

GROTHENDIECK CLASSES AND CHERN CLASSES OF HYPERPLANE ARRANGEMENTS

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ABSTRACT. We show that the characteristic polynomial of a hyperplane arrangement can be recovered from the class in the Grothendieck group of varieties of the complement of the arrangement. This gives a quick proof of a theorem of Orlik and Solomon relating the characteristic polynomial with the ranks of the cohomology of the complement of the arrangement.

We also show that the characteristic polynomial can be computed from the total Chern class of the complement of the arrangement. In the case of free arrangements, we prove that this Chern class agrees with the Chern class of the dual of a bundle of differential forms with logarithmic poles along the hyperplanes in the arrangement; this follows from work of Mustařă and Schenck. We conjecture that this relation holds for any locally quasi-homogeneous free divisor.

We give an explicit relation between the characteristic polynomial of an arrangement and the Segre class of its singularity (‘Jacobian’) subscheme. This gives a variant of a recent result of Wakefield and Yoshinaga, and shows that the Segre class of the singularity subscheme of an arrangement together with the degree of the arrangement determine the ranks of the cohomology of its complement.

We also discuss the positivity of the Chern classes of hyperplane arrangements: we give a combinatorial interpretation of this phenomenon, and we discuss the cases of generic and free arrangements.

1. INTRODUCTION

1.1. The guiding theme of this note is the relation between the combinatorics of a projective hyperplane arrangement and algebro-geometric and intersection-theoretic invariants of the union of the hyperplanes.

Our results are as follows. We work over a field k . We consider a hyperplane arrangement \mathcal{A} in \mathbb{P}^n , and the corresponding central arrangement $\widehat{\mathcal{A}}$ in k^{n+1} . We denote by $A \subseteq \mathbb{P}^n$ the union of the hyperplanes in \mathcal{A} . We let $\chi_{\widehat{\mathcal{A}}}(t)$ be the characteristic polynomial of $\widehat{\mathcal{A}}$, and denote by $\underline{\chi}_{\widehat{\mathcal{A}}}(t)$ the quotient $\chi_{\widehat{\mathcal{A}}}(t)/(t-1)$ (also a polynomial). We let $M(\mathcal{A})$ be the complement $\mathbb{P}^n \setminus A$.

Theorem 1.1. *Let \mathbb{L} be the class of the affine line in the Grothendieck group of k -varieties $K(\text{Var}_k)$. Then $\underline{\chi}_{\widehat{\mathcal{A}}}(\mathbb{L})$ evaluates the class of the complement $M(\mathcal{A})$ in $K(\text{Var}_k)$.*

If the arrangement is defined over a finite field, Theorem 1.1 shows how to recover the Poincaré polynomial of an arrangement by counting points, an observation made by several authors (see e.g., Theorem 2.69 in [OT92]). Placing this elementary result

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in the Grothendieck ring $K(\text{Var}_k)$ for arbitrary k has not-so-elementary applications: for example, we note that the Orlik–Solomon theorem relating the characteristic polynomial to the Poincaré polynomial of the complement of an arrangement ([OS80]) is an immediate consequence of this result (Corollary 2.3). Another application is the determination of the stable rational equivalence class of $M(\mathcal{A})$ (Corollary 2.4).

1.2. As observed in [AM09a], Proposition 2.2, the information carried by the Grothendieck class of a union of linear subspaces is equivalent to the information in the image of its ‘Chern–Schwartz–MacPherson’ (CSM) class in $A_*\mathbb{P}^n$. Therefore, Theorem 1.1 may be recast in terms of CSM classes:

Theorem 1.2. *The Chern–Schwartz–MacPherson class of the complement $M(\mathcal{A})$ may be obtained by replacing t^k with $[\mathbb{P}^k]$ in $\chi_{\widehat{\mathcal{A}}}(t+1)$.*

For a reminder on Chern–Schwartz–MacPherson classes, see §3.1. J. Huh has also observed that the additivity properties of these classes lead to formulas for the characteristic polynomial: see Remark 26 in [Huh]. In the particular case of *free* arrangements, we prove the following:

Theorem 1.3. *Let \mathcal{A} be a projective arrangement such that the corresponding affine arrangement $\widehat{\mathcal{A}}$ is free. Then the Chern–Schwartz–MacPherson class of the complement $M(\mathcal{A})$ equals $c(\Omega_{\mathbb{P}^n}^1(\log A)^\vee) \cap [\mathbb{P}^n]$.*

We view Theorem 1.2 as the primary result, as it holds for arbitrary arrangements, and Theorem 1.3 as a computation in the particular case of free arrangements. We obtain Theorem 1.3 as a consequence of a result of Mustața and Schenck ([MS01]).

It is natural to question whether a similar result may hold for a substantially more general situation. The following conjecture appears to be consistent with all known cases:

Conjecture. *If X is a locally quasi-homogeneous free divisor in a nonsingular variety V , then $c_{SM}(\mathbb{1}_{V \setminus X})$ equals $c(\Omega_V^1(\log X)^\vee) \cap [V]$.*

This statement holds if X is a divisor with simple normal crossings in V , and (as verified in Theorem 1.3) if $V = \mathbb{P}^n$ and X is a free hyperplane arrangement. The restriction to locally quasi-homogeneous divisors is suggested by the case $V = \text{surface}$, studied by Xia Liao [Lia]. (We recall that a hypersurface is ‘locally quasi-homogeneous’ if at each point it admits a weighted homogeneous equation, with positive weights, with respect to some set of analytic parameters.)

1.3. The subtlest part of the information carried by the Chern–Schwartz–MacPherson class of a hypersurface in a nonsingular variety amounts to the *Segre class* of its singularity (‘Jacobian’) subscheme. Extracting this information and applying Theorem 1.2 gives the following (in characteristic 0):

Corollary 1.4. *The characteristic polynomial $\chi_{\widehat{\mathcal{A}}}(t)$ for a hyperplane arrangement \mathcal{A} in \mathbb{P}^n may be recovered from the degree of A and the image in $A_*\mathbb{P}^n$ of the Segre class of the singularity subscheme of A .*

The precise relation between these invariants is given in Theorem 5.1. We note that Wakefield and Yoshinaga have proven ([WY08]) that (essential) hyperplane arrangements may be recovered from their singularity subschemes. Corollary 1.4 shows that the Segre class of the same scheme suffices in order to recover the most basic combinatorial information of the arrangement. Putting together Theorem 5.1 and Corollary 2.3 yields the following surprisingly simple relation: for any hyperplane arrangement in $\mathbb{P}_{\mathbb{C}}^n$ and $k \leq n$

$$(*) \quad \mathrm{rk} H^k(M(\mathcal{A}), \mathbb{Q}) = \sum_{i=0}^k \binom{k}{i} (d-1)^{k-i} \sigma_i \quad ,$$

where d is the degree of the arrangement and the integers σ_i are determined by the push-forward of the Segre class of the singularity subscheme S :

$$\sum_{i=0}^n \sigma_i \cap [\mathbb{P}^{n-i}] = [\mathbb{P}^n] - \iota_* s(S, \mathbb{P}^n) \quad .$$

This equality should be compared with Huh's formula expressing the Betti numbers of the complement as mixed multiplicities; see the proof of Corollary 25 in [Huh]. A referee points out that hyperplane arrangements are the only hypersurfaces for which (*) holds. Indeed, the rank of H^1 of the complement of a hypersurface is one less than the number of distinct irreducible components ([Dim92], Chapter 4, Proposition 1.3). If (*) holds for a degree d hypersurface, then the rank of H^1 equals $d-1$, and it follows that the hypersurface consists of d hyperplanes.

1.4. Finally, we discuss positivity of Chern classes of hyperplane arrangements. Chern–Schwartz–MacPherson classes arising in combinatorial situations have a mysterious tendency to be effective. For example, the Chern–Schwartz–MacPherson class of a toric variety is effective ([Alu06a]); CSM classes of Schubert varieties are conjecturally effective ([AM09b], §4); and all CSM classes of graph hypersurfaces computed to date are effective (cf. [AM09a], Conjecture 1.5). It is natural to inquire about the positivity of CSM classes of hyperplane arrangements.

Theorem 1.5. • *The CSM class of a generic hyperplane arrangement $X \subseteq \mathbb{P}^n$ of degree $d \leq n+3$ (for n even), resp. $d \leq n+4$ (for n odd) is effective.*
 • *The CSM class of every free hyperplane arrangement of degree $\leq n$ in \mathbb{P}^n , $n \leq 8$, is effective.*

However, in §6.4 we exhibit a free arrangement of degree 9 in \mathbb{P}^9 that has non-effective Chern–Schwartz–MacPherson class. The effectivity of the CSM class of an arrangement has a straightforward combinatorial interpretation, which we give in Proposition 6.1.

