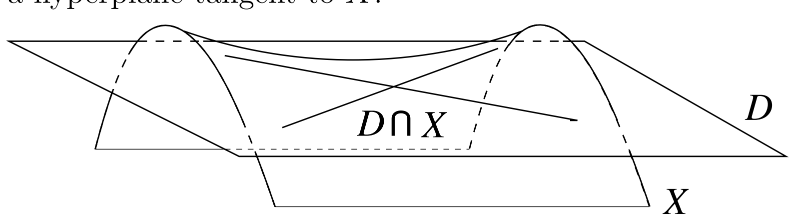
Thus,

$$s(D\cap\partial X, D) = D\cdot s(\partial X, V)\,.$$

Finally, $D\cap\partial X = \partial(D\cap X)$ by hypothesis (1), and the claim follows. This concludes the proof of Theorem 3.1. $\square$

3.3. Theorem 3.1 is sharp, in the sense that both hypotheses are necessary for the stated formula to hold.

To see that the first hypothesis is needed, let X be a nonsingular quadric in $\mathbb{P}^3$, and let D be a hyperplane tangent to X.



Then $D \cap X$ is singular, while $\partial X = \emptyset$; in particular, (1) is not satisfied. Working in the ambient $[\mathbb{P}^3]$ for convenience, we have

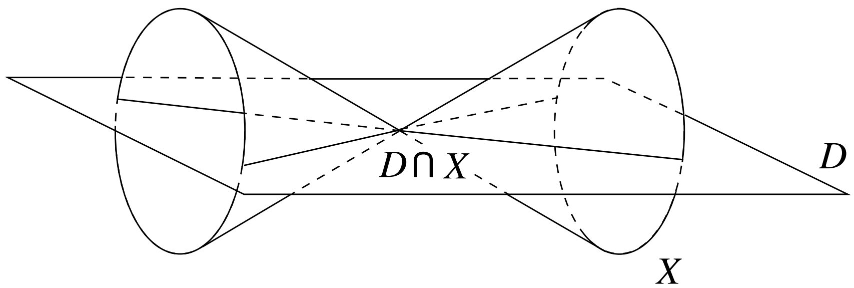
$$c_{\mathrm{SM}}(D \cap X) = 2[\mathbb{P}^1] + 3[\mathbb{P}^0]$$

and

$$\frac{D}{1+D} \cap c_{\mathrm{SM}}(X) = \frac{H}{1+H} \cap (2[\mathbb{P}^2] + 4[\mathbb{P}^1] + 4[\mathbb{P}^0]) = 2[\mathbb{P}^1] + 2[\mathbb{P}^0]$$

(where H is the hyperplane class). Therefore, the stated formula does not hold. Note that the second hypothesis holds (trivially) in this case, since $\partial X = \emptyset$.

To see that the second hypothesis is also necessary, let $X \subseteq \mathbb{P}^3$ be a quadric cone, and let D be a general hyperplane through the vertex.



Then (2) fails as D contains ∂X. Again working in $\mathbb{P}^3$, we have

$$c_{\mathrm{SM}}(D \cap X) = 2[\mathbb{P}^1] + 3[\mathbb{P}^0]$$

and

$$\frac{D}{1+D} \cap c_{\mathrm{SM}}(X) = \frac{H}{1+H} \cap (2[\mathbb{P}^2] + 4[\mathbb{P}^1] + 3[\mathbb{P}^0]) = 2[\mathbb{P}^1] + 2[\mathbb{P}^0]$$

as before. (The coefficient of $[\mathbb{P}^0]$ in $c_{\mathrm{SM}}(X)$ is in fact irrelevant to this computation.) Note that $\partial(D \cap X) = \partial X =$ the vertex of the cone, and in particular $\partial(D \cap X) = D \cap \partial X$, so that the first hypothesis does hold in this case.

4. Deletion-contraction for Chern classes of graph hypersurfaces

4.1. Now we return to the case of graph hypersurfaces. Recall that Γ denotes a graph with n edges; e denotes a regular edge of Γ; $X_\Gamma \subseteq \mathbb{P}^{n-1}$ is the corresponding hypersurface. As $\Gamma \smallsetminus e$ is not a forest, then the point p obtained by setting all coordinates except t_e to zero is a point of X_Γ. We will also implicitly assume that Γ is not disjoinable, and hence that X_Γ is irreducible. This may be done without loss of generality, since C_Γ is multiplicative with respect to joining graphs ([AM11a]).

We consider the blow-up $\widetilde{\mathbb{P}}^{n-1} \to \mathbb{P}^{n-1}$ at p, and the blow-up $\widetilde{X}_\Gamma$ of X_Γ at p realized as the proper transform of X_Γ in $\widetilde{\mathbb{P}}^{n-1}$. We denote D the exceptional divisor in $\widetilde{\mathbb{P}}^{n-1}$, so that $E = D \cap \widetilde{X}_\Gamma$ is the exceptional divisor in $\widetilde{X}_\Gamma$. Applying Theorem 3.1 to this situation yields:

Corollary 4.1. *Assume conditions I and II from §2 hold for* (Γ, e). *Then*

$$c_{SM}(E) = \frac{D}{1+D} \cap c_{SM}(\widetilde{X}_\Gamma)\,.$$

Indeed, as observed in §2.5, conditions I and II imply the hypotheses of Theorem 3.1.

4.2. We are ready to prove a deletion-contraction formula for Chern classes of graph hypersurfaces, subject to the conditions presented in §2.

We can view the blow-up $\widetilde{\mathbb{P}}^{n-1}$ as the graph of the projection $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-2}$ centered at p:

$$\begin{tikzcd} & \widetilde{\mathbb{P}}^{n-1} \arrow[dl] \arrow[dr] & \\ \mathbb{P}^{n-1} \arrow[rr, dashed] & & \mathbb{P}^{n-2} \end{tikzcd}$$

and likewise for $\widetilde{X}_\Gamma$:

$$\begin{tikzcd} & \widetilde{X}_\Gamma \arrow[dl, "\nu"'] \arrow[dr, "\pi"] & \\ X_\Gamma \arrow[rr, dashed] & & \mathbb{P}^{n-2} \end{tikzcd}$$

This situation was briefly described in [AM11b], end of §4: ν is the blow-up of X_Γ at p, while π realizes $\widetilde{X}_\Gamma$ as the blow-up of $\mathbb{P}^{n-2}$ along $X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e}$; the fibers of π are points away from $X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e}$, and $\mathbb{P}^1$ over over points of $X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e}$. The exceptional divisor E of ν is a copy of $X_{\Gamma\smallsetminus e}$ in $D \cong \mathbb{P}^{n-2}$ (as was verified in §2.5). The restriction of $\widetilde{\mathbb{P}}^{n-1} \to \mathbb{P}^{n-2}$ to D gives an isomorphism $D \to \mathbb{P}^{n-2}$.

We are aiming for formulas involving the polynomial 'Feynman rules' $C_\Gamma(t)$ carrying the information of the CSM class of X_Γ. Denote by H the hyperplane class in $\mathbb{P}^{n-1}$. Our main objective is essentially a formula for the polynomial

$$\sum_{i\geq 0} t^i \int H^i \cap c_{\mathrm{SM}}(X_\Gamma)$$

encoding the degrees of the terms in $c_{\mathrm{SM}}(X_\Gamma)$; $C_\Gamma(t)$ may be computed easily from this polynomial.

