CHERN CLASSES OF FREE HYPERSURFACE ARRANGEMENTS

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Abstract. The Chern class of the sheaf of logarithmic derivations along a simple normal crossing divisor equals the Chern-Schwartz-MacPherson class of the complement of the divisor. We extend this equality to more general divisors, which are locally analytically isomorphic to free hyperplane arrangements.

1. Introduction

For us, an *arrangement* in a nonsingular variety V is a reduced divisor D consisting of a union of nonsingular hypersurfaces, such that at each point D is locally analytically isomorphic to a hyperplane arrangement. We say that the arrangement is *free* if all these local models may be chosen to be free hyperplane arrangements. It follows that D is itself a *free divisor* on V: the sheaf of logarithmic differentials $\Omega^1_V(\log D)$ along D is locally free. Equivalently, its dual sheaf of *logarithmic derivations,* $\mathrm{Der}_V(-\log D) := \Omega^1_V(\log D)^\vee$, is locally free. Free hyperplane arrangements in $\mathbb{P}^n$ and divisors with simple normal crossings in a nonsingular variety give examples of free hypersurface arrangements.

In this note we extend to free hypersurface arrangements a result that is known to hold for these examples.

Theorem 1.1. *Let V be a nonsingular complex variety, and let $D \subseteq V$ be a free hypersurface arrangement. Then*

$$c(\mathrm{Der}_V(-\log D)) \cap [V] = c_{SM}(1\!\!1_{V \smallsetminus D}) \quad .$$

Here, $c_{\mathrm{SM}}(1\!\!1_{V \smallsetminus D})$ is the *Chern-Schwartz-MacPherson* class of the constructible function $1\!\!1_{V \smallsetminus D}$, in the sense of [Mac74], see also [Ful84], Example 19.1.7.

For simple normal crossing divisors, the equality of Theorem 1.1 was verified in [GP02] (Proposition 15.3) and [Alu99b]. For free projective hyperplane arrangements, it is Theorem 4.1 in [Alu12], where it is obtained as a simple corollary of a result of Mustaţă and Schenck ([MS01]). Theorem 1.1 will be obtained here by considering the blow-ups giving an embedded resolution of D. Each blow-up will be analyzed by using MacPherson's *graph construction,* showing (Claim 2.4) that the Chern class of the corresponding sheaf of logarithmic derivations is preserved by push-forward. The theorem will then follow from the corresponding behavior of the Chern-Schwartz-MacPherson class and from the case of normal crossing divisors.

In particular, this will give an independent proof (and a substantial generalization) of the case of free hyperplane arrangements treated in [Alu12].

The term 'hypersurface arrangement' is often used in the literature to simply mean a union of hypersurfaces (nonsingular or otherwise). This is a substantially more general notion than the one used in this note. The statement of Theorem 1.1 is *not* true in this generality, even for free divisors. For example, if V is a surface (so that every reduced divisor is free in V), a condition of local homogeneity is necessary for this result to hold, as observed by Xia Liao (cf. [Lia12]).

The paper is organized as follows: in §2 we recall the basic definitions and reduce the main theorem to showing that Chern classes of sheaves of logarithmic derivations are preserved through certain types of blow-ups. This is proven in §3, using the graph construction. In §4 we offer a simple example, and show that the theorem is equivalent to a projection formula for Chern classes of certain coherent sheaves.

A word on the hypotheses: the freeness of the divisor is used crucially in the application of the graph construction; its local analytic structure is less essential, but convenient in some coordinate arguments. It is conceivable that the proof given here may be generalized to divisors satisfying a less restrictive local homogeneity requirement.

The result in this note generalizes Theorem 4.1 in [Alu12]. I presented the results of [Alu12] in my talk at the Hefei conference on Singularity Theory, and I take this opportunity to thank Xiuxiong Chen and Laurentiu Maxim for the invitation to speak at the conference and for organizing a very successful and thoroughly enjoyable meeting. I also thank the referee for valuable suggestions.

2. Set-up

2.1. We work over an algebraically closed field of characteristic 0; the reader is welcome to assume the ground field is $\mathbb{C}$. (Characteristic 0 is required in the theory of Chern-Schwartz-MacPherson classes. See [Ken90] or [Alu06] for a discussion of the theory over algebraically closed fields of characteristic 0.)

Chern-Schwartz-MacPherson classes are classes in the Chow group of a variety V defined for constructible functions on V, and are characterized by the normalization requirement that $c_{\mathrm{SM}}(1\!\!1_V) \in A_*V$ equals $c(TV) \cap [V]$ if V is nonsingular and the covariance property

$$\alpha_* c_{\mathrm{SM}}(\varphi) = c_{\mathrm{SM}}(\alpha_* \varphi)$$

for all proper morphisms $\alpha : V \to V'$ and all constructible functions φ on V. Over $\mathbb{C}$, the push-forward of a constructible function is defined by taking weighted Euler characteristics of fibers: for a subvariety $Z \subseteq V$, $\alpha_*(1\!\!1_Z)(p) = \chi(Z \cap \alpha^{-1}(p))$. Thus, c_{SM} determines a natural transformation from the functor of constructible functions to the Chow functor. The existence of this natural transformation was conjectured by Deligne and Grothendieck, and proved by MacPherson ([Mac74]). Interest in these classes has resurged in the past few years; comparison with other classes for singular varieties gives an intersection theoretic invariant of singularities generalizing directly the Milnor number. A recent survey may be found in [Par06].

The interested reader may consult Example 19.1.7 in [Ful84] for an efficient summary of MacPherson's definition; an alternative construction is presented in [Alu06]. In any case, the details of the definition of these classes are not needed for this paper: only the key covariance property recalled above will be used. Note that if V_1 and V_2

are constructible subsets of V, then

$$c_{\rm SM}(1\!\!1_{V_1\cup V_2}) = c_{\rm SM}(1\!\!1_{V_1}+1\!\!1_{V_2}-1\!\!1_{V_1\cap V_2}) = c_{\rm SM}(1\!\!1_{V_1})+c_{\rm SM}(1\!\!1_{V_2})-c_{\rm SM}(1\!\!1_{V_1\cap V_2}) \quad :$$

the Chern-Schwartz-MacPherson classes satisfy 'inclusion-exclusion'; for example, they are additive on disjoint unions. Also, if V is complete, so that the constant map $\kappa : V \to \mathrm{pt}$ is proper, then by covariance

$$\kappa_* c_{\rm SM}(1\!\!1_U) = c_{\rm SM}(\kappa_* 1\!\!1_U) = c_{\rm SM}(\chi(U)1\!\!1_{\rm pt}) = \chi(U)[\mathrm{pt}]$$

for any constructible U in V. This says that the degree of $c_{\rm SM}(1\!\!1_U)$ equals the Euler characteristic $\chi(U)$, generalizing the Poincaré-Hopf theorem to singular and/or noncomplete varieties. (This was one of the motivations for the original definition of these classes by M.-H. Schwartz, cf. [Sch65a, Sch65b].)

2.2. The covariance property of Chern-Schwartz-MacPherson classes has the following immediate consequence. Let V be a variety, and let $X\subseteq V$ be a subscheme. Let $\rho:\widetilde{V}\to V$ be a proper map, and let $X'\subseteq \widetilde{V}$ be any subscheme such that ρ restricts to an isomorphism $\widetilde{V}\smallsetminus X'\to V\smallsetminus X$. Then

$$\rho_* c_{\rm SM}(1\!\!1_{\widetilde{V}\smallsetminus X'}) = c_{\rm SM}(1\!\!1_{V\smallsetminus X}) \quad .$$

Indeed, $\rho_*(1\!\!1_{\widetilde{V}\smallsetminus X'}) = 1\!\!1_{V\smallsetminus X}$.