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2. GROTHENDIECK CLASSES

2.1. We work over a field k . For considerations involving Chern–Schwartz–MacPherson classes, k will be assumed to be algebraically closed, of characteristic 0 (see the comments at the beginning of §3.1).

We consider an arrangement \mathcal{A} of distinct hyperplanes H_i in \mathbb{P}^n , $i = 1, \dots, d$, $d \geq 2$. We also consider the corresponding arrangement $\widehat{\mathcal{A}}$ of hyperplanes $\widehat{H}_i \subset V = k^{n+1}$; this has the advantage of being *central*. While we are interested in the projective geometry of \mathcal{A} , basic definitions and results in the literature are more often given for the associated affine arrangement $\widehat{\mathcal{A}}$. We will denote by A and \widehat{A} , respectively the union of the hyperplanes in \mathcal{A} and $\widehat{\mathcal{A}}$, respectively. We will view A as a reduced singular hypersurface of \mathbb{P}^n of degree $d \geq 2$.

We will be especially interested in the complements $M(\widehat{\mathcal{A}}) = V \setminus \widehat{A}$, $M(\mathcal{A}) = \mathbb{P}^n \setminus A$. Note that $M(\widehat{\mathcal{A}})$ is a trivial k^* -fibration over $M(\mathcal{A})$.

We will denote by $L(\widehat{\mathcal{A}})$ the poset of intersections $\cap_{i \in J} \widehat{H}_i$, partially ordered by reverse inclusion. (For $x, y \in L(\widehat{\mathcal{A}})$ we will write interchangeably $y \supseteq x$ or $y \leq x$ to denote that the subspace y contains the subspace x , i.e., that y precedes x in the poset $L(\widehat{\mathcal{A}})$.) The space V itself is viewed as the intersection over $J = \emptyset$, and is denoted $0 \in L(\widehat{\mathcal{A}})$. As $\widehat{\mathcal{A}}$ is central, $L(\widehat{\mathcal{A}})$ also has a maximum 1, corresponding to $\cap_i \widehat{H}_i$. The arrangement is *essential* if $\cap_i \widehat{H}_i$ is the origin; this assumption will not be needed in this paper.

The *Möbius function* of $L(\widehat{\mathcal{A}})$ is defined on pairs $x \leq y$ by the following prescription:

$$\begin{aligned} \mu(x, x) &= 1 \quad \text{for all } x \in L(\widehat{\mathcal{A}}) \\ \sum_{x \leq z \leq y} \mu(x, z) &= 0 \quad \text{for all } x < y \text{ in } L(\widehat{\mathcal{A}}). \end{aligned}$$

Write $\mu(x)$ for $\mu(0, x)$. The *characteristic polynomial* of the arrangement $\widehat{\mathcal{A}}$ is

$$\chi_{\widehat{\mathcal{A}}}(t) := \sum_{x \in L(\widehat{\mathcal{A}})} \mu(x) t^{\dim x} = t^{n+1} + \dots \quad .$$

Note that $\chi_{\widehat{\mathcal{A}}}(1) = \sum_{0 \leq z \leq 1} \mu(0, z) = 0$. The *Poincaré polynomial* of the arrangement is

$$\pi_{\widehat{\mathcal{A}}}(t) := \sum_{x \in L(\widehat{\mathcal{A}})} \mu(x) (-t)^{\text{codim } x} = (-t)^{n+1} \cdot \chi_{\widehat{\mathcal{A}}}(-t^{-1}) \quad .$$

In any extension in which $\chi_{\widehat{\mathcal{A}}}(t)$ factors completely, $\chi_{\widehat{\mathcal{A}}}(t) = (t - \alpha_1) \cdots (t - \alpha_{n+1})$, we have

$$\begin{aligned} \pi_{\widehat{\mathcal{A}}}(t) &= (-t)^{n+1} \cdot \chi_{\widehat{\mathcal{A}}}(-t^{-1}) = (-t)^{n+1} \left(-\frac{1}{t} - \alpha_1 \right) \cdots \left(-\frac{1}{t} - \alpha_{n+1} \right) \\ &= (1 + \alpha_1 t) \cdots (1 + \alpha_{n+1} t) \quad . \end{aligned}$$

Note that $\pi_{\widehat{\mathcal{A}}}(-1) = \chi_{\widehat{\mathcal{A}}}(1) = 0$: we may assume that $\alpha_{n+1} = 1$, and we let

$$\underline{\chi}_{\widehat{\mathcal{A}}}(t) = (t - \alpha_1) \cdots (t - \alpha_n) = \frac{\chi_{\widehat{\mathcal{A}}}(t)}{t - 1} \quad , \quad \underline{\pi}_{\widehat{\mathcal{A}}}(t) = (1 + \alpha_1 t) \cdots (1 + \alpha_n t) = \frac{\pi_{\widehat{\mathcal{A}}}(t)}{1 + t} \quad .$$

Thus, $\underline{\chi}_{\widehat{\mathcal{A}}}(t)$ and $\underline{\pi}_{\widehat{\mathcal{A}}}(t)$ are polynomials of degree at most n . The results in this paper are most naturally expressed in terms of these polynomials. As we will see in a moment, $\underline{\pi}_{\widehat{\mathcal{A}}}(t)$ has nonnegative coefficients.

2.2. Our first task is the computation of the Grothendieck class of the complement $M(\mathcal{A})$ in terms of the characteristic polynomial. We denote by $[X]$ the class of a k -variety X in the Grothendieck ring $K(\text{Var}_k)$ of varieties. The class of \mathbb{A}_k^1 is denoted \mathbb{L} .

Theorem 2.1. *With notation as above:*

$$[M(\mathcal{A})] = \underline{\chi}_{\widehat{\mathcal{A}}}(\mathbb{L}) \quad .$$

Proof. For $x \in L(\widehat{\mathcal{A}})$, let

$$x^\circ = x \setminus \bigcup_{y > x} y$$

be the complement of the union of smaller intersections. Write $[x] = \mathbb{L}^{\dim x}$, $[x^\circ]$ for the classes in $K(\text{Var}_k)$ of the corresponding subsets of V . Then

$$[y] = \sum_{x \geq y} [x^\circ] \quad .$$

Applying Möbius inversion ([OT92], Proposition 2.39), this gives

$$[y^\circ] = \sum_{x \subseteq y} \mu(y, x)[x]$$

and in particular

$$[M(\widehat{\mathcal{A}})] = [0^\circ] = \sum_{x \in L(\widehat{\mathcal{A}})} \mu(0, x)[x] = \sum_{x \in L(\widehat{\mathcal{A}})} \mu(x) \mathbb{L}^{\dim x} = \underline{\chi}_{\widehat{\mathcal{A}}}(\mathbb{L}) \quad .$$

Since $M(\widehat{\mathcal{A}})$ fibers over $M(\mathcal{A})$, with k^* fibers, we have $[M(\widehat{\mathcal{A}})] = (\mathbb{L} - 1) \cdot [M(\mathcal{A})]$, and the stated formula follows. \square

2.3. If \mathcal{A} is defined over a finite field \mathbb{F}_q , the content of Theorem 2.1 is that the information carried by the characteristic polynomial of $\widehat{\mathcal{A}}$ may be recovered from counting points of A over \mathbb{F}_{q^r} . This observation is of course not new, see [Ath96] (e.g. Theorem 2.2) for a thorough study of characteristic polynomials of arrangements defined over finite fields.

For complex arrangements, Theorem 2.1 has the following immediate consequence:

Corollary 2.2. *With notation as above, the Deligne-Hodge polynomial of the complement $M(\mathcal{A})$ of a complex hyperplane arrangement in \mathbb{P}^n equals $\underline{\chi}_{\widehat{\mathcal{A}}}(uv)$.*

The Deligne-Hodge polynomial is the polynomial $\sum_{p,q} e^{p,q}(X) u^p v^q$, where $e^{p,q}(X) = \sum_k (-1)^k h^{p,q}(H_c^k(X, \mathbb{Q}))$. It is determined by the Grothendieck class of X , by its well-known additivity/multiplicativity properties ([DK86]). As the polynomial for \mathbb{L} is uv , the statement follows immediately from Theorem 2.1.