4.3. We let H, resp. h, denote the class in $\mathbb{P}^{n-1}$, resp. $\mathbb{P}^{n-2}$. The Chow group of $\widetilde{X}_\Gamma$ is generated by $\pi^*(h)$ and the class E of the exceptional divisor.

Lemma 4.2. *With notation as above,*

$$\sum_{i\geq 0} t^i \int H^i \cap c_{SM}(X_\Gamma) = 1 - \chi(X_{\Gamma\smallsetminus e}) + \int \frac{1+tE}{1-t\,\pi^* h} \cap c_{SM}(\widetilde{X}_\Gamma)\,.$$

Proof. By the functoriality property of CSM classes,

$$\begin{aligned} \nu_*(c_{\mathrm{SM}}(\widetilde{X}_\Gamma)) = \nu_*(c_{\mathrm{SM}}(1\!\!1_{\widetilde{X}_\Gamma})) = c_{\mathrm{SM}}(\nu_* 1\!\!1_{\widetilde{X}_\Gamma}) &= c_{\mathrm{SM}}(1\!\!1_{X_\Gamma} + (\chi(E)-1)1\!\!1_p) \\ &= c_{\mathrm{SM}}(X_\Gamma) + (\chi(E)-1)[p]\,. \end{aligned}$$

Using that $E \cong X_{\Gamma\smallsetminus e}$, and applying the projection formula, this gives

$$\begin{aligned}\sum_{i\ge 0} t^i \int H^i \cap c_{\rm SM}(X_\Gamma) &= 1-\chi(X_{\Gamma\smallsetminus e}) + \sum_{i\ge 0} \nu_*\left(t^i \int (\nu^* H)^i \cap c_{\rm SM}(\widetilde{X}_\Gamma)\right)\\ &= 1-\chi(X_{\Gamma\smallsetminus e}) + \int \frac{1}{1-t\,\nu^*H} \cap c_{\rm SM}(\widetilde{X}_\Gamma)\,,\end{aligned}$$

with the last equality due to the fact that push-forwards preserve degrees, and condensing the summation into a rational function for notational convenience.

Now I claim that $\nu^*H = E + \pi^* h$: indeed, this may be verified by realizing H as the class of a general hyperplane containing p. Also, note that $E\cdot \nu^* H = 0$: realize H as the class of a hyperplane *not* containing p to verify this. Therefore,

$$\frac{1}{1-t\,\nu^*H} - \frac{1+tE}{1-t\,\pi^*h} = \frac{t^2 E\cdot \nu^* H}{(1-t\,\nu^*H)(1-t\,\pi^*h)} = 0\,.$$

The statement follows. □

Lemma 4.3.

$$\int \frac{1}{1-t\,\pi^*h} \cap c_{SM}(\widetilde{X}_\Gamma) = \frac{(1+t)^{n-1}-1}{t} + \sum_{i\ge 0} t^i \int h^i \cap c_{SM}(X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e})\,.$$

Proof. By the projection formula, and since push-forwards preserve degree,

$$\int \frac{1}{1-t\,\pi^*h} \cap c_{\rm SM}(\widetilde{X}_\Gamma) = \sum_{i\ge 0} t^i \int h^i \cap \pi_*(c_{\rm SM}(\widetilde{X}_\Gamma))\,.$$

Applying the functoriality of CSM classes:

$$\begin{aligned}\pi_*(c_{\rm SM}(\widetilde{X}_\Gamma)) &= \pi_*(c_{\rm SM}(1\!\!1_{\widetilde{X}_\Gamma})) = c_{\rm SM}(\pi_* 1\!\!1_{\widetilde{X}_\Gamma}) = c_{\rm SM}(1\!\!1_{\mathbb{P}^{n-2}} + 1\!\!1_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}})\\ &= c(T\mathbb{P}^{n-2})\cap [\mathbb{P}^{n-2}] + c_{\rm SM}(X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e})\,,\end{aligned}$$

where we have used the description of the fibers of π recalled in §4.2. As $c(T\mathbb{P}^{n-2}) = (1+h)^{n-1} - h^{n-1}$, the statement follows. □

4.4. Combining Lemma 4.2 and 4.3, we obtain that *if e is a regular edge on Γ, then*

$$\begin{aligned}\sum_{i\ge 0} t^i \int H^i \cap c_{\rm SM}(X_\Gamma) &= 1-\chi(X_{\Gamma\smallsetminus e}) + \frac{(1+t)^{n-1}-1}{t}\\ &\quad + \sum_{i\ge 0} t^i \int h^i \cap c_{\rm SM}(X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}) + \int \frac{t\,E}{1-t\,\pi^*h} \cap c_{\rm SM}(\widetilde{X}_\Gamma)\,,\end{aligned}$$

The more technical conditions I and II presented in §2 play no role in this statement. They become relevant in evaluating the last term,

$$\int \frac{t\,E}{1-t\,\pi^*h} \cap c_{\rm SM}(\widetilde{X}_\Gamma) = \int \frac{t\,D}{1-t\,h} \cap c_{\rm SM}(\widetilde{X}_\Gamma)\,.$$

here we have replaced E by D (since $E = D\cap \widetilde{X}$, and in particular D restricts to the class of E on $\widetilde{X}_\Gamma$), and π^*h with the corresponding hyperplane class h on $D\cong \mathbb{P}^{n-2}$.

The class $D \cap c_{\mathrm{SM}}(\widetilde{X}_\Gamma)$ is supported on $E \cong X_{\Gamma \smallsetminus e}$. Without further information on e and Γ, it does not seem possible to express this class in more intelligible terms.

Lemma 4.4. *Assume* (Γ, e) *satisfies conditions I and II. Then*

$$\int \frac{t\,D}{1-t\,h} \cap c_{SM}(\widetilde{X}_\Gamma) = \chi(X_{\Gamma\smallsetminus e}) + (t-1)\sum_{i\geq 0} t^i \int h^i \cap c_{SM}(X_{\Gamma\smallsetminus e})$$

Proof. By Corollary 4.1,

$$D \cap c_{\mathrm{SM}}(\widetilde{X}_\Gamma) = (1+D) \cap c_{\mathrm{SM}}(E) = (1-h) \cap c_{\mathrm{SM}}(X_{\Gamma\smallsetminus e})\ :$$

here we have identified $E \subseteq D$ with $X_{\Gamma\smallsetminus e} \subseteq \mathbb{P}^{n-2}$, and used the fact that the class of the exceptional divisor D restricts to $\mathscr{O}(-1)$. The statement is obtained by applying mindless manipulations (and noting $\int c_{\mathrm{SM}}(X_{\Gamma\smallsetminus e}) = \chi(X_{\Gamma\smallsetminus e})$):

$$\begin{aligned}\int \frac{t\,D}{1-t\,h} \cap c_{\mathrm{SM}}(\widetilde{X}_\Gamma) = \int \frac{t(1-h)}{1-t\,h} \cap c_{\mathrm{SM}}(X_{\Gamma\smallsetminus e}) &= \int \left(1+\frac{(t-1)}{1-t\,h}\right) \cap c_{\mathrm{SM}}(X_{\Gamma\smallsetminus e}) \\ &= \chi(X_{\Gamma\smallsetminus e}) + (t-1)\int \frac{1}{1-t\,h} \cap c_{\mathrm{SM}}(X_{\Gamma\smallsetminus e})\end{aligned}$$

with the stated result. □

4.5. Collecting what we have proved at this point:

Proposition 4.5. *Let* Γ *be a graph with* n *edges, and let* e *be a regular edge of* Γ. *Assume* (Γ, e) *satisfies the conditions given in §2. Then*

$$\begin{aligned}\sum_{i\geq 0} t^i \int H^i \cap c_{SM}(X_\Gamma) &= \frac{(1+t)^{n-1}+(t-1)}{t} \\ &\quad + \sum_{i\geq 0} t^i \int h^i \cap c_{SM}(X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e}) + (t-1)\sum_{i\geq 0} t^i \int h^i \cap c_{SM}(X_{\Gamma\smallsetminus e})\end{aligned}$$

This statement improves considerably once it is expressed in terms of the 'polynomial Feynman rules' introduced in [AM11a], §3; we take this as a further indication that the polynomial captures interesting information about Γ. The polynomial essentially evaluates the CSM class of the *complement* of X_Γ in projective space. We will denote the polynomial corresponding to Γ by $C_{X_\Gamma}(t)$, since it depends directly on the graph hypersurface (this makes it an *algebro-geometric* Feynman rule), and since it can be defined for any subset of projective space.