In particular:

Lemma 2.1. *Let V be a nonsingular variety, and let $D\subseteq V$ be a subscheme. Let $\rho:\widetilde{V}\to V$ be a proper morphism such that $\widetilde{V}$ is nonsingular, and the support D' of $\rho^{-1}(D)$ is a divisor with normal crossings and nonsingular components. Then*

$$c_{SM}(1\!\!1_{V\smallsetminus D}) = \rho_*(c(\mathrm{Der}_{\widetilde{V}}(-\log D'))\cap[\widetilde{V}]) \quad .$$

Proof. As recalled in §1, since D' is a simple normal crossing divisor in $\widetilde{V}$, then

$$c_{\rm SM}(1\!\!1_{\widetilde{V}\smallsetminus D'}) = c(\Omega^1_{\widetilde{V}}(\log D')^\vee)\cap[\widetilde{V}] = c(\mathrm{Der}_{\widetilde{V}}(-\log D'))\cap[\widetilde{V}] \quad .$$

This is proved in e.g., [Alu99b], Theorem 1; we quickly recall the argument, for the convenience of the reader. Let D'_i, $i=1,\dots,N$ be the components of D'. Since D' is a divisor with normal crossings, $c(\Omega^1_{\widetilde{V}}(\log D')^\vee)$ equals $c(T\widetilde{V})/\prod_i(1+D'_i)$ (as is well-known, and easily verified). Now the stated equality is clear if $N=0$. For $N>0$:

$$\frac{c(T\widetilde{V})}{\prod_i(1+D'_i)} = \frac{c(T\widetilde{V})}{\prod_{i<N}(1+D'_i)}\left(1-\frac{D'_N}{1+D'_N}\right) = \frac{c(T\widetilde{V})}{\prod_{i<N}(1+D'_i)} - \frac{c(TD'_N)\cdot D'_N}{\prod_{i<N}(1+D'_i)}$$

and therefore

$$\frac{c(T\widetilde{V})}{\prod_i(1+D'_i)}\cap[\widetilde{V}] = \frac{c(T\widetilde{V})}{\prod_{i<N}(1+D'_i)}\cap[\widetilde{V}] - \frac{c(TD'_N)}{\prod_{i<N}(1+D'_i)}\cap[D'_N] \quad .$$

Arguing by induction on N, the first summand equals the $c_{\rm SM}$ class of the complement of the union of the first $N-1$ components, and the second equals the $c_{\rm SM}$ class of the trace of this complement on the N-th component. The equality follows then by the additivity of Chern-Schwartz-MacPherson classes on disjoint unions.

The equality implies the formula stated in the lemma, by covariance: $c_{\mathrm{SM}}(1\!\!1_{V \smallsetminus D}) = \rho_* c_{\mathrm{SM}}(1\!\!1_{\widetilde{V} \smallsetminus D'}) = \rho_*(c(\mathrm{Der}_{\widetilde{V}}(-\log D')) \cap [\widetilde{V}])$. □

2.3. Now let V be a nonsingular variety, and let D be a hypersurface arrangement, as in §1. In particular: at every $p \in D$, there is a choice of analytic coordinates $x_1, \dots, x_n$ such that the ideal of D in the completion $k[[x_1, \dots, x_n]]$ is generated by a product of linear polynomials $\sum \lambda_i x_i$, defining a central hyperplane arrangement $\mathscr{A}_p$.

Lemma 2.2. *The divisor D is free on V if and only if each $\mathscr{A}_p$ is a free central hyperplane arrangement.*

Proof. Recall that a divisor in a nonsingular variety V is free if and only if its singularity subscheme is empty or Cohen-Macaulay of codimension 2 in V at each $p \in D$. It follows that a central hyperplane arrangement is free if and only if its singularity subscheme is empty or Cohen-Macaulay of codimension 2 at the origin. (Cf. [Ter80], Proposition 2.4.)

The statement then follows from the fact that a local ring is Cohen-Macaulay if and only if its completion is ([BH93], Corollary 2.1.8). □

Under the hypotheses of Theorem 1.1, $\mathrm{Der}_V(-\log D)$ is locally free. With $\rho : \widetilde{V} \to V$ as in the statement of Lemma 2.1, $\mathrm{Der}_{\widetilde{V}}(-\log D')$ is also locally free, as D' is a divisor with simple normal crossings. Lemma 2.1 reduces Theorem 1.1 to proving that if D is a free hypersurface arrangement in V, and $\rho : \widetilde{V} \to V$ is as in the statement of Lemma 2.1, then

$$(*) \qquad \rho_*(c(\mathrm{Der}_{\widetilde{V}}(-\log D')) \cap [\widetilde{V}]) = c(\mathrm{Der}_V(-\log D)) \cap [V] \quad .$$

2.4. Next, we observe that an embedded resolution ρ of a hypersurface arrangement D may be obtained by blowing up along the intersections of the components of the arrangement, in order of increasing dimension, and that these intersections are all nonsingular. In order to verify (*), it suffices to verify that the stated equality holds for each of these blow-ups. More precisely: Given a hypersurface arrangement D in a nonsingular variety V, let Z be a component of lowest dimension among the intersections of components of D; let $\pi : \hat{V} \to V$ be the blow-up of V along Z; let E be the exceptional divisor of this blow-up; and let D' be the divisor in $\hat{V}$ consisting of E and the proper transforms of the components of D.

Lemma 2.3. *With notation as above, if D is a free hypersurface arrangement, then so is D'.*

Proof. We can work analytically at a point $p \in V$, so we may assume that D is given by a product of linear forms cutting out the center Z at p. We may in fact assume that there are analytic coordinates $x_1, \dots, x_n$ at p so that Z is given by $x_1 = \cdots = x_r = 0$, and the generator of the ideal of D is a homogeneous polynomial $F(x_1, \dots, x_r) = \prod L_i(\underline{x})$, with L_i linear.

Let $q \in \hat{V}$ be a point over p. We may choose analytic coordinates $(\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$ at q so that $\hat{x}_1 = 0$ is the exceptional divisor, and the blow-up map is given by

$$(x_1, \dots, x_n) = (\hat{x}_1, \hat{x}_1\hat{x}_2, \dots, \hat{x}_1\hat{x}_r, \hat{x}_{r+1}, \dots, \hat{x}_n) \quad .$$

The ideal for D' at q is then generated by

$$\hat{x}_1\, F(1,\hat{x}_2,\dots,\hat{x}_r) \quad ;$$

omitting the factors in $F(1,\hat{x}_2,\dots,\hat{x}_r)$ that do not vanish at q, we write the generator for D' at q as

$$\hat{x}_1\, Q(\hat{x}_2,\dots,\hat{x}_r) \quad ,$$

where Q is a product of linear forms. In particular, D' is a hypersurface arrangement in $\hat{V}$. We have to verify that it is free.

Note that the divisor defined by $Q(\hat{x}_2,\dots,\hat{x}_r)$ is free at q: indeed, the hyperplane arrangement defined by $F(x_1,\dots,x_n)$ is free by assumption, and $Q(x_2,\dots,x_r)$ generates the ideal of this arrangement at points $(t,0,\dots,0)$ with $t\neq 0$. By Saito's criterion ([OT92], Theorem 4.19), $Q(\hat{x}_2,\dots,\hat{x}_r)$ is the determinant of a set of $n-1$ logarithmic derivations $\theta_2,\dots,\theta_n$ at q. Since $\theta_2(\hat{x}_1)=\cdots=\theta_n(\hat{x}_1)=0$, these derivations are logarithmic with respect to $\hat{x}_1\, Q(\hat{x}_2,\dots,\hat{x}_r)$.