Taking into account the fact the mixed Hodge structure on the cohomology of the complement of a hyperplane arrangement is pure ([Sha93]), we obtain the following:

Corollary 2.3. $\underline{\chi}_{\widehat{\mathcal{A}}}(t) = \sum_{k=0}^n (-1)^{n+k} \text{rk } H_c^{n+k}(M(\mathcal{A}), \mathbb{Q}) t^k$.

(To keep track of weights: the Hodge structures of H_c^{n+k} and H^{n-k} are compatible through the Poincaré pairing, see [DK86], §1.4 (f); H^{n-k} is of type $(n-k, n-k)$ by [Sha93]; thus H_c^{n+k} is of type (k, k) .)

As $\text{rk } H_c^{n+k}(M(\mathcal{A}), \mathbb{Q}) = \text{rk } H^{n-k}(M(\mathcal{A}), \mathbb{Q})$, this statement is equivalent to

$$\underline{\pi}_{\widehat{\mathcal{A}}}(t) = \sum_{k=0}^n \text{rk } H^k(M(\mathcal{A}), \mathbb{Q}) t^k \quad .$$

This shows that the coefficients of $\underline{\pi}_{\widehat{\mathcal{A}}}(t)$ are nonnegative. Since $M(\widehat{\mathcal{A}}) \cong M(\mathcal{A}) \times k^*$, this also proves

$$\pi_{\widehat{\mathcal{A}}}(t) = \sum_{k=0}^{n+1} \text{rk } H^k(M(\widehat{\mathcal{A}}), \mathbb{Q}) t^k \quad ,$$

a classic result of Orlik and Solomon ([OT92], Theorem 5.93). The approach presented here appears particularly straightforward, since it shows that this result follows directly from Theorem 2.1, which is a trivial consequence of Möbius inversion.

2.4. Another piece of information that may be derived from the Grothendieck class is the stable birational equivalence class. Two projective nonsingular irreducible varieties X, Y are *stably birational* if $X \times \mathbb{P}^m$ is birational to $Y \times \mathbb{P}^n$ for some m, n . Stable birational equivalence classes of complete nonsingular varieties generate a ring $\mathbb{Z}[SB]$, with addition defined by disjoint union and multiplication by product. Every variety (possibly noncomplete or singular) has a well-defined class in $\mathbb{Z}[SB]$. The ring $\mathbb{Z}[SB]$ is defined and studied in [LL03].

Corollary 2.4. *Let \mathcal{A} be a complex hyperplane arrangement in \mathbb{P}^n , and let $A \subset \mathbb{P}^n$ be the union of its components. Then the class of A in $\mathbb{Z}[SB]$ equals $1 - \underline{\chi}_{\widehat{\mathcal{A}}}(0) = 1 - (-1)^n \text{rk } H^n(M(\mathcal{A}), \mathbb{Q})$.*

Proof. The ring $\mathbb{Z}[SB]$ is isomorphic to $K(\text{Var})/(\mathbb{L})$ ([LL03], Theorem 2.3 and Proposition 2.7). Setting $\mathbb{L} = 0$ in Theorem 2.1 shows that $[M(\mathcal{A})]$ is congruent to the constant term of $\underline{\chi}_{\widehat{\mathcal{A}}}(0)$ in $\mathbb{Z}[SB]$. Since $[\mathbb{P}^n] = 1$ in $\mathbb{Z}[SB]$, it follows that the stable birational equivalence class of A equals $1 - \underline{\chi}_{\widehat{\mathcal{A}}}(0) \in \mathbb{Z} \subseteq \mathbb{Z}[SB]$. The equality $\underline{\chi}_{\widehat{\mathcal{A}}}(0) = (-1)^n \text{rk } H^n(M(\mathcal{A}), \mathbb{Q})$ follows from Corollary 2.3. \square

3. CHERN–SCHWARTZ–MACPHERSON CLASSES

3.1. We now assume the ground field k to be algebraically closed, of characteristic 0. This assumption on the characteristic could likely be relaxed: in any environment in which resolution of singularities is available, one may define a class satisfying inclusion-exclusion and the basic normalization property of Chern–Schwartz–MacPherson classes (see [Alu06b], Definition 4.4 and §3.3); these are the only tools needed in this section. However, characteristic 0 is necessary for the main covariance property of these classes recalled below (see [Alu06b], §5.2), and some results on these classes are currently only known in characteristic zero. Hence, we prefer to conservatively adopt this assumption, at the price of limiting the scope of the results. (Many interesting arrangements can only be realized in positive characteristic.)

Recall that there is a homomorphism from the group of constructible functions on a variety X to the Chow group of X , $\varphi \mapsto c_*(\varphi)$, covariant with respect to proper maps and such that $c_*(\mathbb{1}_X)$ equals the total Chern class of the tangent bundle of X if X is nonsingular ([Mac74]; Example 19.1.7 in [Ful84]; [Alu06b]). The key

covariance property of c_* amounts to the fact that if $\alpha : X \rightarrow Y$ is proper, and φ is a constructible function, then $c_*(\alpha_*(\varphi)) = \alpha_*(c_*(\varphi))$. Here the push-forward of constructible functions is defined by taking Euler characteristics of fibers.

We consider this theory over $X = \mathbb{P}^n$. Every subvariety (possibly singular, non-complete) $Y \subseteq \mathbb{P}^n$ determines a *Chern–Schwartz–MacPherson* (CSM) class $c_{\text{SM}}(Y) := c_*(\mathbb{1}_Y) \in A_*\mathbb{P}^n$.

Theorem 3.1. *Let \mathcal{A} be a hyperplane arrangement in \mathbb{P}^n ; denote by A the union of the hyperplanes in \mathcal{A} , by $M(\mathcal{A})$ the complement of A in \mathbb{P}^n , and let $\underline{\chi}_{\widehat{\mathcal{A}}}(t)$ be the polynomial introduced in §2. Then*

$$c_{\text{SM}}(M(\mathcal{A})) = \underline{\chi}_{\widehat{\mathcal{A}}}(t+1) \quad ,$$

where the right-hand side is interpreted as a class in $A_*\mathbb{P}^n$ by replacing t^k by $[\mathbb{P}^k]$, $k = 0, \dots, n$.

This result may be obtained by applying the formula relating CSM classes and Grothendieck classes for varieties obtained by elementary set-theoretic operations on linear subspaces (Proposition 2.2 in [AM09a]): the CSM class is obtained by viewing the Grothendieck class as a polynomial in $\mathbb{T} = [k^*]$ and replacing \mathbb{T}^r with $[\mathbb{P}^r]$. Since $\mathbb{L} = \mathbb{T} + 1$, Theorem 3.1 follows immediately from Theorem 2.1.

In more conventional notation, Theorem 3.1 states the following:

$$c_{\text{SM}}(M(\mathcal{A})) = \left(h^{n+1} \chi_{\widehat{\mathcal{A}}} \left(1 + \frac{1}{h} \right) \right) \cap [\mathbb{P}^n] = \left(h^n \underline{\chi}_{\widehat{\mathcal{A}}} \left(1 + \frac{1}{h} \right) \right) \cap [\mathbb{P}^n] \quad ,$$

where h denotes the hyperplane class in \mathbb{P}^n , and the expressions on the right should be interpreted as the polynomials obtained by expanding them. For the sake of completeness and ease of reference in later sections, we give here a direct proof of these formulas.