Lemma 4.6. *If* Γ *is not a forest and has* n *edges, then*

$$C_{X_\Gamma}(t) = (1+t)^n - 1 - \sum_{i\geq 0} t^{i+1} \int H^i \cap c_{SM}(X_\Gamma)\,.$$

This is obtained from Proposition 3.7 of [AM11a], by applying simple manipulations. If Γ is a forest, then X_Γ is empty, and the corresponding polynomial is a power of $(t+1)$, cf. Proposition 3.1 in [AM11a].

Consistently with the expression in Lemma 4.6, we set

$$C_Z(t) = (1+t)^{n-1} - 1 - \sum_{i\geq 0} t^{i+1} \int h^i \cap c_{\mathrm{SM}}(Z)$$

for every nonempty subscheme $Z \subseteq \mathbb{P}^{n-2}$, where h denotes the hyperplane class. Note that $C_Z(t)$ depends on the dimension of the space containing Z; this is always clear from the context. If $Z \subset \mathbb{P}^{n-1}$ is empty, we set $C_Z(t) = (1+t)^{n-1}$.

Theorem 4.7 (Deletion-contraction)**.** *Let e be a regular edge of Γ. Assume (Γ, e) satisfies both conditions I and II given in §2. Then*

$$C_{X_\Gamma}(t) = C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t) + (t-1)\, C_{X_{\Gamma\smallsetminus e}}(t)\,.$$

This is the form taken by the formula in Proposition 4.5, once it is written using the notation recalled above. On the right-hand side, both $X_{\Gamma\smallsetminus e}$ and $X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e}$ are viewed as subschemes of $\mathbb{P}^{n-2}$.

The deletion-contraction formula of Theorem 4.7 holds also if $\Gamma \smallsetminus e$ is a forest, provided that $C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t)$ and $C_{X_{\Gamma\smallsetminus e}}(t)$ are both taken to equal $(t+1)^{n-1}$.

Remark 4.8. Differentiating the formula in Theorem 4.5 and setting t to 0 gives

$$C'_{X_\Gamma}(0) = C'_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(0) + C_{X_{\Gamma\smallsetminus e}}(0) - C'_{X_{\Gamma\smallsetminus e}}(0)\,.$$

The value of the derivative at 0 equals the Euler characteristic of the complement ([AM11a], Proposition 3.1); and as $\Gamma \smallsetminus e$ is not a forest, then $C_{X_{\Gamma\smallsetminus e}}(0) = 0$. In this case,

$$n - \chi(X_\Gamma) = n - 1 - \chi(X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e}) - n + 1 + \chi(X_{\Gamma\smallsetminus e})\,,$$

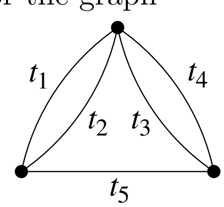
i.e.,

$$\chi(X_\Gamma) = n + \chi(X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e}) - \chi(X_{\Gamma\smallsetminus e})\,.$$

Remarkably, this formula holds as soon as e is a regular edge on Γ, as verified in [AM11b], (3.20). In fact ([AM11b], Theorem 3.8) this formula follows from an analogous formula at the level of Grothendieck classes which holds if e is a regular edge on Γ, regardless of whether conditions I and II are verified. We find it very mysterious that these conditions should affect the CSM classes involved in the deletion-contraction formula, but not affect their zero-dimensional terms. ⌟

4.6. While the argument proving the deletion-contraction formula requires the technical conditions I and II to hold for (Γ, e), there could be a legitimate doubt that the formula itself may hold for more general edges; after all, deletion-contraction for Grothendieck classes does hold in a more general situation (cf. Remark 4.8). The example that follows shows that the formula does not necessarily hold if the second condition fails.

Example 4.9. Condition II fails for the graph



with respect to the bottom edge e (Example 2.2; also cf. §2.6). Labeling the edges by coordinates as indicated, the graph polynomial is

$$\Psi_\Gamma = \quad t_5(t_1+t_2)(t_3+t_4)+(t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4)\,.$$

The corresponding hypersurface X_Γ is singular along two nonsingular conics meeting at the point $p=(0:0:0:0:1)$. The blow-up $\widetilde{X}_\Gamma$ is singular along the proper transforms of these two conics, and along a curve contained in the exceptional divisor. As the exceptional divisor contains a component of $\partial\widetilde{X}_\Gamma$, it is clear that condition II is not satisfied, cf. Claim 2.6. It is equally straightforward to verify that condition I *is* satisfied in this case.

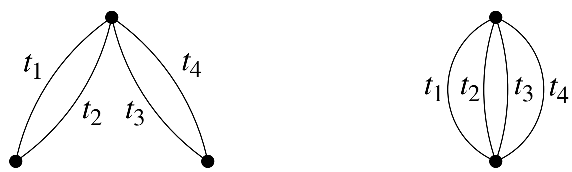
Since X_Γ is nonsingular in codimension 1, its codimension-0 and 1 terms must agree with the Chern class of its virtual tangent bundle, i.e., with the class for a nonsingular hypersurface of degree 3 in $\mathbb{P}^4$:

$$c_{\mathrm{SM}}(X_\Gamma)=3[\mathbb{P}^3]+6[\mathbb{P}^2]+\dots\,.$$

This observation suffices to determine

$$C_{X_\Gamma}(t)=t^5+2\,t^4+\underline{4}\,t^3+\text{l.o.t.}\,.$$

Deletion and contraction:



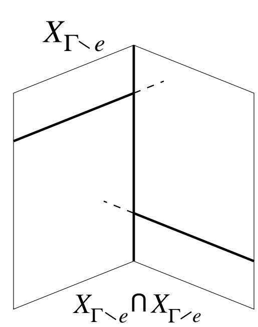
have polynomials

$$\begin{aligned}\Psi_{\Gamma\smallsetminus e} &= \quad (t_1+t_2)(t_3+t_4)\\ \Psi_{\Gamma/e} &= \quad t_1t_2t_3+t_1t_2t_4+t_1t_3t_4+t_2t_3t_4\,.\end{aligned}$$

We have

$$C_{X_{\Gamma\smallsetminus e}}(t)=t^2(t+1)^2\ :$$

this is easy to obtain directly, as $X_{\Gamma\smallsetminus e}$ consists of the union of two planes in $\mathbb{P}^3$; and it also follows from the formularium in Proposition 3.1 and Theorem 3.6 of [AM11a]. As for $X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}$, this is easily checked to consist of three lines in $\mathbb{P}^3$, meeting at two points.