On the other hand the Euler derivation $\theta_1=\hat{x}_1\,\partial/\partial\hat{x}_1+\hat{x}_2\,\partial/\partial\hat{x}_2+\cdots+\hat{x}_n\,\partial/\partial\hat{x}_n$ is logarithmic with respect to $\hat{x}_1 Q$ as this is homogeneous (cf. [OT92], Definition 4.7), and $\det(\theta_1,\dots,\theta_n)$ is a unit multiple of $\hat{x}_1\, Q(\hat{x}_2,\dots,\hat{x}_r)$. This shows that D' is free, again by Saito's criterion. □

2.5. By Lemma 2.3, $\mathrm{Der}_{\hat{V}}(-\log D')$ is locally free if $\mathrm{Der}_V(-\log D)$ is, and we may consider its ordinary Chern classes. We have reduced the proof of Theorem 1.1 to the following statement.

Claim 2.4. *Let D be a free hypersurface arrangement on a nonsingular variety V; let $\pi:\hat{V}\to V$ be the blow-up of V along a component of lowest dimension of the intersection of components of D, and let $D'=(\pi^{-1}(D))_{red}$, as above. Then*

$$\pi_*(c(\mathrm{Der}_{\hat{V}}(-\log D'))\cap[\hat{V}])=c(\mathrm{Der}_V(-\log D))\cap[V] \quad .$$

The next section is devoted to the proof of this claim, and this will complete the proof of Theorem 1.1.

3. Proof of Theorem 1.1

3.1. We will prove Claim 2.4 as an application of MacPherson's *graph construction.* Given a homomorphism $\sigma:\mathscr{E}\to\mathscr{F}$ of vector bundles on a variety Y, consider the graph of $\lambda\sigma$ for $\lambda\in k$, as a subbundle of $\mathscr{E}\oplus\mathscr{F}$. For all λ, this defines an embedding of Y in the Grassmannian $G=\mathrm{Grass}_{\mathrm{rk}\,\mathscr{E}}(\mathscr{E}\oplus\mathscr{F})$, such that the pull-back of the universal subbundle ζ of G is isomorphic to $\mathscr{E}$. The graph construction describes the limit 'as $\lambda\to\infty$' of this embedding as a cycle in G, using which one may compare the Chern classes of $\mathscr{E}$ and $\mathscr{F}$. We refer the reader to Example 18.1.6 in [Ful84] for the details and key properties of this useful construction. We will use the fact that if σ restricts to an isomorphism on a subbundle $\mathscr{K}$ of $\mathscr{E}$ over a subvariety E of Y, then $\mathscr{K}$ embeds as a subbundle of $\zeta|_E$ over the cycle at infinity; and an analogous dual statement concerning epimorphisms. These facts are straightforward consequences of the construction.

As in §2, we denote by Z the center of the blow-up, E the exceptional divisors; and the natural morphisms as in this diagram:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "p"'] & \hat{V} \arrow[d, "\pi"] \\ Z \arrow[r, "\iota"'] & V \end{tikzcd}$$

By assumption Z is a nonsingular subvariety of V; we let r be its codimension. In a neighborhood of Z, Z is the transversal intersection of r components of D: indeed, if $D_1, \dots, D_r$ cut out Z at a point, then Z is contained in a connected component of $D_1 \cap \cdots \cap D_r$, so it must be equal to it as $D_1 \cap \cdots \cap D_r$ is nonsingular by our hypothesis on D. The key lemma will be the following:

Lemma 3.1. *Under the hypotheses of Claim 2.4:*

- *There is a vector bundle homomorphism* $\sigma : \pi^* \operatorname{Der}_V(-\log D) \to \operatorname{Der}_{\hat{V}}(-\log D')$ *that is an isomorphism in the complement of* E.
- *The restriction of* σ *to* E *induces a morphism of complexes of vector bundles*

$$\begin{tikzcd} \mathscr{O}_E \arrow[r, hook] \arrow[d, "="'] & \pi^* \operatorname{Der}_V(-\log D)|_E \arrow[r, two heads] \arrow[d, "\sigma|_E"] & p^* \operatorname{Der}_Z \arrow[d, "="'] \\ \mathscr{O}_E \arrow[r, hook] & \operatorname{Der}_{\hat{V}}(-\log D')|_E \arrow[r, two heads] & p^* \operatorname{Der}_Z \end{tikzcd}$$

The monomorphisms and epimorphisms shown in this diagram will be defined in the course of the proof of Lemma 3.1, in §3.3; the monomorphisms will be monomorphisms of vector bundles.

Claim 2.4 follows from Lemma 3.1, as we now show. Applying the graph construction to σ yields a cycle $\sum_i a_i[W_i]$ of dimension $n = \dim \hat{V}$ in the Grassmannian $G = \operatorname{Grass}_n(\pi^* \operatorname{Der}_V(-\log D) \oplus \operatorname{Der}_{\hat{V}}(-\log D'))$ over $\hat{V}$. The details of the construction of the subvarieties W_i are immaterial here; the key property of this cycle is that since σ is an isomorphism off E,

$$c(\pi^* \operatorname{Der}_V(-\log D)) \cap [\hat{V}] - c(\operatorname{Der}_{\hat{V}}(-\log D')) \cap [\hat{V}] = \sum_{W_i \to E} a_i \eta_{i*}(c(\zeta) \cap [W_i]) \quad ,$$

where $\eta_i : W_i \to \hat{V}$ are the maps induced by projection, and ζ is the rank-n universal bundle on G. (See (c) in Example 18.1.6 of [Ful84].) Pushing forward to V, and since $\operatorname{Der}_V(-\log D)$ is assumed to be locally free,

$$c(\operatorname{Der}_V(-\log D)) \cap [V] - \pi_*(c(\operatorname{Der}_{\hat{V}}(-\log D')) \cap [\hat{V}]) = \sum_{W_i \to E} a_i \pi_* \eta_{i*}(c(\zeta) \cap [W_i]) \quad .$$

Therefore, in order to verify Claim 2.4 it suffices to prove that $\pi_* \eta_*(c(\zeta) \cap [W]) = 0$ for every component $W = W_i$ projecting into E via $\eta = \eta_i$. We let $\underline{\eta}$ be the morphism

$W \to E$:

$$\begin{tikzcd}
W \arrow[d, "\underline{\eta}"'] \arrow[dr, "\eta"] & \\
E \arrow[r, "j"'] \arrow[d, "p"'] & \hat{V} \arrow[d, "\pi"] \\
Z \arrow[r, "\iota"'] & V
\end{tikzcd}$$

We have $\pi \circ \eta = \iota \circ p \circ \underline{\eta}$. Thus it suffices to show that

$$p_* \underline{\eta}_* (c(\zeta) \cap [W]) = 0 \quad .$$

The component W lies in $G|_E = \mathrm{Grass}_n(\pi^* \mathrm{Der}_V(-\log D)|_E \oplus \mathrm{Der}_{\hat{V}}(-\log D')|_E)$, and ζ restricts to the universal bundle $\zeta|_E$ on $G|_E$; $c(\zeta) \cap [W] = c(\zeta|_E) \cap [W]$ by functoriality of Chern classes. By the second part of Lemma 3.1, the restriction of $\zeta|_E$ to each component W is the middle term in a complex of vector bundles

$$\mathscr{O}_E \hookrightarrow \zeta|_E \twoheadrightarrow p^* \mathrm{Der}_Z \quad ;$$

this follows from the facts recalled at the beginning of this section. We obtain then that $c(\zeta|_E) = c(p^* \mathrm{Der}_Z)\, c(\xi)$, where $\xi = \ker(\zeta|_E \to p^* \mathrm{Der}_Z)/\mathscr{O}_E$ is the homology of this complex. By the projection formula,

$$p_* \underline{\eta}_* (c(\zeta) \cap [W]) = c(\mathrm{Der}_Z) \cap p_* \underline{\eta}_* (c(\xi) \cap [W]) \quad .$$

Since $\dim W = \dim V$ and ξ has rank $= \mathrm{codim}_Z V - 1$, the nonzero components of $c(\xi) \cap [W]$ have dimension $> \dim Z$, and therefore $p_* \underline{\eta}_* (c(\xi) \cap [W]) = 0$. It follows that $\pi_* \eta_* (c(\zeta) \cap [W]) = 0$ as needed. □

3.2. We are thus reduced to proving Lemma 3.1. Recall that Z denotes the codimension r, nonsingular center of the blow-up. We will use the following notation:

- By assumption, there exist r components $D_1, \dots, D_r$ of D such that Z is a connected component of $D_1 \cap \cdots \cap D_r$. We will denote by D^+ the union of $D_1, \dots, D_r$. Note that D^+ is a divisor with normal crossings in a neighborhood of Z.
- $\hat{D}$ will denote $\pi^{-1}(D)$, so that $D' = \hat{D}_{\mathrm{red}}$.
- Similarly, $\hat{D}^+$ will be $\pi^{-1}(D^+)$.