Proof. The second formula is just a restatement of the first one. To obtain the first one, consider the functions from $L(\widehat{\mathcal{A}})$ to the abelian group of constructible functions on k^{n+1} , defined by $x \mapsto \mathbb{1}_x$, $x \mapsto \mathbb{1}_{x^\circ}$. (Here x° is as in the proof of Theorem 2.1.) We have

$$\mathbb{1}_y = \sum_{x \geq y} \mathbb{1}_{x^\circ} \quad ,$$

and hence

$$\mathbb{1}_{y^\circ} = \sum_{x \subseteq y} \mu(y, x) \mathbb{1}_x$$

by Möbius inversion. In particular,

$$\mathbb{1}_{M(\widehat{\mathcal{A}})} = \sum_{x \in L(\widehat{\mathcal{A}})} \mu(x) \mathbb{1}_x \quad .$$

For positive dimensional x , the characteristic functions $\mathbb{1}_x$ on k^{n+1} are pull-backs of the corresponding characteristic functions from \mathbb{P}^n . It follows that

$$\mathbb{1}_{M(\mathcal{A})} = \sum_{x \in L(\widehat{\mathcal{A}})} \mu(x) \mathbb{1}_x$$

on \mathbb{P}^n , where \underline{x} denotes the projective subspace corresponding to $x \subset V$ (and $\underline{x} = \emptyset$ if x is the origin in V). Applying MacPherson's natural transformation, this shows that

$$c_{\text{SM}}(M(\mathcal{A})) = \sum_{x \in L(\widehat{\mathcal{A}})} \mu(x) c_{\text{SM}}(\underline{x}) \in A_* \mathbb{P}^n \quad .$$

Now $\underline{x} \cong \mathbb{P}^{\dim x - 1}$, and hence

$$c_{\text{SM}}(\underline{x}) = (1 + h)^{\dim x} h^{n+1 - \dim x} \cap [\mathbb{P}^n]$$

as a class in \mathbb{P}^n . (In particular $c_{\text{SM}}(\underline{x}) = 0$ if x is the origin in V , as it should, since $h^{n+1} \cap [\mathbb{P}^n] = 0$.) It follows that

$$c_{\text{SM}}(M(\mathcal{A})) = \left(h^{n+1} \sum_{x \in L(\widehat{\mathcal{A}})} \mu(x) \frac{(1 + h)^{\dim x}}{h^{\dim x}} \right) \cap [\mathbb{P}^n] = \left(h^{n+1} \chi_{\widehat{\mathcal{A}}} \left(1 + \frac{1}{h} \right) \right) \cap [\mathbb{P}^n] \quad ,$$

as stated. □

Corollary 3.2. *With the same notation:*

$$c_{\text{SM}}(M(\mathcal{A})) = (1 + h)^n \pi_{\widehat{\mathcal{A}}} \left(\frac{-h}{1 + h} \right) \cap [\mathbb{P}^n] = \pi_{\widehat{\mathcal{A}}} \left(\frac{-h}{1 + h} \right) \cap (c(T\mathbb{P}^n) \cap [\mathbb{P}^n]) \quad .$$

Example 3.3. Suppose \mathcal{A} is *generic*, i.e., it consists of d hyperplanes meeting with normal crossings. Then

$$c_{\text{SM}}(M(\mathcal{A})) = c(\Omega(\log A)^\vee) \cap [A] = \frac{1}{(1 + h)^d} \cap (c(T\mathbb{P}^n) \cap [\mathbb{P}^n]) \quad ,$$

see e.g., Theorem 1 in [Alu99b]. By Corollary 3.2, then:

$$\pi_{\widehat{\mathcal{A}}} \left(\frac{-h}{1 + h} \right) = \frac{1}{(1 + h)^d}$$

modulo h^{n+1} . Setting $t = \frac{-h}{1+h}$, i.e., $h = \frac{-t}{1+t}$, gives

$$\pi_{\widehat{\mathcal{A}}}(t) \equiv (1 + t)^d \pmod{t^{n+1}} \quad .$$

The coefficient of t^{n+1} in $\pi_{\widehat{\mathcal{A}}}(t)$ is then determined by the fact that $\pi_{\widehat{\mathcal{A}}}(-1) = 0$.

For instance, $\pi_{\widehat{\mathcal{A}}}(t) = (1 + t)^d$ for $d \leq n + 1$. The *Boolean arrangement*, where $\widehat{\mathcal{A}}$ consists of the $n + 1$ coordinate hyperplanes in $V \cong k^{n+1}$, is of this type; cf. §2.3 in [OT92]. ┘

4. FREE ARRANGEMENTS

4.1. The key feature of Example 3.3 is the (well known) fact that if D is a divisor with normal crossings in a nonsingular variety, then the bundle $\Omega^1(\log D)$ is locally free and its total Chern class computes the Chern–Schwartz–MacPherson class of the complement of D . Thus, for normal crossings arrangements the left-hand side of the formulas in Theorem 3.1 and Corollary 3.2 may be viewed as the Chern class of a bundle, and the formulas may be interpreted as an alternative computation of this class.

In this section we show that this interpretation extends to *free arrangements*, by which we mean projective arrangements \mathcal{A} such that the corresponding affine arrangements $\widehat{\mathcal{A}}$ are free in the sense of [OT92]. For these arrangements, the sheaf $\Omega_{\mathbb{P}^n}^1(\log A)$ of differential 1-forms with logarithmic poles along the union A of the hyperplanes in \mathcal{A} is locally free. We will prove:

Theorem 4.1. *For free arrangements \mathcal{A} in \mathbb{P}^n ,*

$$(\dagger) \quad c_{SM}(M(\mathcal{A})) = c(\Omega_{\mathbb{P}^n}^1(\log A)^\vee) \cap [\mathbb{P}^n] \quad .$$

We recall that sections of the logarithmic sheaf $\Omega_{\mathbb{P}^n}^1(\log A)$ are differential forms ω such that both $f\omega$ and $fd\omega$ are regular, where $f = 0$ is the equation for A . This definition, due to Saito, generalizes Deligne’s definition for normal crossing divisors. In general, a divisor D of a variety X is said to be ‘free’ if this sheaf is locally free.

4.2. Taken together, Theorem 3.1 and Theorem 4.1 give a relation between the characteristic polynomial of an arrangement \mathcal{A} and the total Chern class of $\Omega_{\mathbb{P}^n}^1(\log A)$ in the case of free arrangements. Theorem 3.1 may be viewed as a generalization of this relation, as it holds for arbitrary projective arrangements—notwithstanding the fact that its proof is completely trivial, while the proof of the particular case of free arrangements in the form of Theorem 4.1 requires some actual work. By our good fortune, the main ingredient in this work may be found in a paper of M. Mustața and H. Schenck; Theorem 4.1 will be obtained as a consequence of Theorem 4.1 in [MS01].

This is one instance in which MacPherson’s functorial theory of Chern classes clearly provides the ‘right’ generalization of Chern classes to noncomplete (and/or singular) varieties, and it is tempting to guess that the equality in Theorem 4.1 may hold for more general free divisors. Work of Xia Liao ([Lia]) shows that for a reduced curve X in a nonsingular surface V , the Chern–Schwartz–MacPherson class of the complement equals the Chern class of the corresponding bundle of logarithmic derivations only if the Milnor and Tjurina numbers of the singularities of X agree. This indicates that a hypothesis of local quasi-homogeneity is likely necessary for a generalization of Theorem 4.1; we conjecture as much in the introduction.

In any case, the question of comparing the CSM class of the complement of a divisor D and the Chern classes of the corresponding sheaf of differential forms with logarithmic poles along D appears to be interesting and approachable. Theorem 5.13 in [DS12] indicates that a correction term will be necessary if this sheaf is not locally free, as it provides such a term for the corresponding generalization of Theorem 4.1 in [MS01] to the ‘locally tame’ case.

4.3. The formula in Theorem 4.1 holds if A is any divisor with simple normal crossings in a nonsingular variety. For a proof of this elementary fact, see e.g., Theorem 1 in [Alu99b] or Proposition 15.3 in [GP02]; this observation may in fact be used to give an alternative treatment of CSM classes ([Alu06b]). The obvious strategy to prove Theorem 4.1 would therefore be to apply resolution of singularities and reduce to the case of normal crossing divisors. The behavior of the left-hand side of (\dagger) through a resolution is controlled by the covariance of CSM classes:

Lemma 4.2. *Let X be a variety, and let $Y \subseteq X$ be a subscheme. Let $\rho : \tilde{X} \rightarrow X$ be a proper map, and $Y' \subseteq \tilde{X}$ any subscheme such that ρ restricts to an isomorphism of the complements $M(Y')$ of Y' in \tilde{X} and $M(Y)$ of Y in X . Then*

$$c_{SM}(M(Y)) = \rho_* c_{SM}(M(Y')) \quad .$$

Proof. Under the hypotheses specified in the statement, $\rho_*(\mathbb{1}_{M(Y')}) = \mathbb{1}_{M(Y)}$, and the equality follows then immediately from the covariance property of Chern–Schwartz–MacPherson classes (recalled in §3.1). \square

Lemma 4.2 reduces the proof of Theorem 4.1 to verifying that the Chern class of the bundle of differential forms with logarithmic poles is preserved under push-forward for suitable blow-ups. The difficulty lies in the fact that the bundle itself is *not* preserved under blow-ups. However, Silvotti ([Sil97]) has analyzed the behavior of the logarithmic bundle under blow-ups in the case of arrangements, and we feel that his analysis should suffice in order to obtain a proof of Theorem 4.1.