It follows that $c_{\mathrm{SM}}(X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e})=3([\mathbb{P}^1]+2[\mathbb{P}^0])-2[\mathbb{P}^0]=3[\mathbb{P}^1]+4[\mathbb{P}^0]$, and hence

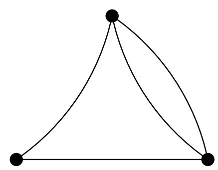
$$C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t)=t^4+4\,t^3+3\,t^2\,.$$

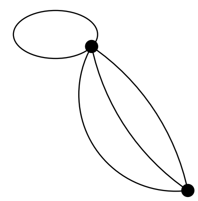
Thus, we see that

$$C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t)+(t-1)C_{X_{\Gamma\smallsetminus e}}(t)=t^5+2\,t^4+\underline{3}\,t^3+2\,t^2\neq C_{X_\Gamma}(t)\,,$$

verifying that the formula in Theorem 4.7 need not hold if condition II fails.

A more thorough analysis shows that $C_{X_\Gamma}(t)=t^5+2\,t^4+4\,t^3+2\,t^2$, so that the underlined coefficient is the only discrepancy. In fact, this may be checked by using Theorem 4.7, using deletion-contraction with respect to a diagonal edge e'; both conditions I and II hold in this case by Lemma 2.3. Deletion and contraction are:





The reader can check that the graph hypersurface for the deletion is a quadric cone in $\mathbb{P}^3$, and hence $c_{\rm SM}(X_{\Gamma\smallsetminus e'})=2[\mathbb{P}^2]+4[\mathbb{P}^1]+3[\mathbb{P}^0]$. The intersection $X_{\Gamma\smallsetminus e'}\cap X_{\Gamma/e'}$ consists of the union of a nonsingular conic and a line meeting at a point, hence $c_{\rm SM}(X_{\Gamma\smallsetminus e'}\cap X_{\Gamma/e'})=3[\mathbb{P}^1]+3[\mathbb{P}^0]$. This yields

$$C_{X_{\Gamma\smallsetminus e'}}(t)=t^4+2\,t^3+2\,t^2+t\quad,\quad C_{X_{\Gamma\smallsetminus e'}\cap X_{\Gamma/e'}}(t)=t^4+4\,t^3+3\,t^2+t$$

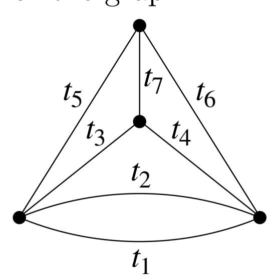
and hence

$$C_{X_\Gamma}(t)=(t^4+4\,t^3+3\,t^2+t)+(t-1)(t^4+2\,t^3+2\,t^2+t)=t^5+2\,t^4+4\,t^3+2\,t^2$$

as claimed, by Theorem 4.7. ⌟

As pointed out in Remark 2.8, we do not know an example for which the first condition fails while the second one holds. We list the relevant classes in an example for which both conditions fail.

Example 4.10. Condition I fails for the graph



with respect to the vertical edge e (Example 2.1). Labeling the edges by coordinates as indicated, the graph polynomial is

$$\begin{aligned}\Psi_\Gamma=\quad & t_7t_6t_4t_2+t_7t_6t_4t_1+t_7t_6t_3t_2+t_7t_6t_3t_1+t_7t_6t_1t_2+t_7t_5t_4t_2+t_7t_5t_4t_1+t_7t_5t_3t_2\\ &+t_7t_5t_3t_1+t_7t_5t_1t_2+t_7t_4t_1t_2+t_7t_1t_2t_3+t_6t_5t_4t_2+t_6t_5t_4t_1+t_6t_5t_3t_2+t_6t_5t_3t_1\\ &+t_6t_5t_1t_2+t_6t_4t_3t_2+t_6t_4t_3t_1+t_6t_1t_2t_3+t_5t_4t_3t_2+t_5t_4t_3t_1+t_5t_4t_1t_2+t_1t_2t_3t_4\,,\end{aligned}$$

and the corresponding X_Γ is a hypersurface of degree 4 in $\mathbb{P}^6$. The computation of the terms needed to verify the formula in Theorem 4.7 for this case is more involved than in Example 4.9, and we omit the details. (The Macaulay2 procedure accompanying [Alu03] was used for these computations.) We obtain:

$$c_{\rm SM}(X_\Gamma) = 4[\mathbb{P}^5] + 12[\mathbb{P}^4] + 26[\mathbb{P}^3] + 29[\mathbb{P}^2] + 21[\mathbb{P}^1] + 7[\mathbb{P}^0]\,,$$

yielding

$$C_{X_\Gamma}(t) = t^7 + 3\,t^6 + 9\,t^5 + 9\,t^4 + \underline{6}\,t^3\,.$$

As for deletion and contraction:

$$c_{\rm SM}(X_{\Gamma\smallsetminus e}) = 3[\mathbb{P}^4] + 9[\mathbb{P}^3] + 14[\mathbb{P}^2] + 14[\mathbb{P}^1] + 7[\mathbb{P}^0]$$
$$c_{\rm SM}(X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}) = 6[\mathbb{P}^3] + 10[\mathbb{P}^2] + 13[\mathbb{P}^1] + 7[\mathbb{P}^0]$$

from which

$$C_{X_{\Gamma\smallsetminus e}}(t) = t^6 + 3\,t^5 + 6\,t^4 + 6\,t^3 + t^2 - t$$
$$C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t) = t^6 + 6\,t^5 + 9\,t^4 + 10\,t^3 + 2\,t^2 - t$$

and therefore

$$C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t) + (t-1)C_{X_{\Gamma\smallsetminus e}}(t) = t^7 + 3\,t^6 + 9\,t^5 + 9\,t^4 + \underline{5}\,t^3\,.$$

This differs from $C(X_\Gamma)$ by t^3, and shows that the formula in Theorem 4.7 need not hold if condition I is not satisfied. (But note that Condition II also fails in this example.) ⌟

4.7. If e is *not* a regular edge of Γ, then the corresponding deletion-contraction formulas are trivial consequences of properties of $C_{X_\Gamma}(t)$ listed in [AM11a], §3 (especially Proposition 3.1).

- If e is a bridge in Γ, then
$$C_{X_\Gamma}(t) = (t+1)C_{X_{\Gamma\smallsetminus e}}(t)\,.$$
- If e is a looping edge in Γ, then
$$C_{X_\Gamma}(t) = t\,C_{X_{\Gamma\smallsetminus e}}(t)\,.$$

If e is a looping edge, then $\Gamma\smallsetminus e = \Gamma/e$, and hence $X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e} = X_{\Gamma\smallsetminus e}$. Thus, the formula given above matches the formula obtained by applying Theorem 4.7. The formula in Theorem 4.7 is also trivially satisfied if $\Gamma\smallsetminus e$ is a forest.

The formula for bridges has a transparent geometric explanation. If e is a bridge, then $\Psi_\Gamma = \Psi_{\Gamma\smallsetminus e}$; thus X_Γ is a cone over $X_{\Gamma\smallsetminus e}$, and the formula given above follows easily from this fact.