Remark 3.2. The difference between a divisor and its reduction is immaterial here (in characteristic zero). For a divisor A in a nonsingular variety V, the sections of the sheaf $\mathrm{Der}_V(-\log A)$ may be defined as those derivation which send a section F corresponding to A to a multiple of F: in other words, there is an exact sequence

$$0 \longrightarrow \mathrm{Der}_V(-\log A) \longrightarrow \mathrm{Der}_V \longrightarrow \mathscr{O}_A(A)$$

where (locally) the last map applies a given derivation to F (see e.g. [Dol07], §2). It is straightforward to verify that if ∂ is a derivation, and F_{red} consists of the factors of F taken with multiplicity 1, then $\partial(F) \in (F)$ if and only if $\partial(F_{\mathrm{red}}) \in (F_{\mathrm{red}})$. Thus $\mathrm{Der}_V(-\log A)$ and $\mathrm{Der}_V(-\log A_{\mathrm{red}})$ coincide as subsheaves of Der_V. Therefore, we may use $\hat{D}$ in place of D', and we don't need to make a distinction between $\hat{D}^+$ and its reduction. ⌟

Remark 3.3. We recall the following useful description of $\mathrm{Der}_V(-\log D)$ (cf. [OT92], Proposition 4.8): if D is the union of distinct components D_i, then $\mathrm{Der}_V(-\log D) = \cap_i \mathrm{Der}_V(-\log D_i)$ within Der_V. Indeed, it suffices to prove that $\mathrm{Der}_V(-\log(A\cup B)) = \mathrm{Der}_V(-\log A) \cap \mathrm{Der}_V(-\log B)$ if A and B have no components in common. This amounts to the statement that if F and G have no common factors, then $\partial(FG) \in (FG)$ if and only if $\partial(F) \in (F)$ and $\partial(G) \in (G)$ for all derivations ∂, which is immediate. ⌟

Remark 3.4. In particular, if a divisor A consists of a selection of the components of D, then $\mathrm{Der}_V(-\log D) \subseteq \mathrm{Der}_V(-\log A)$. Therefore, we have inclusions

$$\mathrm{Der}_V(-\log D) \subseteq \mathrm{Der}_V(-\log D^+) \quad , \quad \mathrm{Der}_{\hat{V}}(-\log \hat{D}) \subseteq \mathrm{Der}_{\hat{V}}(-\log \hat{D}^+) \quad .$$

Further, the monomorphism $\mathrm{Der}_V(-\log D) \hookrightarrow \mathrm{Der}_V(-\log D^+)$ of locally free sheaves remains a monomorphism after pull-back via π: the determinant of this morphism is nonzero on V, and it remains nonzero on the blow-up $\hat{V}$. ⌟

Lemma 3.5. *The (reduction of the) divisor $\hat{D}^+$ is a divisor with normal crossings in a neighborhood of E, and $\pi^* \mathrm{Der}_V(-\log D^+) \cong \mathrm{Der}_{\hat{V}}(-\log \hat{D}^+)$.*

Proof. The first assertion is a simple verification in local coordinates (cf. Lemma 2.3). The second assertion only need be verified in a neighborhood of E, so it reduces to the case of normal crossings, where it is straightforward. More details may be found in Theorem 4.1 of [AM09]. (Also cf. Lemma 1.3 in [Alu10].) □

We may use the isomorphism obtained in Lemma 3.5 to identify $\pi^* \mathrm{Der}_V(-\log D^+)$ and $\mathrm{Der}_{\hat{V}}(-\log \hat{D}^+)$. Via this identification, we will verify that $\pi^* \mathrm{Der}_V(-\log D)$ is contained in $\mathrm{Der}_{\hat{V}}(-\log \hat{D})$. The corresponding monomorphism of locally free sheaves $\pi^* \mathrm{Der}_V(-\log D) \hookrightarrow \mathrm{Der}_{\hat{V}}(-\log \hat{D}) = \mathrm{Der}_{\hat{V}}(-\log D')$ will give the homomorphism σ whose existence is claimed in Lemma 3.1.

Note that the sought-for σ appears to go in the wrong direction. The differential of π maps $\mathrm{Der}_{\hat{V}}$ to $\pi^* \mathrm{Der}_V$, and restricts to a homomorphism $\mathrm{Der}_{\hat{V}}(-\log D^+) \to \pi^* \mathrm{Der}_V(-\log D^+)$. This is an isomorphism as observed in Lemma 3.5, and the claim here is that its *inverse* restricts to a morphism

$$\sigma : \pi^* \mathrm{Der}_V(-\log D) \longrightarrow \mathrm{Der}_{\hat{V}}(-\log \hat{D}) \quad ,$$

which will then clearly be an isomorphism off E as needed in §3.1.

Lemma 3.6. *Via the isomorphism $\pi^* \mathrm{Der}_V(-\log D^+) \cong \mathrm{Der}_{\hat{V}}(-\log \hat{D}^+)$, we have the inclusion $\pi^* \mathrm{Der}_V(-\log D) \subseteq \mathrm{Der}_{\hat{V}}(-\log \hat{D})$.*

Proof. By definition of $\mathrm{Der}_V(-\log D)$ there is an exact sequence

$$(\dagger) \qquad \mathrm{Der}_V(-\log D) \longrightarrow \mathrm{Der}_V(-\log D^+) \longrightarrow \mathscr{O}_D(D)$$

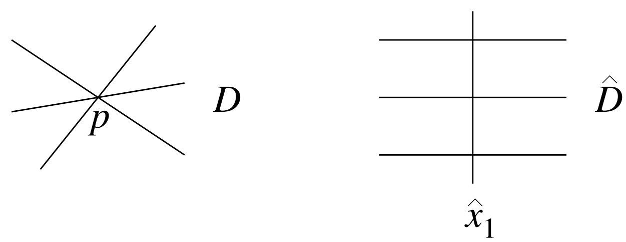
where the first map is a monomorphism, and the second applies a given logarithmic derivation to a section F defining D. Pulling back to $\hat{V}$ gives a complex

(‡)
$$\pi^* \operatorname{Der}_V(-\log D) \longrightarrow \pi^* \operatorname{Der}_V(-\log D^+) \cong \operatorname{Der}_{\hat{V}}(-\log \hat{D}^+) \longrightarrow \pi^* \mathscr{O}_D(D) \cong \mathscr{O}_{\hat{D}}(\hat{D})$$

The first map remains a monomorphism (Remark 3.4), and maps $\pi^* \operatorname{Der}_V(-\log D)$ into the kernel of the second map, which is $\operatorname{Der}_{\hat{V}}(-\log \hat{D})$ by definition of the latter. □

This completes the proof of the first part of Lemma 3.1. Note that σ is a monomorphism of sheaves, not of vector bundles.

Example 3.7. Let $V = \mathbb{P}^2$, and let D be the divisor consisting of three distinct concurrent lines. We blow-up at the point of intersection p:



In affine coordinates centered at p, we may assume D has equation $F = x_1 x_2 (x_1 + x_2) = 0$. We choose coordinates $\hat{x}_1, \hat{x}_2$ in an affine chart in the blow-up $\hat{V}$ so that the blow-up map is given by
$$x_1 = \hat{x}_1 \quad , \quad x_2 = \hat{x}_1 \hat{x}_2 \quad ;$$
the exceptional divisor E has equation $\hat{x}_1 = 0$, and $\hat{D}$ is given by the vanishing of $\hat{F} = \hat{x}_1^3 \hat{x}_2 (1 + \hat{x}_2)$ (the fourth component is at ∞ in this chart); it is a divisor with normal crossings.