In fact, Silvotti proves ([Sil97], Proposition 4.5) that if $\Omega_X^1(\log D)$ splits as a direct sum of line bundles $\mathcal{L}_1, \dots, \mathcal{L}_n$, then the corresponding bundle $\Omega_{X'}^1(\log D')$ in the blow-up along a subvariety Y also splits, and in fact

$$\Omega_{X'}^1(\log D') \cong (\sigma^* \mathcal{L}_1 \otimes (-\mu_1 E)) \oplus \cdots \oplus (\sigma^* \mathcal{L}_n \otimes (-\mu_n E))$$

for non-negative integers μ_1, \dots, μ_n , where E denotes the exceptional divisor. (Terao proved that the splitting does occur in the case of free hyperplane arrangements, cf. Proposition 5.1 in [Sil97].) A proof of Theorem 4.1 follows easily if one could show that *at most* $\text{codim } Y - 1$ of the numbers μ_i are nonzero.

In fact, given that Theorem 4.1 does hold, it seems that this must indeed be the case. It would be nice to have a direct proof of this fact.

4.4. Theorem 4.1 in [MS01] provides us with an alternative approach to Theorem 4.1. Denote by Ω^1 the *module* of differential forms with logarithmic poles along the central arrangement $\widehat{\mathcal{A}}$ in k^{n+1} . This is a graded module, hence it defines a coherent sheaf $\widetilde{\Omega}^1$ on \mathbb{P}^n . Under the assumption that the arrangement is free, $\widetilde{\Omega}^1$ is a rank- $(n+1)$ locally free sheaf on \mathbb{P}^n .

Theorem 4.3 (Mustață–Schenck). *If $\widehat{\mathcal{A}}$ is an essential free arrangement, then*

$$c(\widetilde{\Omega}^1) = \pi_{\widehat{\mathcal{A}}}(h) \quad ,$$

where $h = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$.

Remark 4.4. In fact, Mustață and Schenck prove the equality in Theorem 4.3 under the weaker hypothesis that the arrangement is *locally* free. Also, we note that while the statement of our Theorem 4.1 assumes the ground field to be algebraically closed of characteristic 0, this assumption is not needed in the result of Mustață and Schenck.

In Theorem 4.3, the right-hand side should be parsed as the truncation of the Poincaré polynomial modulo h^{n+1} , as $h^{n+1} = 0$ in \mathbb{P}^n . Also, we recall that ‘essential’ means that the intersection of all hyperplanes in the projective arrangement \mathcal{A} is empty. \lrcorner

We now prove Theorem 4.1 as a corollary of Theorems 3.1 and 4.3.

Lemma 4.5. *Let \mathcal{A} be a free arrangement. Then there is an exact sequence*

$$0 \longrightarrow \Omega_{\mathbb{P}^n}^1(\log A) \longrightarrow \widetilde{\Omega}^1 \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0 \quad .$$

Proof. The Euler derivation $x_0 \frac{\partial}{\partial x_0} + \cdots + x_n \frac{\partial}{\partial x_n}$ defines an epimorphism $\widetilde{\Omega}^1 \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)$ (cf. Proposition 4.27 in [OT92] for the dual statement). The shifted epimorphism $\widetilde{\Omega}^1(-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}$ is the natural extension of the standard epimorphism $\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)} \rightarrow \mathcal{O}_{\mathbb{P}^n}$ whose kernel defines the sheaf of differential forms over \mathbb{P}^n (as in [Har77], II.8.13). The kernel of this epimorphism is then the sheaf of meromorphic differential forms satisfying the same conditions as the forms in Ω^1 , and this is the definition of $\Omega_{\mathbb{P}^n}^1(\log A)$. \square

Proof of Theorem 4.1. First, we note that we may assume the arrangement to be essential. Indeed, every arrangement \mathcal{A} in \mathbb{P}^n is a cone over an essential arrangement \mathcal{A}' in \mathbb{P}^{n-k} , for some $k \geq 0$. Assume the formula (†) in Theorem 4.1 is known for \mathcal{A}' :

$$c_{\text{SM}}(M(\mathcal{A}')) = c(\Omega_{\mathbb{P}^{n-k}}^1(\log A')^\vee) \cap [\mathbb{P}^{n-k}] \quad .$$

Inductively, it suffices to show that the formula for \mathcal{A} follows from this for $k = 1$. Write $c_{\text{SM}}(M(\mathcal{A}')) = g(h) \cap [\mathbb{P}^{n-1}]$, for a polynomial g of degree $\leq n-1$; we are assuming that $c(\Omega_{\mathbb{P}^{n-1}}^1(\log A')^\vee) = g(h)$. We have $c_{\text{SM}}(A) = f(h) \cap [\mathbb{P}^{n-1}]$ for $f(h) = (1+h)^n - h^n - g(h)$, hence by Proposition 5.2 in [AM09a]

$$c_{\text{SM}}(A) = (1+h)f(h) \cap [\mathbb{P}^n] + [\mathbb{P}^0] \quad ,$$

and therefore

$$\begin{aligned} c_{\text{SM}}(M(\mathcal{A})) &= c_{\text{SM}}(\mathbb{P}^n) - c_{\text{SM}}(A) \\ &= (((1+h)^{n+1} - h^{n+1}) - ((1+h)((1+h)^n - h^n - g(h)) + h^n)) \cap [\mathbb{P}^n] \\ &= (1+h)g(h) \cap [\mathbb{P}^n] \end{aligned}$$

(The more combinatorially minded reader may reach the same conclusion as a consequence of Theorem 3.1.) On the other hand, by Lemma 4.5 we have

$$c(\Omega_{\mathbb{P}^n}^1(\log A)) = c(\widetilde{\Omega}^1(-1))$$

where Ω^1 is the module of differentials with logarithmic poles along \mathcal{A}' . Under the assumption that \mathcal{A} is free so is \mathcal{A}' , and $\Omega^1 = \widetilde{\Omega}^1 \oplus k$. Therefore $c(\widetilde{\Omega}^1(-1)) = (1-h)c(\widetilde{\Omega}^1(-1))$, and again by Lemma 4.5 we get $c(\Omega_{\mathbb{P}^n}^1(\log A)) = (1-h)c(\widetilde{\Omega}^1(-1))$, and hence

$$c(\Omega_{\mathbb{P}^n}^1(\log A))^\vee = (1+h)c(\widetilde{\Omega}^1(-1)^\vee) = (1+h)g(h) \quad .$$

It follows that

$$c_{\text{SM}}(M(\mathcal{A})) = (1+h)g(h) \cap [\mathbb{P}^n] = c(\Omega_{\mathbb{P}^n}^1(\log A))^\vee \cap [\mathbb{P}^n] \quad ,$$

which is (†) for \mathcal{A} , as claimed.

Therefore, we may assume that the arrangement is essential. By Theorem 4.3,

$$c(\widetilde{\Omega}^1) = \pi_{\widehat{\mathcal{A}}}(h) = (1+h)\pi_{\widehat{\mathcal{A}}}(h)$$

in $A^*\mathbb{P}^n$ (that is, modulo h^{n+1}). Using Lemma 4.5, it follows that

$$\underline{\pi}_{\widehat{\mathcal{A}}}(h) \equiv (1+h)^{-1}c(\widetilde{\Omega}^1) \pmod{h^{n+1}} \equiv c(\Omega_{\mathbb{P}^n}^1(\log A) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) \pmod{h^{n+1}} ,$$

and hence

$$\underline{\pi}_{\widehat{\mathcal{A}}}(h) = c(\Omega_{\mathbb{P}^n}^1(\log A) \otimes \mathcal{O}_{\mathbb{P}^n}(1))$$

as polynomials in h , since both sides have degree $\leq n$. Now (as in §2.1) we factor $\underline{\chi}_{\widehat{\mathcal{A}}}(t) = (t - \alpha_1) \cdots (t - \alpha_n)$ over an extension¹ of \mathbb{Q} , and note that $\underline{\pi}_{\widehat{\mathcal{A}}}(t) = (1 + \alpha_1 t) \cdots (1 + \alpha_n t)$. With this notation, we have shown

$$c(\Omega_{\mathbb{P}^n}^1(\log A) \otimes \mathcal{O}_{\mathbb{P}^n}(1)) = (1 + \alpha_1 h) \cdots (1 + \alpha_n h) ,$$

and it follows that

$$\begin{aligned} c(\Omega_{\mathbb{P}^n}^1(\log A)^\vee) &= (1 - \alpha_1 h + h) \cdots (1 - \alpha_n h + h) \\ &= h^n \left(1 + \frac{1}{h} - \alpha_1\right) \cdots \left(1 + \frac{1}{h} - \alpha_n\right) \\ &= h^n \underline{\chi}_{\widehat{\mathcal{A}}}\left(1 + \frac{1}{h}\right) \end{aligned}$$

with the usual caveat that the right-hand side must be interpreted as the polynomial obtained by expanding it. By Theorem 3.1 this proves

$$c(\Omega_{\mathbb{P}^n}^1(\log A)^\vee) \cap [\mathbb{P}^n] = c_{\text{SM}}(M(\mathcal{A})) ,$$

and we are done. \square

Remark 4.6. The projective version of Theorem 4.1 from [MS01] used above is also given in [DS12], §5, and generalized to locally tame arrangements. \lrcorner