5. Multiple-edge formulas

5.1. Deletion-contraction formulas may be used to obtain formulas for the operation of 'multiplying edges', i.e., inserting edges parallel to a given edge of a graph. If e is an edge of a graph Γ and $m \geq 1$, we will denote by Γ_{me} the graph obtained by replacing e with m edges connecting the same vertices. (In particular, $\Gamma = \Gamma_e$.) Multiple-edge formulas are obtained in [AM11b], §5, for the case of the Grothendieck class. As in Theorem 4.7, the deletion-contraction formula involves a 'non-combinatorial' term (the Grothendieck class of the intersection $X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e}$). By virtue of a propitious cancellation, the resulting formula for doubling an edge only relies on combinatorial data:

$$\mathbb{U}(\Gamma_{2e}) = (\mathbb{T}-1)\,\mathbb{U}(\Gamma) + \mathbb{T}\,\mathbb{U}(\Gamma\smallsetminus e) + (\mathbb{T}+1)\,\mathbb{U}(\Gamma/e)\,.$$

([AM11b], Proposition 5.2), provided e is a regular edge on Γ.

In this section we show that a similar situation occurs for the CSM invariant. Again, the deletion-contraction formula (Theorem 4.7) involves a summand which we are not able to interpret directly in combinatorial terms (that is, $C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t)$); and again a fortunate cancellation leads to a purely combinatorial formula for doubling edges. We will prove:

Theorem 5.1. *Let Γ be a graph, and let e be a regular edge of Γ such that (Γ, e) satisfies conditions I and II of §2. Then*

$$C_{\Gamma_{2e}}(t) = (2t-1)\,C_\Gamma(t) - t(t-1)\,C_{\Gamma\smallsetminus e}(t) + C_{\Gamma/e}(t)$$

(Here we write $C_\Gamma(t)$ for $C_{X_\Gamma}(t)$, etc.)

By Lemma 2.3, this formula applies, in particular, if e is already multiple in Γ. This fact will be used in the proof of Theorem 5.1, and will lead to multiple-edge formulas in §5.3.

The formula also holds if e is a looping edge, interpreting $C_{\Gamma/e}(t)$ to be 0 in this case (cf. §4.7).

5.2. *Proof of Theorem 5.1.* By Theorem 4.7, under the hypothesis of the theorem, we have

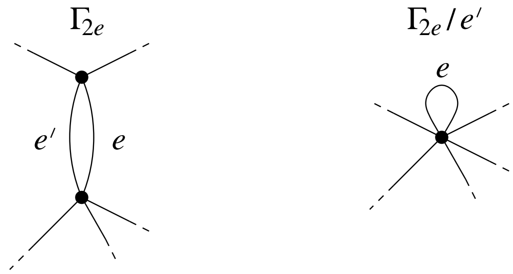
$$\begin{aligned} C_{X_\Gamma}(t) &= C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t) + (t-1)\,C_{X_{\Gamma\smallsetminus e}}(t)\,, \\ C_{X_{\Gamma_{2e}}}(t) &= C_{X_\Gamma\cap X_{\Gamma_{2e}/e'}}(t) + (t-1)\,C_{X_\Gamma}(t)\,, \end{aligned}$$

where e' denotes the edge parallel to e in Γ_{2e}. Indeed, the first formula holds as (Γ, e) satisfies conditions I and II by hypothesis, and the second formula holds since (Γ_{2e}, e') satisfies conditions I and II by Lemma 2.3; note that $\Gamma_{2e} \smallsetminus e' = \Gamma$. The theorem will be obtained by comparing the two intersections $X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e} \subseteq \mathbb{P}^{n-2}$ and $X_\Gamma \cap X_{\Gamma_{2e}/e'} \subseteq \mathbb{P}^{n-1}$ (where $n =$ number of edges of Γ).

Let t_e, $t_{e'}$ be the variables corresponding to the two parallel edges e, e' in Γ_{2e}, and let $t_1, \dots, t_{n-1}$ be the variables corresponding to the other edges. We have

$$\Psi_{\Gamma_{2e}} = t_{e'}\Psi_\Gamma + \Psi_{\Gamma_{2e}/e'}\,.$$

The graph Γ_{2e}/e' may be obtained by attaching a looping edge marked e at the vertex obtained by contracting e in Γ:



As a consequence,

$$\Psi_{X_{\Gamma_{2e/e'}}} = t_e \Psi_{X_{\Gamma/e}} ,$$

and $X_{\Gamma \smallsetminus e} \cap X_{\Gamma/e}$, $X_\Gamma \cap X_{\Gamma_{2e/e'}}$ have ideals

$$\left(\Psi_{\Gamma\smallsetminus e}, \Psi_{\Gamma/e}\right) \quad , \quad \left(\Psi_\Gamma, t_e \Psi_{\Gamma/e}\right)$$

respectively. The first should be viewed as an ideal in $k[t_1, \dots, t_{n-1}]$, and the second as an ideal in $k[t_1, \dots, t_{n-1}, t_e]$. Denoting by $V(f_1, f_2, \dots)$ the locus $f_1 = f_2 = \cdots = 0$, we have

$$V(\Psi_\Gamma, t_e\Psi_{\Gamma/e}) = V(\Psi_\Gamma, t_e) \cup V(\Psi_\Gamma, \Psi_{\Gamma/e}) = V(\Psi_{\Gamma/e}, t_e) \cup V(t_e \Psi_{\Gamma\smallsetminus e}, \Psi_{\Gamma/e})$$

using the fact that $\Psi_\Gamma = t_e \Psi_{\Gamma\smallsetminus e} + \Psi_{\Gamma/e}$,

$$= V(\Psi_{\Gamma/e}, t_e) \cup V(\Psi_{\Gamma\smallsetminus e}, \Psi_{\Gamma/e}) .$$

This shows that $X_\Gamma \cap X_{\Gamma_{2e/e'}}$ is the union of a copy of $X_{\Gamma/e} \subseteq \mathbb{P}^{n-2}$ and a cone in $\mathbb{P}^{n-1}$ over $X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e} \subseteq \mathbb{P}^{n-2}$. The intersection of these two loci is $V(\Psi_{\Gamma\smallsetminus e}, \Psi_{\Gamma/e}, t_e)$, that is, a copy of $X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e} \subseteq \mathbb{P}^{n-2}$.

The invariant $C_-(t)$ satisfies an inclusion-exclusion property because so does the Chern-Schwartz-MacPherson class, and its behavior with respect to taking a cone amounts to multiplication by $(t+1)$: this follows easily from the formula for CSM classes of cones, see Proposition 5.2 in [AM09]. Therefore,

$$\begin{aligned} C_{X_\Gamma \cap X_{\Gamma_{2e/e'}}}(t) &= C_{X_{\Gamma/e}}(t) + (t+1)\, C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t) - C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t) \\ &= C_{X_{\Gamma/e}}(t) + t\, C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t) . \end{aligned}$$

Using this fact together with the two formulas given at the beginning of the proof, we get

$$\begin{aligned} C_{X_{\Gamma_{2e}}}(t) - (t-1)\, C_{X_\Gamma}(t) &= C_{X_{\Gamma/e}}(t) + t\, C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t) \\ &= C_{X_{\Gamma/e}}(t) + t\, \left(C_{X_\Gamma}(t) - (t-1)\, C_{X_{\Gamma\smallsetminus e}}(t)\right) , \end{aligned}$$

which yields immediately the stated formula. □

5.3. By Lemma 2.3, a multiple regular edge satisfies both conditions I and II, and hence the doubling edge formula of Theorem 5.1 applies to it.