We work in the local rings R, $\hat{R}$ at $(0,0)$ in both V and $\hat{V}$. We can let D^+ be the divisor $x_1 x_2 = 0$, so that $\hat{D}^+$ has ideal $(\hat{x}_1^2 \hat{x}_2)$. Bases for $\operatorname{Der}_V(-\log D^+)$, $\operatorname{Der}_{\hat{V}}(-\log \hat{D}^+)$ are
$$\langle x_1 \partial_1, x_2 \partial_2 \rangle \quad , \quad \langle \hat{x}_1 \hat{\partial}_1, \hat{x}_2 \hat{\partial}_2 \rangle$$
where $\partial_i = \partial/\partial x_i$, $\hat{\partial}_i = \partial/\partial \hat{x}_i$, and, as the reader may verify, the isomorphism $\pi^* \operatorname{Der}_V(-\log D^+) \xrightarrow{\sim} \operatorname{Der}_{\hat{V}}(-\log \hat{D}^+)$ maps $\pi^*(x_1\partial_1)$ to $\hat{x}_1 \hat{\partial}_1 - \hat{x}_2 \hat{\partial}_2$ and $\pi^*(x_2 \partial_2)$ to $\hat{x}_2 \hat{\partial}_2$. A derivation $a_1(x) x_1 \partial_1 + a_2(x) x_2 \partial_2$ is in $\operatorname{Der}_V(-\log D)$ iff
$$(a_1(x)x_1\partial_1 + a_2(x)\partial_2)(x_1x_2(x_1+x_2)) = a_1(x)F + a_1(x)x_1^2x_2 + a_2(x)F + a_2(x)x_1x_2^2 \in (F),$$
that is, iff
$$a_1(x)x_1 + a_2(x)x_2 \in (x_1 + x_2) \quad .$$
It follows that a basis for $\operatorname{Der}_V(-\log D)$ is
$$\langle x_1\partial_1 + x_2\partial_2, (x_1+x_2)x_2\partial_2 \rangle \quad ,$$
and we may represent sequence (†) at $(0,0)$ as
$$R \oplus R \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 1 & x_1+x_2 \end{smallmatrix}\right)} R \oplus R \xrightarrow{\left(\begin{smallmatrix} x_1^2x_2 & x_1x_2^2 \end{smallmatrix}\right)} R/(F)$$

Tensoring by $\hat{R}$ gives the corresponding sequence (‡):

$$\hat{R}\oplus\hat{R} \xrightarrow{\left(\begin{smallmatrix}1 & 0\\ 1 & \hat{x}_1(1+\hat{x}_2)\end{smallmatrix}\right)} \hat{R}\oplus\hat{R} \xrightarrow{\left(\begin{smallmatrix}\hat{x}_1^3\hat{x}_2 & \hat{x}_1^3\hat{x}_2^2\end{smallmatrix}\right)} \hat{R}/(\hat{F})$$

which realizes $\pi^*\operatorname{Der}_V(-\log D)$ as a submodule of

$$\operatorname{Der}_{\hat{V}}(-\log\hat{D}) = \ker\left(\begin{pmatrix}\hat{x}_1^3\hat{x}_2 & \hat{x}_1^3\hat{x}_2^2\end{pmatrix}\right) = \operatorname{im}\left(\begin{pmatrix}1 & 0\\ 1 & 1+\hat{x}_2\end{pmatrix}\right)$$

In these coordinates, a matrix representation for $\pi^*\operatorname{Der}_V(-\log D)\hookrightarrow\operatorname{Der}_{\hat{V}}(-\log\hat{D})$ is evidently $\left(\begin{smallmatrix}1 & 0\\ 0 & \hat{x}_1\end{smallmatrix}\right)$. ⌟

Remark 3.8. Example 3.7 illustrates the general local situation: dualizing Proposition 4.5 in [Sil97], one sees that one may always choose local coordinates and bases for $\operatorname{Der}_V(-\log D)$, $\operatorname{Der}_{\hat{V}}(-\log\hat{D})$ so that the matrix of $\pi^*\operatorname{Der}_V(-\log D)\hookrightarrow\operatorname{Der}_{\hat{V}}(-\log\hat{D})$ is diagonal, with entries given by powers of the equation for the exceptional divisor. This will not be needed in the following, but it is a useful model to keep in mind in reading what follows. ⌟

3.3. We are left with the task of proving the second part of Lemma 3.1, which amounts to the existence of a certain trivial subbundle and an epimorphism to $p^*\operatorname{Der}_Z$ for both $\pi^*\operatorname{Der}_V(-\log D)$ and $\operatorname{Der}_{\hat{V}}(-\log\hat{D})$. We will prove that there is a commutative diagram of locally free sheaves on E:

$$\begin{tikzcd}
\mathscr{O}_E \arrow[rr, hook] \arrow[dr] & & \operatorname{Der}_{\hat{V}}(-\log\hat{D})|_E \arrow[dr] & \\
& \pi^*\operatorname{Der}_V(-\log D)|_E \arrow[ur, "\sigma|_E"'] \arrow[rr, two heads] & & p^*\operatorname{Der}_Z
\end{tikzcd}$$

such that the composition $\mathscr{O}_E\to p^*\operatorname{Der}_Z$ is the zero morphism. The top horizontal morphism will be a monomorphism of vector bundles, and it follows from the commutativity of the diagram that so is the leftmost slanted morphism. Similarly, the bottom horizontal morphism will be an epimorphism, and it follows that so is the rightmost slanted morphism. Thus, the full statement of Lemma 3.1 follows from the existence of this diagram.

3.3.1. We deal with the epimorphism side first. According to our hypotheses, the center Z of the blow-up is the transversal intersection (in a neighborhood of Z) of the r components of D^+, and is contained in the other components of D. As $Z\subseteq V$, we have a natural embedding of $Der_Z\cong TZ$ as the kernel of the natural map from $\operatorname{Der}_V|_Z\cong TV|_Z$ to the normal bundle N_ZV.

Lemma 3.9. *There is an exact sequence of vector bundles*

$$0\longrightarrow\mathscr{O}_Z^{\oplus r}\longrightarrow\operatorname{Der}_V(-\log D^+)|_Z\longrightarrow\operatorname{Der}_V|_Z\longrightarrow N_ZV\longrightarrow 0\quad.$$

In particular, there is an epimorphism $\operatorname{Der}_V(-\log D^+)|_Z\twoheadrightarrow\operatorname{Der}_Z$.