5. SEGRE CLASSES OF SINGULARITY SUBSCHEMES

5.1. In [WY08], M. Wakefield and M. Yoshinaga prove that any (essential) projective arrangement \mathcal{A} may be reconstructed from the *singularity subscheme* S of the hypersurface $A \subseteq \mathbb{P}^n$, that is, the subscheme defined by the partial derivatives of an equation for A . In this section we prove that the polynomial $\underline{\pi}_{\widehat{\mathcal{A}}}(t)$ determines and is determined by the degree of the arrangement and the Segre class (cf. [Ful84], Chapter 4) of the singularity subscheme in \mathbb{P}^n . As we will use Chern–Schwartz–MacPherson classes for this result, we still work over algebraically closed fields of characteristic 0. The following statement makes sense over any field, but we do not know whether it holds in such generality.

Theorem 5.1. *Let $\iota : S \hookrightarrow \mathbb{P}^n$ be the singularity subscheme of an arrangement \mathcal{A} in \mathbb{P}^n . (That is, S is defined by the partial derivatives of an equation for the hypersurface A .) Let $\sigma_i \in \mathbb{Z}$ be such that*

$$[\mathbb{P}^n] - \iota_* s(S, \mathbb{P}^n) = \sum_{i=0}^n \sigma_i h^i \cap [\mathbb{P}^n] \in A_* \mathbb{P}^n .$$

¹According to a theorem of Terao, the polynomial of a free arrangement actually factors over \mathbb{Z} ; cf. §6.4. This is not needed here.

Then

$$\underline{\pi}_{\widehat{\mathcal{A}}}(t) = \sum_{k=0}^n \left(\sum_{i=0}^k \binom{k}{i} (d-1)^{k-i} \sigma_i \right) t^k \quad .$$

Matching this formula with the expression for $\underline{\pi}_{\widehat{\mathcal{A}}}(t)$ obtained in the wake of Corollary 2.3 yields the formula given in the introduction for the ranks of the cohomology of the complement.

Proof. By Corollary 3.2,

$$(\dagger) \quad c_{\text{SM}}(A) = c(T\mathbb{P}^n) \cap [\mathbb{P}^n] - (1+h)^n \underline{\pi}_{\widehat{\mathcal{A}}} \left(\frac{-h}{1+h} \right) \cap [\mathbb{P}^n] \quad .$$

On the other hand, by Theorem I.4 in [Alu99a],

$$c_{\text{SM}}(A) = c(T\mathbb{P}^n) \cap \left(\frac{dh}{1+dh} \cap [\mathbb{P}^{n-1}] + \frac{1}{1+dh} \cap ((\iota_* s(S, \mathbb{P}^n))^\vee \otimes_{\mathbb{P}^n} \mathcal{O}(dh)) \right)$$

where d is the degree of the arrangement, and this expression uses notation given in [Alu99a], §1.4. Writing $\iota_* s(S, \mathbb{P}^n) = \sum_{i=0}^n s_i [\mathbb{P}^i]$, this means

$$(\ddagger) \quad c_{\text{SM}}(A) = (1+h)^{n+1} \left(\frac{dh}{1+dh} + \frac{1}{1+dh} \sum_{i=0}^n \frac{s_i \cdot (-h)^{n-i}}{(1+dh)^{n-i}} \right) \cap [\mathbb{P}^n]$$

Comparing (\dagger) and (\ddagger) gives the following equality of series modulo h^{n+1} :

$$\underline{\pi}_{\widehat{\mathcal{A}}} \left(\frac{-h}{1+h} \right) = \frac{1+h}{1+dh} - \frac{1+h}{1+dh} \sum_{i=0}^n \frac{s_i \cdot (-h)^{n-i}}{(1+dh)^{n-i}} \quad ,$$

and setting $t = -h/(1+h)$ yields

$$\underline{\pi}_{\widehat{\mathcal{A}}}(t) \equiv \frac{1}{1-(d-1)t} \left(1 - \sum_{i=0}^n s_i \cdot \left(\frac{t}{1-(d-1)t} \right)^{n-i} \right) \pmod{t^{n+1}} \quad .$$

Now with notation as in the statement we have $\sigma_0 = 1$ and $\sigma_i = -s_{n-i}$ for $i > 0$, and therefore

$$\underline{\pi}_{\widehat{\mathcal{A}}}(t) \equiv \frac{1}{1-(d-1)t} \sum_{i=0}^n \sigma_i \cdot \left(\frac{t}{1-(d-1)t} \right)^i \pmod{t^{n+1}} \quad .$$

The statement follows immediately from this equality. \square

Using notation as in §1.4 of [Alu99a], the formula given in Theorem 5.1 may be rewritten as

$$(*) \quad \underline{\pi}_{\widehat{\mathcal{A}}}(h) \cap [\mathbb{P}^n] = \frac{1}{1-(d-1)h} \cap (([\mathbb{P}^n] - \iota_* s(S, \mathbb{P}^n)) \otimes_{\mathbb{P}^n} \mathcal{O}(-(d-1)h)) \quad .$$

This is occasionally convenient in concrete computations, see e.g., Example 5.4.

5.2. We illustrate Theorem 5.1 with a few examples, in which the computation of the Poincaré polynomial of the arrangement can also be performed easily with standard techniques. Algorithms computing Segre classes may be implemented in software systems such as Macaulay2 ([GS]); one such implementation is described in [Alu03]. See Example 6.6 for an illustration of the use of such a routine.

Example 5.2. The three transversal intersections of the configuration of Example 6.2 count for one point each in the Segre class of the singularity subscheme. To evaluate the contribution of the triple intersection, write it in local coordinates as the singularity subscheme of $xy(x+y) = 0$; the Jacobian ideal is then

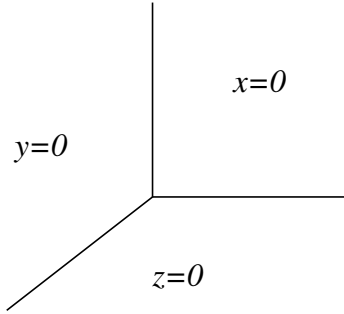
$$(2xy + y^2, x^2 + 2xy)$$

and it follows that the contribution to the Segre class is 4 points. Thus $(\sigma_0, \sigma_1, \sigma_2) = (1, 0, -7)$, and Theorem 5.1 gives

$$\pi_{\widehat{\mathcal{A}}}(t) = 1 + 3t + t^2 \quad .$$

Therefore $\pi_{\widehat{\mathcal{A}}}(t) = (1+t)\pi_{\widehat{\mathcal{A}}}(t) = 1 + 4t + 5t^2 + 2t^3$, i.e., $\chi_{\widehat{\mathcal{A}}}(t) = t^3 - 4t^2 + 5t - 2$, as it should. \lrcorner

Example 5.3. Let \mathcal{A} consist of three planes in \mathbb{P}^3 , with equation $xyz = 0$.



The singularity subscheme S is supported on three lines; it is defined by the ideal (yz, xz, xy) , so it is the intersection of three quadrics Q_1, Q_2, Q_3 . The Segre class $s(S, \mathbb{P}^3)$ is $3[\mathbb{P}^1] + m[\mathbb{P}^0]$ for some integer m . One way to evaluate m is the following: the intersection product of the three quadrics must be 8 by Bézout's theorem, and can be evaluated by applying the 'basic construction' (Proposition 6.1 (a) in [Ful84]) to the fiber diagram

$$\begin{array}{ccc} S = Q_1 \cap Q_2 \cap Q_3 & \longrightarrow & \mathbb{P}^3 \\ \downarrow & & \downarrow \\ Q_1 \times Q_2 \times Q_3 & \longrightarrow & \mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3 \end{array}$$

Writing h_i for the hyperplane class in the i -th copy of \mathbb{P}^3 , this gives

$$8 = \int (1 + 2h_1)(1 + 2h_2)(1 + 2h_3) \cap (3[\mathbb{P}^1] + m[\mathbb{P}^0]) = 18 + m \quad ,$$

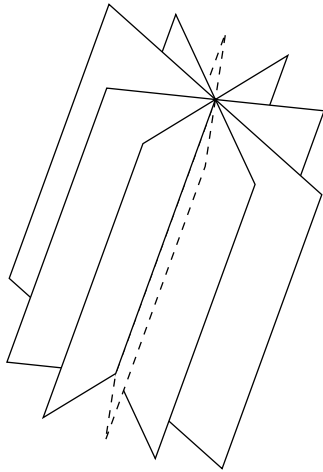
from which $m = -10$.