Theorem 5.2. *Let e be a regular edge of Γ. Then*

$$\sum_{m\ge 0} C_{\Gamma_{(m+1)e}}(t)\, \frac{s^m}{m!} = e^{ts} \left(K(t)\, C_\Gamma(t) - K'(t)\, C_{\Gamma_{2e}}(t) + \frac{K''(t)}{2}\, C_{\Gamma_{3e}}(t)\right) ,$$

where

$$K(t) = t^2 e^{-s} + (t-1)(ts - t - 1)\,.$$

Equivalently,

$$\begin{aligned} C_{\Gamma_{(m+1)e}}(t) &= \big(t^2\, C_\Gamma(t) - 2t\, C_{\Gamma_{2e}}(t) + C_{\Gamma_{3e}}(t)\big)\,(t-1)^{m-1} \\ &\quad - \big((t^2-1)\, C_\Gamma(t) - 2t\, C_{\Gamma_{2e}}(t) + C_{\Gamma_{3e}}(t)\big)\, t^{m-1} \\ &\quad + \big((t^2-t)\, C_\Gamma(t) - (2t-1)\, C_{\Gamma_{2e}}(t) + C_{\Gamma_{3e}}(t)\big)\,(m-1)\, t^{m-2}\,. \end{aligned}$$

The fact that the coefficient of $C_\Gamma(t)$ determines the others by taking derivatives is a consequence of the analogous feature displayed by the coefficients of the basic recursion for $C_{\Gamma_{me}}$:

Lemma 5.3. *Let e be a regular edge of Γ. Then for $m \geq 1$*

$$C_{\Gamma_{(m+3)e}}(t) = (3t-1)\, C_{\Gamma_{(m+2)e}}(t) - (3t^2 - 2t)\, C_{\Gamma_{(m+1)e}}(t) + (t^3 - t^2)\, C_{\Gamma_{me}}(t)\,.$$

This feature reflects the fact that the characteristic polynomial for the recursion is a function of $t - x$:

$$x^3 - (3t-1)\, x^2 + t(3t-2)\, x - t^2(t-1) = -(t-x)^2(t-x-1)\,.$$

This fact is intriguing, and we do not have a conceptual explanation for it. (Note that the corresponding fact is not verified for the recursion computing the Grothendieck class. Using (5.8) in [AM11b], it is immediate to show that

$$\mathbb{U}(\Gamma_{(m+3)e}) = (2\mathbb{T} - 1)\, \mathbb{U}(\Gamma_{(m+2)e}) - \mathbb{T}\,(\mathbb{T}-2)\, \mathbb{U}(\Gamma_{(m+1)e}) - \mathbb{T}^2\, \mathbb{U}(\Gamma_{me})\,,$$

with notation as in [AM11b], if e is regular in Γ. The characteristic polynomial factors as $(x+1)\,(\mathbb{T} - x)^2$.) Lemma 5.3 is an immediate consequence of Theorem 5.1:

Proof. If e is a regular edge, then e satisfies conditions I and II in Γ_{ne} for all $n \geq 2$, by Lemma 2.3. Apply Theorem 5.1 to obtain

$$\begin{aligned} C_{\Gamma_{(m+2)e}}(t) &= (2t-1)\, C_{\Gamma_{(m+1)e}}(t) - t(t-1)\, C_{\Gamma_{me}}(t) + t^m\, C_{\Gamma/e}(t) \\ C_{\Gamma_{(m+3)e}}(t) &= (2t-1)\, C_{\Gamma_{(m+2)e}}(t) - t(t-1)\, C_{\Gamma_{(m+1)e}}(t) + t^{m+1}\, C_{\Gamma/e}(t) \end{aligned}$$

for $m \geq 1$. The stated recursion follows immediately, by eliminating $C_{\Gamma/e}(t)$ from these expressions. $\square$

Theorem 5.2 follows directly from this lemma, by standard methods.

Remark 5.4. Theorem 5.2 may be rewritten in the following fashion: for all regular edges e of Γ,

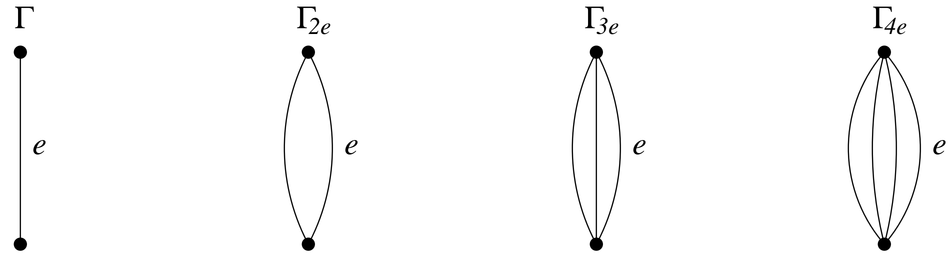
$$\begin{aligned} \sum_{m\geq 0} C_{\Gamma_{(m+1)e}}(t)\, \frac{s^m}{m!} &= \big(e^{ts} - e^{(t-1)s}\big)\, C_{\Gamma_{2e}}(t) \\ &\quad - \big((t-1)\, e^{ts} - t\, e^{(t-1)s}\big)\, C_\Gamma(t) + t\,\big((s-1)\, e^{ts} + e^{(t-1)s}\big)\, C_{\Gamma/e}(t)\,. \end{aligned}$$

That is, $C_{\Gamma_{me}}(t)$ is given by the expression

$$\big(C_{\Gamma_{2e}}(t) - t\, C_\Gamma(t) - t\, C_{\Gamma/e}(t)\big)\,\big(t^{m-1} - (t-1)^{m-1}\big) + \big(C_\Gamma(t) + (m-1)\, C_{\Gamma/e}(t)\big)\, t^{m-1}$$

for all $m \geq 1$. These expressions are simpler to apply than Theorem 5.2, since $C_{\Gamma/e}(t)$ is usually more immediately accessible than $C_{\Gamma_{3e}}$. ⌟

Example 5.5. The n-edge *banana graph* consists of n edges connecting two distinct vertices.



We have

$$C_\Gamma(t) = t + 1 \quad , \quad C_{\Gamma_{2e}}(t) = t\,(t+1) \quad , \quad C_{\Gamma/e}(t) = 1 \ :$$

indeed, Γ is a single bridge, and Γ_{2e} is a 2-polygon (see Proposition 3.1 in [AM11a]). The formula in Remark 5.4 yields the following pretty generating function:

$$\sum_{m\geq 0} C_{\Gamma_{(m+1)e}}(t)\,\frac{s^m}{m!} = (1+ts)\,e^{ts} + t\,e^{(t-1)s} \,,$$

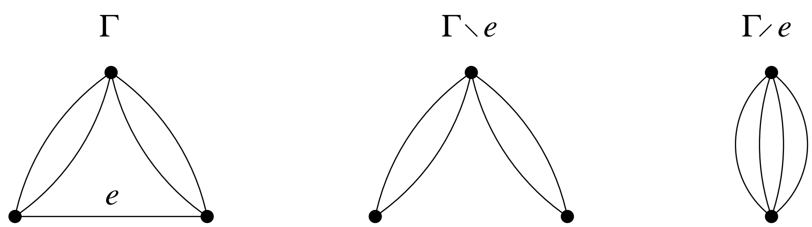
or equivalently

$$C_{\Gamma_{ne}}(t) = n\,t^{n-1} + t\,(t-1)^{n-1}$$

for $n \geq 1$. This agrees with the formula given in Example 3.8 of [AM11a], which was obtained by a very different method. ⌟

5.4. The coefficients obtained in Remark 5.4 agree with the ones conjectured in [AM11b], §6.2, with the difference that the formulas given here are applied to Γ_{2e} rather than $\Gamma = \Gamma_e$; this accounts for the extra factor of t in the last coefficient. The point is that the hypotheses of Theorem 5.1 are automatically satisfied for Γ_{2e} since e has parallel edges in Γ_{2e}, while they are not necessarily satisfied for Γ. Accordingly, while it is tempting to interpret Γ_{0e} as $\Gamma \smallsetminus e$, the generating function given in Remark 5.4 cannot be extended in general to provide information about this graph.