Proof. We have (see Remark 3.2) an exact sequence

$$0 \longrightarrow \mathrm{Der}_V(-\log D^+) \longrightarrow \mathrm{Der}_V \longrightarrow \mathscr{O}_{D^+}(D^+)$$

The image of the rightmost map is the ideal of $\mathscr{O}_{D^+}(D^+)$ defined locally by the partials of a generator for the ideal of D^+. Near Z, where Z is the complete intersection of $D_1, \dots, D_r$, it is easy to verify that this ideal is isomorphic to $\oplus_{i=1}^r \mathscr{O}_{D_i}(D_i)$. Thus, tensoring by $\mathscr{O}_Z$ gives an exact sequence

$$0 \to \mathrm{Tor}_1(\mathscr{O}_Z, \oplus_{i=1}^r \mathscr{O}_{D_i}(D_i)) \to \mathrm{Der}_V(-\log D^+)|_Z \to \mathrm{Der}_V|_Z \to \oplus_{i=1}^r \mathscr{O}_{D_i}(D_i)|_Z \to 0$$

(The leftmost term is 0 as Der_V is locally free.) The term $\oplus_{i=1}^r \mathscr{O}_{D_i}(D_i)|_Z$ is $N_Z V$, and the map from $\mathrm{Der}_V|_Z$ is the standard projection $TV|_Z \to N_Z V$. The Tor on the left is the direct sum of $\mathrm{Tor}_1(\mathscr{O}_Z, \mathscr{O}_{D_i}(D_i))$, and it is easy to verify that each such term is $\cong \mathscr{O}_Z$, as claimed. □

Remark 3.10. We can choose local parameters $x_1, \dots, x_n$ for V at a point of Z such that x_i is a generator for the ideal of D_i for $i = 1, \dots, r$. Then $\mathrm{Der}_V(-\log D^+)$ has a basis given by derivations

$$x_1\partial_1\,,\ \dots, x_r\partial_r\,,\ \partial_{r+1}\,,\ \dots\,,\ \partial_n$$

where $\partial_i = \partial/\partial x_i$. With the same coordinates, $\partial_{r+1}, \dots \partial_n$ restrict to a basis for Der_Z, and the epimorphism found in Lemma 3.9 acts in the evident way. The kernel is spanned by the restrictions of $x_i\partial_i$, $i = 1, \dots, r$; these are the r trivial factors appearing on the left in the sequence in Lemma 3.9.

Also note that the 'Euler derivation' $x_1\partial_1 + \cdots + x_r\partial_r$ spans a trivial subbundle $\mathscr{O}_Z \hookrightarrow \mathscr{O}_Z^{\oplus r}$ of the kernel. Thus, we have a complex of vector bundles

$$\mathscr{O}_Z \hookrightarrow \mathrm{Der}_V(-\log D^+)|_Z \twoheadrightarrow \mathrm{Der}_Z$$

on Z. Pulling back to E, this gives a complex of vector bundles on E:

$$\mathscr{O}_E \hookrightarrow \pi^* \mathrm{Der}_V(-\log D^+)|_E \twoheadrightarrow \pi^* \mathrm{Der}_Z \quad .$$

We have to verify that the same occurs for $\pi^* \mathrm{Der}_V(-\log D)$ and $\mathrm{Der}_{\hat{V}}(-\log \hat{D})$. ⌟

Consider $\mathrm{Der}_V(-\log D)$. We have (Remark 3.4) inclusions

$$\mathrm{Der}_V(-\log D) \subseteq \mathrm{Der}_V(-\log D^+) \subseteq \mathrm{Der}_V \quad .$$

Restricting to Z, and in view of Lemma 3.9, we get morphisms

$$\mathrm{Der}_V(-\log D)|_Z \longrightarrow \mathrm{Der}_V(-\log D^+)|_Z \twoheadrightarrow \mathrm{Der}_Z \quad .$$

Claim 3.11. *The composition* $\mathrm{Der}_V(-\log D)|_Z \to \mathrm{Der}_Z$ *is an epimorphism.*

Proof. Working with local parameters as in Remark 3.10, it suffices to note that the derivations $\partial_{r+1}, \dots, \partial_n$ are in $\mathrm{Der}_V(-\log D)$: this is clear, since by assumption D admits a local generator of the form $x_1 \cdots x_r\, G(x_1, \dots, x_r)$. □

Pulling back to E and using Lemma 3.6 we get morphisms

$$\begin{tikzcd} \pi^* \operatorname{Der}_V(-\log D)|_E \arrow[r, "\sigma|_E"] \arrow[rrr, bend right, two heads] & \operatorname{Der}_{\hat{V}}(-\log \hat{D})|_E \arrow[r] & \pi^* \operatorname{Der}_V(-\log D^+)|_E \arrow[r, two heads] & p^* \operatorname{Der}_Z \end{tikzcd}$$

and this yields the commutative triangle on the right in the diagram at the beginning of the section.

3.3.2. Finally, we have to deal with the triangle on the left.

Lemma 3.12. *Let A be a nonsingular hypersurface of a nonsingular variety V. Then there is an exact sequence of vector bundles*

$$0 \longrightarrow \mathscr{O}_A \longrightarrow \operatorname{Der}_V(-\log A)|_A \longrightarrow \operatorname{Der}_A \longrightarrow 0 \quad .$$

Proof. This is a particular case of Lemma 3.9. □

Remark 3.13. Applying this lemma to $E \subseteq \hat{V}$ gives a distinguished copy of $\mathscr{O}_E$ in $\operatorname{Der}_{\hat{V}}(-\log E)|_E$. Adopting local parameters at a point of Z as in Remark 3.10, we can choose coordinates

$$\hat{x}_1\,,\, \hat{x}_2\,,\, \dots\,,\, \hat{x}_r\,,\, \hat{x}_{r+1}\,,\, \dots\,,\, \hat{x}_n$$

at a point of E in a chart of the blow-up $\hat{V}$ so that the blow-up map is given by

$$\begin{cases} x_1 = \hat{x}_1 & \\ x_i = \hat{x}_1\hat{x}_i & i = 2, \dots, r \\ x_j = \hat{x}_j & j = r+1, \dots, n \end{cases}$$

The exceptional divisor is given by $\hat{x}_1 = 0$. Then a basis for $\operatorname{Der}_{\hat{V}}(-\log E)$ at this point is

$$\hat{x}_1\hat{\partial}_1\,,\, \hat{\partial}_2\,,\, \dots\,,\, \hat{\partial}_n$$

where $\hat{\partial}_i = \partial/\partial\hat{x}_i$. The distinguished copy $\mathscr{O}_E \subseteq \operatorname{Der}_{\hat{V}}(-\log E)$ found in Lemma 3.12 is spanned by $\hat{x}_1\hat{\partial}_1$. ⌟

Now recall (Remark 3.4) that $\operatorname{Der}_{\hat{V}}(-\log \hat{D}) \subseteq \operatorname{Der}_{\hat{V}}(-\log E)$.

Claim 3.14. *The distinguished $\mathscr{O}_E \subseteq \operatorname{Der}_{\hat{V}}(-\log E)|_E$ is contained in $\operatorname{Der}_{\hat{V}}(-\log \hat{D})|_E$.*

Proof. We work in coordinates as in Remark 3.10 and 3.13. By hypothesis, D is given analytically by the vanishing of $F = x_1 \cdots x_r \cdot G(x_1 \cdots x_r)$, where G is homogeneous. In the chart considered above in the blow-up, $\hat{D}$ is therefore given by the vanishing of

$$\hat{F} = \hat{x}_1^m \hat{x}_2 \cdots \hat{x}_r\, G(1, \hat{x}_2, \dots, \hat{x}_r)$$

where m is the multiplicity of D along Z. We then see that

$$(\hat{x}_1\hat{\partial}_1)\hat{F} = m\, \hat{x}_1^m \hat{x}_2 \cdots \hat{x}_r\, G(1, \hat{x}_2, \dots, \hat{x}_r) = m\hat{F} \in (\hat{F}) \quad :$$

this shows that $\hat{x}_1\hat{\partial}_1 \in \operatorname{Der}_{\hat{V}}(-\log \hat{D})$, as claimed. □

Since $\mathrm{Der}_{\hat{V}}(-\log \hat{D}) \subseteq \mathrm{Der}_{\hat{V}}(-\log \hat{D}^+)$, and the latter is $\cong \pi^* \mathrm{Der}_V(-\log D^+)$, we can view the distinguished $\mathscr{O}_E$ as a subsheaf of $\pi^* \mathrm{Der}_V(-\log D^+)|_E$. Chasing coordinates, it is straightforward to check that

$$\hat{x}_1 \hat{\partial}_1 \mapsto x_1 \partial_1 + \cdots + x_r \partial_r \quad :$$

that is, this copy of $\mathscr{O}_E$ corresponds to the 'Euler derivation' identified in Remark 3.10. Further, we see that it is also contained in $\pi^* \mathrm{Der}_V(-\log D)|_E$: indeed, since in the chosen analytic coordinates F is homogeneous (up to factors not vanishing along Z), the Euler derivation acts on F by multiplying it by its degree.