With notation as in Theorem 5.1 we have $(\sigma_0, \dots, \sigma_3) = (1, 0, -3, 10)$, from which

$$\pi_{\widehat{\mathcal{A}}}(t) = 1 + 2t + (4 - 3)t^2 + (8 - 3 \cdot 2 \cdot 3 + 10)t^3 = 1 + 2t + t^2 = (1+t)^2 \quad .$$

Therefore $\pi_{\widehat{\mathcal{A}}}(t) = (1+t)^3$. Of course this agrees with Example 3.3, since \mathcal{A} is a generic arrangement. \lrcorner

Example 5.4. Let \mathcal{A} consist of d hyperplanes in the pencil of hyperplanes containing a fixed codimension-2 subspace in \mathbb{P}^n .



The singularity subscheme S is supported on \mathbb{P}^{n-2} . To evaluate its Segre class, blow-up along this subspace; if E is the exceptional divisor, the Segre class of the latter pushes forward to the Segre class of \mathbb{P}^{n-2} , by the birational invariance of Segre classes: $\frac{E}{1+E} \mapsto s(\mathbb{P}^{n-2}, \mathbb{P}^n) = \frac{1}{(1+h)^2} \cap [\mathbb{P}^{n-2}]$. A straightforward computation shows that the singularity subscheme pulls back to $(d-1)$ times the exceptional divisor. Therefore (again by birational invariance) $s(S, \mathbb{P}^n)$ is the push-forward of $(d-1)E/(1+(d-1)E)$, and matching terms gives

$$\iota_* s(S, \mathbb{P}^n) = \frac{1}{(1+(d-1)h)^2} \cap (d-1)^2 [\mathbb{P}^{n-2}] \quad .$$

Using (*), we get that

$$\begin{aligned} \pi_{\widehat{\mathcal{A}}}(h) \cap [\mathbb{P}^n] &= \frac{1}{1-(d-1)h} \cap \left([\mathbb{P}^n] - \frac{(d-1)[\mathbb{P}^{n-2}]}{(1+(d-1)h)^2} \otimes \mathcal{O}(-(d-1)h) \right) \\ &= \frac{1}{1-(d-1)h} \cap ([\mathbb{P}^n] - (d-1)^2 [\mathbb{P}^{n-2}]) \\ &= (1+(d-1)h) \cap [\mathbb{P}^n] \quad . \end{aligned}$$

Therefore $\pi_{\widehat{\mathcal{A}}}(t) = 1 + (d-1)t$. It follows that $\pi_{\widehat{\mathcal{A}}}(t) = (1+(d-1)t)(1+t) = 1 + dt + (d-1)t^2$, and $\chi_{\widehat{\mathcal{A}}}(t) = t^{n+1} - dt^n + (d-1)t^{n-1}$. (This is of course also evident from the poset associated with this arrangement.) \square

6. POSITIVITY

6.1. One problem that prompted us to take a more careful look at hyperplane arrangements is the issue of *positivity* of Chern–Schwartz–MacPherson classes. In the nonsingular case, positivity of Chern classes is well understood; for example, $c(TX) \cap [X]$ is effective if TX is generated by global sections (cf. [Ful84], Example 12.1.7). We know of no such statement for Chern classes of singular varieties, and preciously few examples are known: the CSM class of a toric variety is represented by an effective cycle (this follows from “Ehlers’ formula”, see e.g., [BBF92]), and CSM classes of Schubert varieties are conjecturally effective. It is natural to ask the following

Question. Denote by $A \subseteq \mathbb{P}^n$ the union of the hyperplanes of an arrangement \mathcal{A} . For which arrangements \mathcal{A} is $c_{SM}(A)$ effective?

Here, by ‘effective’ we mean that $c_{\text{SM}}(A) \in A_*\mathbb{P}^n$ should be represented by an effective cycle. By Theorem 3.1, $c_{\text{SM}}(A)$ is determined by the characteristic polynomial of $\widehat{\mathcal{A}}$, so this is a combinatorial question.

6.2. Here is the explicit translation of effectivity in combinatorial terms:

Proposition 6.1. *Let \mathcal{A} be a hyperplane arrangement in \mathbb{P}^n , and let μ be the corresponding Möbius function, as in §2.1. Then $c_{\text{SM}}(A)$ is effective if and only if all coefficients of the polynomial*

$$-\sum_{x \neq 0} \mu(x)(t+1)^{\dim x}$$

are nonnegative.

In fact, the coefficient of t^k in this expression equals the coefficient of $[\mathbb{P}^{k-1}]$ in $c_{\text{SM}}(A)$, for $k \geq 1$; the constant term equals 1.

Proof. In the (direct) proof of Theorem 3.1 we obtained the equality

$$c_{\text{SM}}(M(\mathcal{A})) = \sum_{x \in L(\widehat{\mathcal{A}})} \mu(x)c_{\text{SM}}(\underline{x}) \quad ,$$

where \underline{x} denotes the projective subspace of \mathbb{P}^n determined by x . The summand corresponding to $x = 0$ is $\mu(0)c_{\text{SM}}(\underline{0}) = c(T\mathbb{P}^n) \cap [\mathbb{P}^n]$. Thus

$$c_{\text{SM}}(A) = c(T\mathbb{P}^n) \cap [\mathbb{P}^n] - c_{\text{SM}}(M(\mathcal{A})) = -\sum_{x \neq 0} \mu(x)c_{\text{SM}}(\underline{x}) \quad .$$

Now $\underline{x} \cong \mathbb{P}^{\dim x - 1}$, so

$$c_{\text{SM}}(\underline{x}) = \sum_{k=1}^{\dim x} \binom{\dim x}{k} [\mathbb{P}^{k-1}] \quad .$$

Hence the coefficient of $[\mathbb{P}^{k-1}]$ in $c_{\text{SM}}(A)$ equals

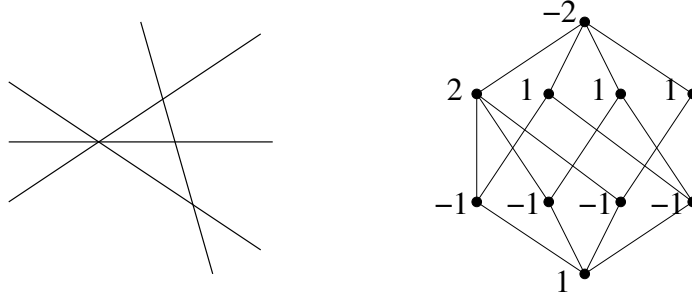
$$-\sum_{x \neq 0 : \dim x \geq k} \mu(x) \binom{\dim x}{k} \quad ,$$

that is, the coefficient of t^k in

$$-\sum_{x \neq 0} \mu(x)(t+1)^{\dim x} \quad .$$

This holds for $k \geq 1$. On the other hand, since $\sum_x \mu(x) = 0$ and $\mu(0) = 1$, the constant term in this expression is $1 > 0$. It follows that $c_{\text{SM}}(A)$ is effective if and only if all coefficients of this polynomial are nonnegative, which is the statement. \square

Example 6.2. We illustrate Proposition 6.1 with a simple example.



The poset and Möbius function for the arrangement on the left (in \mathbb{P}^2) are given on the right; the 0 element of the lattice, i.e., k^3 , is at the bottom. The polynomial appearing in Proposition 6.1 is

$$-(-4(t + 1)^2 + 5(t + 1) - 2) = 4t^2 + 3t + 1 \quad .$$

As the coefficients are all positive, the CSM class of this arrangement is effective; this class equals $4[\mathbb{P}^1] + 3[\mathbb{P}^0]$. □

6.3. Heuristically, arrangements of low degree should be more likely to have effective CSM class. This is the case for generic arrangements:

Proposition 6.3. *Let \mathcal{A} be a generic arrangement of $d \geq 1$ distinct hyperplanes in \mathbb{P}^n . Then $c_{SM}(A)$ is effective for $n = 1$ and all d , and for $n > 1$ and*

- n even, $d \leq n + 3$,
- n odd, $d \leq n + 4$,

and it is not effective otherwise.