Example 5.6. We verify that Theorem 5.1 does not necessarily hold if e does not satisfy the hypotheses presented in §2. For this, we return to the graph of Examples 2.2 and 4.9.



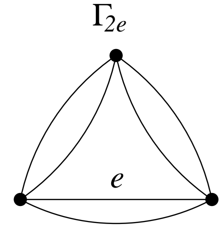
We have (cf. Examples 4.9 and 5.5)

$$\begin{aligned} C_\Gamma(t) &= t^5 + 2\,t^4 + 4\,t^3 + 2\,t^2 \\ C_{\Gamma\smallsetminus e}(t) &= (t+1)^2\,t^2 \\ C_{\Gamma/e}(t) &= 4\,t^3 + t(t-1)^3\,; \end{aligned}$$

the formula in Theorem 5.1 would give

$$(*) \qquad (2t-1)\,C_\Gamma(t) - t(t-1)\,C_{\Gamma\smallsetminus e}(t) + C_{\Gamma/e}(t) = t^6 + 2\,t^5 + \underline{8}\,t^4 + 2\,t^3 + t^2 - t$$

and we can verify that this does *not* equal $C_{\Gamma_{2e}}(t)$.



Indeed, the graph polynomial for Γ_{2e} is

$$\Psi_{\Gamma_{2e}} = t_6t_5t_4t_2 + t_6t_5t_4t_1 + t_6t_5t_3t_2 + t_6t_5t_3t_1 + t_6t_4t_3t_2 + t_6t_4t_3t_1 \\ + t_6t_4t_1t_2 + t_6t_1t_2t_3 + t_5t_4t_3t_2 + t_5t_4t_3t_1 + t_5t_4t_1t_2 + t_5t_1t_2t_3$$

(where the pairs of variables (t_1, t_2), (t_3, t_4), (t_5, t_6) correspond to the three pairs of parallel edges). Macaulay2 confirms that $X_{\Gamma_{2e}}$ is nonsingular in codimension 1, hence

$$c_{\mathrm{SM}}(X_{\Gamma_{(2)}}) = 4\,[\mathbb{P}^4] + 8\,[\mathbb{P}^3] + \cdots$$

from which

$$C_{\Gamma_{2e}}(t) = t^6 + 2\,t^5 + \underline{7}\,t^4 + \text{l.o.t.}\,,$$

differing from (*). In fact, a computation using the code from [Alu03] shows that

$$C_{\Gamma_{2e}}(t) = t^6 + 2\,t^5 + 7\,t^4 + 2\,t^3 + t^2 - t\,;$$

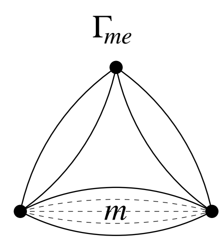
this differs from (*) by exactly t^4. The corresponding CSM class is

$$4[\mathbb{P}^4] + 8[\mathbb{P}^3] + 18[\mathbb{P}^2] + 14[\mathbb{P}] + 7[\mathbb{P}^0]\,,$$

correcting the formula given at the end of §6 in [AM11b]. (Incidentally, the fact that in these examples the discrepancies occurring when conditions I and II fail are pure powers of t is also intriguing, and calls for an explanation.) By Remark 5.4, we also obtain that

$$C_{\Gamma_{me}}(t) = (t^2 - t + 1)^2\,t\,(t-1)^{m-1} + \big(4t^3 + t^2 + 4t - 1 + (m-1)\,(t^3 + t^2 + 3t - 1)\big)\,t^m$$

for all $m \geq 1$.



6. Alternative approach, via specialization

6.1. In this section we explain briefly a different approach to the question studied in this paper, based on the theory of specialization of Chern classes; this theory is originally due to Verdier ([Ver81]). For simplicity, we now work over $\mathbb{C}$.

Let $\{X_u\}$ be a family of hypersurfaces over a disk; assume that the fibers over $u \neq 0$ are all isomorphic. Under suitable (and mild) hypotheses, one may define a specific constructible function σ on the central fiber $X = X_0$, with the property that $c_{\mathrm{SM}}(\sigma)$ equals the specialization (in the sense of intersection theory) of the CSM class of the general fiber. The value $\sigma(p)$ at a point $p \in X$ equals the Euler characteristic of the intersection of the ϵ-ball centered at p with nearby fibers X_u, as $\epsilon \to 0$ and $|u| \ll \epsilon$. See [Ver81], Proposition 4.1. It may also be computed in terms of an embedded resolution of X, see [Alu].

In the situation we will consider here, the hypersurfaces X_u will be elements of a pencil in projective space. It is easy to verify that $\sigma(p) = 1$ if X is nonsingular at p, and $\sigma(p) = 0$ if p is a point of transversal intersection of two nonsingular components of X, provided that p does not belong to the base locus of the pencil. Parusiński and Pragacz ([PP01], Proposition 5.1) prove that $\sigma(p) = 1$ if $p \in X_u \cap X$ $(u \neq 0)$ *is* a point of the base locus, if X_u is *smooth and transversal to the strata of a fixed Whitney stratification of* X at p. This beautiful observation will be used below.

6.2. Returning to graph hypersurfaces, recall that if e is a regular edge of Γ, then

$$\Psi_\Gamma = t_e\, \Psi_{\Gamma \smallsetminus e} + \Psi_{\Gamma/e}\,;$$

as $\Gamma \smallsetminus e$ is not a forest, both $\Psi_{\Gamma\smallsetminus e}$ and $\Psi_{\Gamma/e}$ are polynomials of positive degree. Thus, we may view Ψ_Γ as an element of the pencil spanned by $t_e\,\Psi_{\Gamma\smallsetminus e}$ and $\Psi_{\Gamma/e}$: $u = 1$ in

$$(*) \qquad \Psi_u = t_e\, \Psi_{\Gamma\smallsetminus e} + u\, \Psi_{\Gamma/e}\,.$$

We note that $X_u : \{\Psi_u = 0\}$ is isomorphic to $X_\Gamma : \{\Psi_1 = 0\}$ for $u \neq 0$: indeed, with $d = \deg \Psi_\Gamma$,

$$\begin{aligned}\Psi_u(u^2 t_e, u\mathbf{t}_{\neq e}) &= u^2\, t_e \Psi_{\Gamma\smallsetminus e}(u\mathbf{t}_{\neq e}) + u\, \Psi_{\Gamma/e}(u\mathbf{t}_{\neq e}) = u^{d+1}\, t_e \Psi_{\Gamma\smallsetminus e}(\mathbf{t}_{\neq e}) + u^{d+1}\, \Psi_{\Gamma/e}(\mathbf{t}_{\neq e}) \\ &= u^{d+1}\, \Psi_\Gamma(t_e, \mathbf{t}_{\neq e})\,,\end{aligned}$$

where $\mathbf{t}_{\neq e}$ denotes the variables corresponding to edges other than e. View (*) as defining a family as in §6.1, with central fiber $X : \{\Psi_0 = 0\}$. That is,

$$X = X^\wedge_{\Gamma\smallsetminus e} \cup H\,,$$

where n = number of edges in the graph Γ, $H \cong \mathbb{P}^{n-2}$ is the hyperplane defined by $t_e = 0$, and $X^\wedge_{\Gamma\smallsetminus e}$ denotes the cone over $X_{\Gamma\smallsetminus e} \subseteq H$ with vertex at the point $p = (t_e : \mathbf{t}_{\neq e}) = (1 : 0 : \cdots : 0)$. As a consequence of Verdier's theorem (Théorème 5.1 in [Ver81]),

$$c_{\rm SM}(1\!\!1_{X_\Gamma}) = c_{\rm SM}(\sigma)\,,$$

where these classes are taken in $A_*\mathbb{P}^{n-1}$, and σ is the specialization function on X defined in §6.1.