At this point we have the following situation:

$$\begin{tikzcd} \mathscr{O}_E \arrow[r, hook] \arrow[rrr, bend left] & \pi^* \mathrm{Der}_V(-\log D)|_E \arrow[r, "\sigma|_E"] & \mathrm{Der}_{\hat{V}}(-\log \hat{D})|_E \arrow[r] & \pi^* \mathrm{Der}_V(-\log \hat{D}^+)|_E \end{tikzcd}$$

This yields the commutative triangle on the left in the diagram at the beginning of the section. (The above construction shows that the monomorphisms from $\mathscr{O}_E$ to $\mathrm{Der}_{\hat{V}}(-\log \hat{D})|_E$ and $\pi^* \mathrm{Der}_V(-\log \hat{D}^+)|_E$ are monomorphisms of vector bundles, as needed.) The composition with the projection to $p^* \mathrm{Der}_Z$ is 0 as noted in Remark 3.10, so this completes the proof of Lemma 3.1. Claim 2.4 follows from Lemma 3.1 as shown in §3.1, so this concludes the proof of Theorem 1.1.

4. Further remarks and examples

4.1. An example. We illustrate Theorem 1.1 by computing the Chern class of a sheaf of logarithmic derivations in a simple case. Any value Theorem 1.1 may have lies in the contrast between the standard computation, by means of the basic sequence defining the sheaf, and the computation using Chern-Schwartz-MacPherson classes, which has a very different, 'combinatorial' flavor.

We assume D consists of $m \geq 2$ nonsingular components D_i, each of class X, meeting pairwise transversally along a codimension-2 nonsingular complete subvariety Z.

—Computation using c_{SM}-classes. As $D = \cup_i D_i$, and since all components meet along Z, we have

$$1\!\!1_D = 1\!\!1_Z + \sum_i 1\!\!1_{D_i \smallsetminus Z} = (\sum_i 1\!\!1_{D_i}) - (m-1) 1\!\!1_Z \quad ,$$

and hence

$$1\!\!1_{V \smallsetminus D} = 1\!\!1_V - \sum_i 1\!\!1_{D_i} + (m-1) 1\!\!1_Z \quad .$$

Since V, all D_i, and Z are nonsingular, the basic normalization property of c_{SM} classes (§2.1) gives

$$c_{\mathrm{SM}}(1\!\!1_{V \smallsetminus D}) = c(TV) \cap [V] - \sum_i c(TD_i) \cap [D_i] + (m-1) c(TZ) \cap [Z] \quad .$$

We are assuming that all components have the same class X, and hence Z has class $X \cdot X$. Thus, this gives

$$c_{\rm SM}(1\!\!1_{V\smallsetminus D}) = c(TV)\left(1-\sum_{i=1}^m \frac{X}{1+X}+(m-1)\frac{X^2}{(1+X)^2}\right)\cap [V]$$

According to Theorem 1.1, this class equals $c(\mathrm{Der}_V(-\log D))\cap[V]$. That is,

$$c(\mathrm{Der}_V(-\log D)) = \frac{c(TV)\,(1-(m-2)X)}{(1+X)^2} \quad .$$

Standard computation. The basic sequence recalled in Remark 3.2 may be completed to

$$0 \longrightarrow \mathrm{Der}_V(-\log D) \longrightarrow \mathrm{Der}_V \longrightarrow \mathscr{O}_D(D) \longrightarrow \mathscr{O}_{JD}(D) \longrightarrow 0 \quad ,$$

where JD is the *singularity subscheme* (or *Jacobian subscheme*) of D. Therefore,

$$c(\mathrm{Der}_V(-\log D)) = \frac{c(\mathrm{Der}_V)}{c(\mathscr{O}_D(D))}\,c(\mathscr{O}_{JD}(D)) = \frac{c(TV)}{1+D}\,c(\mathscr{O}_{JD}(D)) \quad .$$

In the case at hand, we are assuming that D is defined by a section $f_1\cdots f_m$ of $\mathscr{O}(mX)$; Z is defined by (say) $f_1=f_2=0$, meeting transversally at every point of Z; and $f_i=(a_if_1+b_if_2)$ for $i\ge 3$, without multiple components. Thus, $f_1\cdots f_m = P(f_1,f_2)$ for a homogeneous polynomial $P(s,t)$ with constant coefficients. As the differentials df_1 and df_2 are assumed to be linearly independent everywhere along Z, the ideal of JD is generated by $\frac{\partial P}{\partial s}(f_1,f_2)$ and $\frac{\partial P}{\partial t}(f_1,f_2)$, and these have no component in common. It follows that JD is a complete intersection of two sections of $\mathscr{O}((m-1)X)$, so $\mathscr{O}_{JD}$ is resolved by a Koszul complex:

$$0\to\mathscr{O}_V(-2(m-1)X)\to\mathscr{O}_V(-(m-1)X)\oplus\mathscr{O}_V(-(m-1)X)\to\mathscr{O}_V\to\mathscr{O}_{JD}\to 0$$

and twisting by $\mathscr{O}_V(D)=\mathscr{O}_V(mX)$ gives the exact sequence

$$0\longrightarrow\mathscr{O}_V((-m+2)X)\longrightarrow\mathscr{O}_V(X)\oplus\mathscr{O}_V(X)\longrightarrow\mathscr{O}_V(D)\longrightarrow\mathscr{O}_{JD}(D)\longrightarrow 0 \quad .$$

Thus

$$c(\mathscr{O}_{JD}(D)) = \frac{c(\mathscr{O}_V(D))\,c(\mathscr{O}(-(m-2))X)}{c(\mathscr{O}_V(X))^2} = \frac{(1+D)(1-(m-2)X)}{(1+X)^2} \quad .$$

Taking this into account in the expression for $c(\mathrm{Der}_V(-\log D))$ given above, we recover the result of the $c_{\rm SM}$ computation. ⌟

While this may be largely a matter of taste, the standard computation appears to us to involve subtler information than the alternative combinatorial computation via $c_{\rm SM}$ classes afforded by applying Theorem 1.1. The point is that the $c_{\rm SM}$ class already includes information on the singularity subscheme JD: see [Alu99a] for the precise relation. Computing the $c_{\rm SM}$ class, which is straightforward for a hypersurface arrangement, takes automatically care of accounting for the total Chern class of $\mathscr{O}_{JD}(D)$.

4.2. A projection formula. If $\mathscr{E}$ is a vector bundle on a scheme X, and $\alpha : Y \to X$ is a proper morphism, then for any class A in the Chow group of Y we have

$$\alpha_*(c(\alpha^*\mathscr{E}) \cap A) = c(\mathscr{E}) \cap \alpha_*(A) \quad .$$

This is a basic result on Chern classes, see Theorem 3.2 (c) in [Ful84]. On a nonsingular variety, a notion of total Chern class is available for all coherent sheaves: this follows from the isomorphism $K_0(V) \cong K^0(V)$ for V nonsingular ([Ful84], §15.1) and the Whitney formula. However, a straightforward projection formula as in the case of vector bundles does not hold for arbitrary coherent sheaves.

