LOG CANONICAL THRESHOLD AND SEGRE CLASSES OF MONOMIAL SCHEMES

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Abstract. We express the Segre class of a monomial scheme in projective space in terms of log canonical thresholds of associated ideals. Explicit instances of the relation amount to identities involving the classical polygamma functions.

1. Introduction

The *log canonical threshold* of an ideal is a measure of the singularity of the corresponding scheme. It can be defined in a broad variety of ways, relating it to many different areas of algebraic geometry and singularity theory; a survey of this notion and of its ubiquity may be found in [Mus12]. The purpose of this short note is to point out another unexpected connection: we will show that, in the particular case of *monomial schemes,* the *Segre class* may be computed from the log canonical threshold of certain related ideals.

By a result of J. Howald ([How01]), the log canonical threshold of a monomial ideal I in a polynomial ring has a very simple expression in terms of the Newton diagram of the ideal: it measures the distance of the diagram from the origin along the main diagonal. This is a straightforward consequence of Howald's realization of the multiplier ideal of a monomial ideal and the fact that the log canonical threshold equals the smallest jumping number of an ideal. It easily follows that the whole diagram for I may be reconstructed from knowledge of the log canonical thresholds of suitable extensions of the ideal. We apply this observation to obtain the Segre class of the scheme defined by I in projective space. Segre classes are basic invariants in intersection theory; Chapter 4 in [Ful84] is the standard reference for this notion. They are characterized by the fact that they are invariant under birational maps (in the sense of [Ful84], Proposition 4.2) and that if S is a local complete intersection in V, then the Segre class $s(S,V)$ equals the inverse Chern class of the normal bundle of S in V: $s(S,V)=c(N_SV)^{-1}\cap[S]$.

The result of this note is the following.

Theorem 1.1. *Let I be a proper monomial ideal in the variables $x_1,\dots,x_n$, and let S be the subscheme defined by I in $\mathbb{P}^M$, $M\geq n-1$. For $r_i>0$, denote by $I_{r_1,\dots,r_n}$ the extension of I under the homomorphism defined by $x_i\mapsto x_i^{r_i}$, $i\leq n$. Then*

$$s(S,\mathbb{P}^M)=1-\lim_{m\to\infty}\sum\frac{m\,n!\,X_1\cdots X_n}{(m+a_1X_1+\cdots+a_nX_n)^{n+1}} \tag{1}$$

where the sum is taken over all $(a_1,\dots,a_n)\in\mathbb{Z}^n_{>0}$ such that

$$\operatorname{lct}(I_{a_2\cdots a_n,\dots,a_1\cdots a_{n-1}})\geq\frac{m}{a_1\cdots a_n}\quad,$$

and X_i denotes the hyperplane $x_i=0$.

The limit appearing in the statement should be interpreted as follows. When the parameters $X_1,\dots,X_n$ are set to complex numbers (say, with positive real part), the given limit converges to, and hence determines, a rational function of $X_1,\dots,X_n$, with a well-defined expansion as a series in $X_1, \dots, X_n$. The statement is that evaluating the terms of this series as intersection products with X_i = the i-th coordinate hyperplane in $\mathbb{P}^M$, the right-hand side equals the Segre class of S in $\mathbb{P}^M$. (Each of the terms is supported on a subscheme of S, cf. Lemma 2.10 in [Alu], hence this computation determines a class in A_*S.)

Theorem 1.1 is proved in §3. In §2 we illustrate the result in simple examples. In the case of ideals generated by a pure power x_1^ℓ, the statement reduces to an elementary limit of polygamma functions. In general, every independent computation of the Segre class of a monomial ideal would give rise, via (1), to an identity involving limits and series of such functions. We find this observation intriguing, but we hasten to add that the shape of the formulas, more than their algebro-geometric content, seems to be responsible for this phenomenon. The role played by the log canonical threshold is limited to the demarcation of the Newton polytope of I in the positive orthant in $\mathbb{R}^n$ (Lemma 3.1).

Our main interest in Theorem 1.1 stems from the fact that both sides of (1) are defined for arbitrary homogeneous ideals in a polynomial ring. It is natural to ask to what extent formulas such as (1) may hold for non-monomial schemes, perhaps after a push-forward to projective space.

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2. EXAMPLES

Let $n = 1$, and $I = (x_1^\ell)$ for some $\ell \geq 1$. Then $I_{a_2\cdots a_n,\dots,a_1\cdots a_{n-1}} = I$, $\mathrm{lct}(I) = \frac{1}{\ell}$, and the range of summation specified in Theorem 1.1 is $\mathrm{lct}(I) \geq \frac{m}{a_1}$, that is, $a_1 \geq m\ell$. Thus, the summation in the statement of the theorem is

$$\sum_{a \geq m\ell} \frac{mX_1}{(m + aX_1)^2} \quad .$$

Recall that the r-th *polygamma function* $\Psi^{(r)}(x)$, defined for $r > 0$ as the r-th derivative of the *digamma* function $\frac{d}{dx}\ln(\Gamma(x)) = \frac{\frac{d}{dx}\Gamma(x)}{\Gamma(x)}$, admits the series representation

$$\Psi^{(r)}(x) = (-1)^{r+1} r! \sum_{a \geq 0} \frac{1}{(a+x)^{r+1}}$$

for x complex, not equal to a negative integer. We have

$$\sum_{a \geq m\ell} \frac{x^2}{(m+ax)^2} = \sum_{a \geq 0} \frac{x^2}{(m + (a + m\ell)x)^2} = \sum_{a \geq 0} \frac{1}{(a + m\ell + \frac{m}{x})^2} = \Psi^{(1)}\left(m\ell + \frac{m}{x}\right) \quad .$$