Proof. The arrangement \mathcal{A} is generic precisely when A is a divisor with simple normal crossings. As seen in Example 3.3, the CSM class of the complement is $c_{SM}(M(\mathcal{A})) = \frac{1}{(1+h)^d} \cap (c(T\mathbb{P}^n) \cap [\mathbb{P}^n])$, and hence $c_{SM}(A)$ equals

$$\left(1 - \frac{1}{(1+h)^d}\right) (1+h)^{n+1} \cap [\mathbb{P}^n] = \sum_{k=0}^n \left(\binom{n+1}{k} - (-1)^k \binom{k+d-n-2}{k} \right) h^k \cap [\mathbb{P}^n] \quad .$$

The statement is easy to verify from this expression. If n is even and $d \geq n + 4$, the coefficient of $[\mathbb{P}^0]$ (i.e., the Euler characteristic of A) is bound by

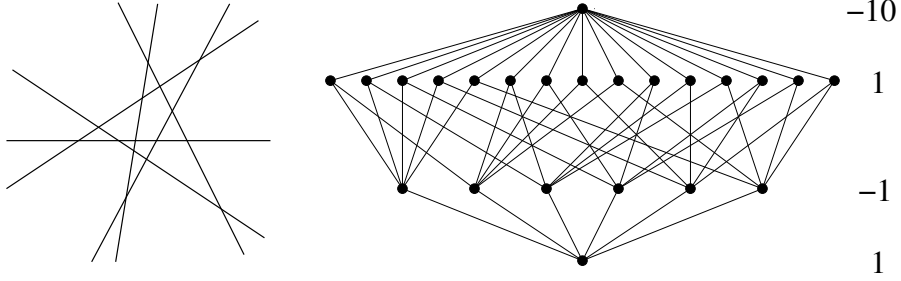
$$(n + 1) - \binom{n + 2}{2} < 0 \quad ;$$

if $n > 1$ is odd and $d \geq n + 5$, the coefficient of $[\mathbb{P}^1]$ is bound by

$$\binom{n + 1}{2} - \binom{n + 2}{3} < 0 \quad ,$$

so the class is not effective outside of the specified range. □

Example 6.4. The smallest generic arrangement with non-effective CSM class consists of six lines in \mathbb{P}^2 :



The polynomial in the statement of Proposition 6.1 is

$$-(6 \cdot (-1) \cdot (t+1)^2 + 15 \cdot 1 \cdot (t+1) - 10) = 6t^2 - 3t + 1 \quad ,$$

and not all its coefficients are nonnegative. The CSM class of this arrangement is $6[\mathbb{P}^1] - 3[\mathbb{P}^0]$; its Euler characteristic is -3 . \square

6.4. Another source of interesting examples comes from free arrangements.

Proposition 6.5. *For $n \leq 8$, every free arrangement \mathcal{A} of $d \leq n$ hyperplanes in \mathbb{P}^n has effective CSM class.*

Proof. According to a theorem of Terao ([Ter81]; see also [OT92], Chapter 4), the characteristic polynomial of a free central arrangement $\widehat{\mathcal{A}}$ in k^{n+1} factors over \mathbb{Z} :

$$\chi_{\widehat{\mathcal{A}}}(t) = (t - d_1) \cdots (t - d_n) \cdot (t - d_{n+1}) \quad ,$$

where the d_i 's are the 'exponents' of the arrangement, i.e., the degrees of the generators of the (free) module of $\widehat{\mathcal{A}}$ -derivations (Definitions 4.5 and 4.25 in [OT92]). One of the exponents necessarily equals 1 (cf. §2.1); the sum of the exponents equals the number of hyperplanes in the arrangement ([OT92], Proposition 4.26).

Thus, we may assume that the characteristic polynomial of the arrangement is

$$(t - d_1) \cdots (t - d_n)(t - 1)$$

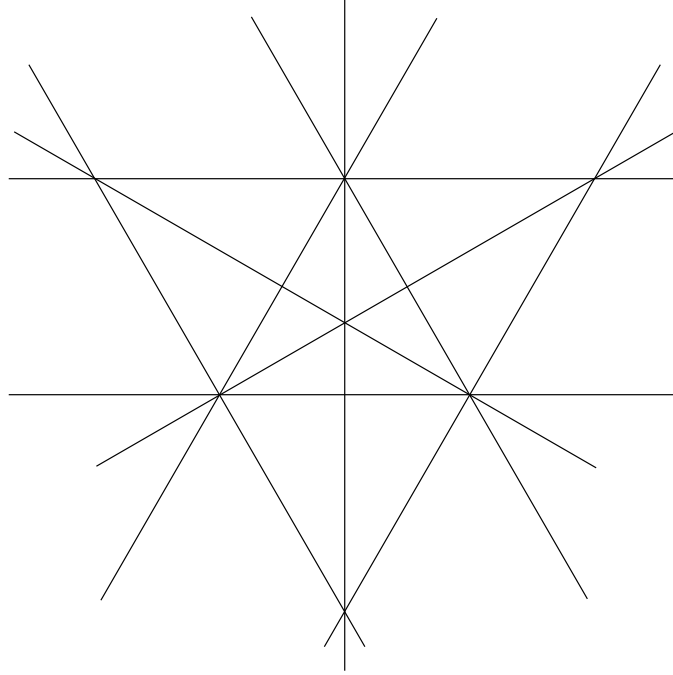
with $n \leq 8$, $d_i \in \mathbb{N}$, $d_1 + \cdots + d_n \leq n - 1$. With the aid of a computer, applying Proposition 6.1 to all these cases is straightforward. \square

Based on Proposition 6.5 and the case of generic arrangements, one may be tempted to guess that arrangements of $d \leq n$ hyperplanes in \mathbb{P}^n have effective CSM class. The following is the smallest counterexample to this statement, for free arrangements.

Example 6.6. The polynomial

$$\begin{vmatrix} x_0 & x_0^3 & x_0^5 \\ x_1 & x_1^3 & x_1^5 \\ x_2 & x_2^3 & x_2^5 \end{vmatrix} = x_0 x_1 x_2 (x_0 - x_1)(x_0 - x_2)(x_1 - x_2)(x_0 + x_1)(x_0 + x_2)(x_1 + x_2)$$

defines a free arrangement of 9 lines in \mathbb{P}^2 , with exponents 1, 3, 5. The corresponding 9 planes in k^3 meet along 13 distinct lines; 6 of these lines lie on 2 planes, 4 on 3, and 3 on 4 planes. It follows that the Möbius function takes values 1 at 6 lines, 2 at 4, and 3 at 3. It also follows that the value of the Möbius function for the affine arrangement at the origin is -15 .



The cone over this projective arrangement in \mathbb{P}^9 is a free arrangement \mathcal{A} of 9 hyperplanes, with characteristic polynomial

$$\chi_{\widehat{\mathcal{A}}}(t) = t^{10} - 9t^9 + (6 \cdot 1 + 4 \cdot 2 + 3 \cdot 3)t^8 - 15t^7 = (t - 5)(t - 3)(t - 1)t^7 \quad .$$

Using the criterion in Proposition 6.1, we compute

$$\begin{aligned} & -(-9(t+1)^9 + 23(t+1)^8 - 15(t+1)^7) \\ & = 9t^9 + 58t^8 + 155t^7 + 217t^6 + 161t^5 + 49t^4 - 7t^3 - 5t^2 + 2t + 1 \quad , \end{aligned}$$

verifying that the Chern–Schwartz–MacPherson class of this arrangement is not effective.

We end by remarking that the CSM routine described in [Alu03] and implemented in Macaulay2 offers a quick verification of this computation: the calculation of this CSM class from the equation of the arrangement takes about .1 seconds on a laptop computer. The same routine may be used to compute the Segre class of the singularity subscheme, giving with notation as in Theorem 5.1

$$(\sigma_0, \dots, \sigma_9) = (1, 0, -49, 664, -6528, 54272, -389120, 2260992, -7340032, -58720256).$$

Applying Theorem 5.1 gives then

$$\pi_{\widehat{\mathcal{A}}}(t) = 1 + 8t + 15t^2 = (1 + 3t)(1 + 5t) \quad ,$$

in agreement with the combinatorial computation shown above. ┘

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