Example 4.1. Let V be nonsingular, let $X, Y \hookrightarrow V$ be irreducible hypersurfaces, and let $i : X \hookrightarrow V$ be the inclusion. From the exact sequence

$$0 \longrightarrow \mathscr{O}_V(-X) \longrightarrow \mathscr{O}_V \longrightarrow \mathscr{O}_X \longrightarrow 0$$

it follows that $c(\mathscr{O}_X) = \frac{1}{1-X}$, and similarly $c(\mathscr{O}_Y) = \frac{1}{1-Y}$. As $i^*(\mathscr{O}_X) = \mathscr{O}_X$, we see that

$$i_*(c(i^*\mathscr{O}_X) \cap [X]) = i_*([X]) \neq c(\mathscr{O}_X) \cap i_*([X]) = \frac{[X]}{1-X} \quad :$$

the projection formula does not hold in this case. On the other hand, $i^*(\mathscr{O}_Y) = \mathscr{O}_{X\cap Y}$, and $X \cap Y$ is a divisor in X with bundle $\mathscr{O}_X(X \cap Y) = i^*\mathscr{O}_V(Y)$; therefore,

$$i_*(c(i^*\mathscr{O}_Y) \cap [X]) = i_*\left(\frac{[X]}{1 - i^*Y}\right) = \frac{[X]}{1-Y} = c(\mathscr{O}_Y) \cap i_*([X]) \quad :$$

the projection formula does hold in this case. ⌟

The difference between the two cases considered in this example is a matter of Tor functors: $\mathrm{Tor}_1^{\mathscr{O}_V}(\mathscr{O}_X, \mathscr{O}_X) \cong \mathscr{O}_X(-X)$ is not trivial, while $\mathrm{Tor}_1^{\mathscr{O}_V}(\mathscr{O}_X, \mathscr{O}_Y)$ vanishes. It is essentially evident from the definitions that for a coherent sheaf $\mathscr{F}$ on V, and a morphism $\alpha : W \to V$,

$$\alpha^* c(\mathscr{F}) = \prod_{i\geq 0} c(\mathrm{Tor}_i^{\mathscr{O}_V}(\mathscr{O}_W, \mathscr{F}))^{(-1)^i} = c(\alpha^*\mathscr{F}) \cdot \prod_{i\geq 1} c(\mathrm{Tor}_i^{\mathscr{O}_V}(\mathscr{O}_W, \mathscr{F}))^{(-1)^i} \quad ,$$

and in particular $c(\alpha^*\mathscr{F}) = \alpha^* c(\mathscr{F})$ if the higher Tors vanish. In this case (for example, in the case of vector bundles) the projection formula holds if α is proper. More generally, the projection formula holds if α_* maps to 1 the total Chern classes of the higher Tors.

4.3. Now recall the situation of this paper, and particularly the blow-up considered in Claim 2.4 and §3: D is a hypersurface arrangement in a nonsingular variety V, and Z is an intersection of minimal dimension of components of D. In fact, $Z = D_1 \cap \cdots \cap D_r$, where $D_1, \dots, D_r$ are components of D meeting with normal crossings in a neighborhood of Z. We denote by D^+ the union of these components, and we have observed (Remark 3.4) that $\mathrm{Der}_V(-\log D) \subseteq \mathrm{Der}_V(-\log D^+)$.

The sections obtained by applying the derivations in $\mathrm{Der}_V(-\log D^+)$ to a section F defining D, together with F, define a subscheme J^+D of $\mathscr{O}_D$, which should be viewed as a 'modified Jacobian subscheme' of the hypersurface arrangement D (depending on the choice of the subdivisor D^+). We consider the coherent sheaf $\mathscr{O}_{J^+D}(D)$ in V.

Finally, recall that $\pi : \hat{V} \to V$ denotes the blow-up of V along Z.

Claim 4.2. *The formula in Theorem 1.1 is implied by the statement that, for all blow-ups as above, $\mathscr{O}_{J^+D}(D)$ satisfies the projection formula with respect to the blow-up map π:*

$$\pi_*(c(\pi^*\mathscr{O}_{J^+D}(D)) \cap [\hat{V}]) = c(\mathscr{O}_{J^+D}(D)) \cap [V] \quad .$$

Proof. Arguing as in §2, we only need to deal with the case of a single blow-up; we will show that the given formula is equivalent to the formula in Claim 2.4.

Restricting the basic sequence recalled in Remark 3.2 to D^+ gives an exact sequence

$$0 \longrightarrow \mathrm{Der}_V(-\log D) \longrightarrow \mathrm{Der}_V(-\log D^+) \longrightarrow \mathscr{O}_D(D)$$

and the image of the last morphism is the ideal generated by applying the derivations from $\mathrm{Der}_V(-\log D^+)$ to F; this ideal defines J^+D, so we have an exact sequence

$$0 \longrightarrow \mathrm{Der}_V(-\log D) \longrightarrow \mathrm{Der}_V(-\log D^+) \longrightarrow \mathscr{O}_D(D) \longrightarrow \mathscr{O}_{J^+D}(D) \longrightarrow 0$$

on V. Notice that this implies that

$$c(\mathrm{Der}_V(-\log D)) = \frac{c(\mathrm{Der}_V(-\log D^+))}{1+D}\, c(\mathscr{O}_{J^+D}(D)) \quad .$$

Now we claim that (with notation as in §3) there is an exact sequence
($\diamond$)

$$0 \longrightarrow \mathrm{Der}_{\hat{V}}(-\log \hat{D}) \longrightarrow \pi^*\,\mathrm{Der}_V(-\log D^+) \longrightarrow \pi^*\mathscr{O}_D(D) \longrightarrow \pi^*\mathscr{O}_{J^+D}(D) \longrightarrow 0 \quad .$$

Indeed, pulling back the last terms of the previous sequence to $\hat{V}$ gives the last terms of ($\diamond$), by right-exactness of $-\otimes_{\mathscr{O}_V}\mathscr{O}_{\hat{V}}$; via the isomorphisms $\pi^*\,\mathrm{Der}_V(-\log D^+) \cong \mathrm{Der}_{\hat{V}}(-\log \hat{D}^+)$ (Lemma 3.5) and $\pi^*\mathscr{O}_D(D) \cong \mathscr{O}_{\hat{D}}(\hat{D})$, the morphism in the middle is seen to act by applying derivations from $\mathrm{Der}_{\hat{V}}(-\log \hat{D})$ to a section defining $\hat{D}$. Hence its kernel is $\mathrm{Der}_{\hat{V}}(-\log \hat{D})$, as needed for ($\diamond$). From ($\diamond$), it follows that

$$c(\mathrm{Der}_{\hat{V}}(-\log \hat{D})) = \frac{c(\pi^*\,\mathrm{Der}_V(-\log D^+))}{1+\pi^*D}\, c(\pi^*\mathscr{O}_{J^+D}(D)) \quad ,$$

and hence, applying the ordinary projection formula (as $\mathrm{Der}_V(-\log D^+)$ is locally free)

$$\pi_*(c(\mathrm{Der}_{\hat{V}}(-\log \hat{D})) \cap [\hat{V}]) = \frac{c(\mathrm{Der}_V(-\log D^+))}{1+D}\, \pi_*(c(\pi^*\mathscr{O}_{J^+D}(D)) \cap [\hat{V}]) \quad .$$

Comparing with the previous equality of Chern classes, we see that the projection formula for $\mathscr{O}_{J^+D}(D)$,

$$\pi_*(c(\pi^*\mathscr{O}_{J^+D}(D)) \cap [\hat{V}]) = c(\mathscr{O}_{J^+D}(D)) \cap [V]$$

is equivalent to

$$\pi_*(c(\mathrm{Der}_{\hat{V}}(-\log \hat{D})) \cap [\hat{V}]) = c(\mathrm{Der}_{\hat{V}}(-\log \hat{D})) \cap [V] \quad ,$$

that is the formula in Claim 2.4, as claimed. (As pointed out in Remark 3.2, we can replace $\hat{D}$ for D' in Claim 2.4.) □

By Claim 4.2, an independent proof of the projection formula for $\mathscr{O}_{J+D}(D)$ would give an alternative proof of Theorem 1.1. Note that the relevant Tor does not vanish in general; the task amounts to showing that its Chern class pushes forward to 1. We were not able to construct a more direct proof of this fact.

Remark 4.3. Xia Liao has shown ([Lia12]) that the equality in Theorem 1.1, for any divisor D, is equivalent to a projection formula involving the blow-up along the (ordinary) Jacobian subscheme of D. ⌟

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