6.3. The difficulty with this approach lies in the explicit computation of σ. Again we indicate two conditions under which a result may be obtained, matching the result of the more algebraic approach taken in the rest of the paper.

The first condition is a 'set-theoretic' version of condition I:

$$\text{(Condition I')} \qquad \partial X_{\Gamma\smallsetminus e} \subseteq X_{\Gamma/e}\,.$$

As pointed out in §2.2, Condition I implies this inclusion at the level of schemes; here, we are only requiring it at the level of sets. If this condition is satisfied, then the value of σ may be determined for all points of $X \smallsetminus X_\Gamma$. This consists of the complements of X_Γ in $X^\wedge_{\Gamma\smallsetminus e} \smallsetminus H$, in $X^\wedge_{\Gamma\smallsetminus e} \cap H$, and in $H \smallsetminus X^\wedge_{\Gamma\smallsetminus e}$.

Lemma 6.1. *If condition I' holds, then*

- *For* $q \in (X^\wedge_{\Gamma\smallsetminus e} \smallsetminus H) \smallsetminus X_\Gamma$, $\sigma(q) = 1$;
- *For* $q \in (X^\wedge_{\Gamma\smallsetminus e} \cap H) \smallsetminus X_\Gamma$, $\sigma(q) = 0$.
- *For* $q \in (H \smallsetminus X^\wedge_{\Gamma\smallsetminus e}) \smallsetminus X_\Gamma$, $\sigma(q) = 1$;

In fact, $\sigma(q) = 1$ *for all* $q \in H \smallsetminus X^\wedge_{\Gamma\smallsetminus e}$.

Proof. If condition I' holds, then the singularities of $X_{\Gamma\smallsetminus e}$ are contained in $X_{\Gamma/e}$, and it follows easily that the singular locus of the cone $X^\wedge_{\Gamma\smallsetminus e}$ is contained in X_Γ.

On the other hand, note that $\Psi_{\Gamma\smallsetminus e}$ is one of the partial derivatives of Ψ_Γ; hence, the singularities of X_Γ are contained in $X^\wedge_{\Gamma\smallsetminus e}$.

Thus: the first statement holds since $X^\wedge_{\Gamma\smallsetminus e}$ is nonsingular at q, and q does not belong to the base locus of the pencil. The third statement likewise holds because q is not in the base locus, and $H \cong \mathbb{P}^2$ is nonsingular. In the second statement, q is nonsingular on both $X^\wedge_{\Gamma\smallsetminus e}$ and H, and these hypersuraces intersect transversally at p, so $\sigma(q) = 0$ as recalled in §6.1.

To prove the last assertion, we have to consider $q \in (H \cap X_\Gamma) \smallsetminus X^\wedge_{\Gamma\smallsetminus e}$. (We have already dealt with the other points in $H \smallsetminus X^\wedge_{\Gamma\smallsetminus e}$.) At these points, both X and X_Γ are nonsingular (by condition I'). Further, the intersection $H \cap X_\Gamma = X_{\Gamma/e}$ is also nonsingular at such points, again by condition I'. Thus the hypotheses of the result of Parusiński and Pragacz recalled in §6.1 are satisfied, and it follows that $\sigma(q) = 1$. □

6.4. The locus unaccounted for in Lemma 6.1 is $X_\Gamma \cap X^\wedge_{\Gamma\smallsetminus e}$. These are points in the base locus which are contained in the component $X^\wedge_{\Gamma\smallsetminus e}$. *If* X_Γ were nonsingular and transversal to the strata of $X^\wedge_{\Gamma\smallsetminus e}$ at these points, then (according to the formula of Parusiński and Pragacz) we would have $\sigma(q) = 1$ for $q \in X_\Gamma \cap X^\wedge_{\Gamma\smallsetminus e}$; the deviation of σ from 1 at such points is a measure of non-transversality of the two hypersurfaces. Also note that this locus equals the cone over $X_{\Gamma\smallsetminus e} \cap X_{\Gamma/e}$ with vertex at p. The following condition should again be interpreted as a subtle notion of 'transversality' of the intersection of the various loci considered here.

(Condition II') $$\sigma(q) = 1 \text{ for } q \in X^\wedge_{\Gamma\smallsetminus e} \cap X^\wedge_{\Gamma/e}\,.$$

Lemma 6.2. *Assume both conditions I' and II' hold. Then*

$$\sigma = 1\!\!1_{X^\wedge_{\Gamma\smallsetminus e}} + 1\!\!1_H - 2 1\!\!1_{X_{\Gamma\smallsetminus e}} + 1\!\!1_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}$$

where the latter two mentioned loci are viewed as subsets of $H \cong \mathbb{P}^{n-2}$.

Proof. This follows from Lemma 6.1, condition II', and elementary bookkeeping. □

6.5. Applying Verdier's theorem now recovers the same formula we obtained in Theorem 4.7

Theorem 6.3. *Let* e *be a regular edge of* Γ. *Assume* (Γ, e) *satisfies both conditions I' and II' given above. Then*

$$C_{X_\Gamma}(t) = C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t) + (t-1)\, C_{X_{\Gamma\smallsetminus e}}(t)\,.$$

Proof. Applying Verdier's theorem and additivity of CSM classes, we get

$$c_{\mathrm{SM}}(X) = c_{\mathrm{SM}}(\sigma) = c_{\mathrm{SM}}(1\!\!1_{X^\wedge_{\Gamma\smallsetminus e}}) + c_{\mathrm{SM}}(1\!\!1_H) - 2\, c_{\mathrm{SM}}(1\!\!1_{X_{\Gamma\smallsetminus e}}) + c_{\mathrm{SM}}(1\!\!1_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}})\,.$$

Recalling $H \cong \mathbb{P}^{n-2}$, and expressing in terms of polynomial Feynman rules, this gives

$$C_{X_\Gamma}(t) = (t+1)C_{X_{\Gamma\smallsetminus e}}(t) - 2\,C_{X_{\Gamma\smallsetminus e}}(t) + C_{X_{\Gamma\smallsetminus e}\cap X_{\Gamma/e}}(t)\,,$$

with the stated result. Here we also used the fact that $C_{X^\wedge_{\Gamma\smallsetminus e}}(t) = (t+1)\,C_{X_{\Gamma\smallsetminus e}}$, an easy consequence of Proposition 5.2 in [AM09]. □

6.6. In conclusion, we have verified that the deletion-contraction formula obtained under the 'algebraic' conditions I and II described in §2 also holds under the conditions I' and II' given in §§6.3-6.4.

These latter conditions have a more 'geometric' flavor: condition I' is a set-theoretic statement, and condition II' amounts to a statement on the Euler characteristics of intersections of an ϵ-ball with nearby fibers in a fibration naturally associated with the pair (Γ, e). In this sense they are perhaps easier to appreciate, although in practice the algebraic counterparts I and II are more readily verifiable in given cases, by means of tools such as [GS]. It would be interesting to relate these two sets of conditions more precisely: does (Γ, e) satisfy I and II if and only if it satisfies I' and II'? Do these conditions admit a transparent combinatorial interpretation?

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