Thus, formally

$$\sum_{a \in \mathbb{Z}_{>0}, a \geq m\ell} \frac{mX_1}{(m + aX_1)^2} = \frac{m\Psi^{(1)}(m\ell + \frac{m}{X_1})}{X_1} \quad ,$$

and the right-hand side in (1) may be rewritten as

$$1 - \lim_{m \to \infty} \frac{m}{X_1} \Psi^{(1)}\left(m\ell + \frac{m}{X_1}\right) \quad .$$

The asymptotic behavior of $\Psi^{(r)}(x)$ is well-known: as $x \to \infty$ in any fixed sector not including the negative real axis,

$$\Psi^{(r)}(x) \sim (-1)^{r+1} r! \left(\frac{x^{-r}}{r} + \frac{x^{-r-1}}{2} + \sum_{k\geq 1} \frac{B_{2k}}{(2k)!} \frac{\Gamma(r+2k)}{\Gamma(r+1)} x^{-r-2k} \right)$$

(see for instance [Apo13], (25.11.43)). In particular, for fixed ℓ and x

$$\Psi^{(1)}\left(m\left(\ell+\frac{1}{x}\right)\right) \sim \left(m\left(\ell+\frac{1}{x}\right)\right)^{-1} = \frac{x}{m(1+\ell x)}$$

as $m \to \infty$ in $\mathbb{Z}_{>0}$. Therefore,

$$\lim_{m\to\infty} \frac{m}{X_1} \Psi^{(1)}\left(m\ell + \frac{m}{X_1}\right) = \lim_{m\to\infty} \frac{m}{X_1} \frac{X_1}{m(1+\ell X_1)} = \frac{1}{1+\ell X_1} \quad .$$

Theorem 1.1 asserts that

$$s(S, \mathbb{P}^M) = 1 - \frac{1}{1+\ell X_1} = \frac{\ell X_1}{1+\ell X_1} = c(N_S \mathbb{P}^M)^{-1} \cap [S] \quad ,$$

as it should, since S is a divisor in this case.

The assumption $n = 1$ in this computation must be irrelevant, since the Segre class is not affected by this choice. The computation itself *is,* however, affected by the choice of n. Viewing the monomial x_1^ℓ as a monomial in (for example) two variables x_1, x_2 leads via Theorem 1.1 to the formula

$$s(S, \mathbb{P}^M) = 1 - \lim_{m\to\infty} \sum \frac{2mX_1X_2}{(m+a_1X_1+a_2X_2)^3} \quad ,$$

where the summation is over all positive integers a_1, a_2 such that $\mathrm{lct}(I_{a_2,a_1}) \geq \frac{m}{a_1a_2}$. Since $I_{a_2,a_1} = (x_1^{\ell a_2})$, this amounts to the requirement that $a_1 \geq m\ell, a_2 \geq 1$, so the summation may be rewritten

$$\sum_{a_1\geq m\ell, a_2\geq 1} \frac{2mX_1X_2}{(m+a_1X_1+a_2X_2)^3} = \frac{2mX_1X_2}{X_2^3} \sum_{a_1\geq m\ell} \sum_{a_2\geq 0} \frac{1}{(a_2+1+\frac{m+a_1X_1}{X_2})^3} \quad .$$

After performing the second summation, we see that the content of Theorem 1.1 in this case is

$$s(S, \mathbb{P}^M) = 1 - \lim_{m\to\infty} \frac{-mX_1}{X_2^2} \sum_{a_1\geq m\ell} \Psi^{(2)}\left(\frac{m+a_1X_1+X_2}{X_2}\right) \quad . \tag{2}$$

Heuristically, we can now argue that, as $m \to \infty$,

$$\Psi^{(2)}\left(\frac{m+a_1X_1+X_2}{X_2}\right) \sim -\left(\frac{m+a_1X_1+X_2}{X_2}\right)^{-2}$$

so that, again as $m \to \infty$,

$$\begin{aligned}\sum_{a_1\geq m\ell} \Psi^{(2)}\left(\frac{m+a_1X_1+X_2}{X_2}\right) &\sim -\sum_{a_1\geq m\ell} \frac{X_2^2}{(m+a_1X_1+X_2)^2} = -\frac{X_2^2}{X_1^2} \sum_{a\geq 0} \frac{1}{(a+m\ell+\frac{m+X_2}{X_1})^2} \\ &= -\frac{X_2^2}{X_1^2} \Psi^{(1)}\left(m\ell + \frac{m+X_2}{X_1}\right) \sim -\frac{X_2^2}{X_1^2}\left(m\ell+\frac{m+X_2}{X_1}\right)^{-1} = -\frac{X_2^2}{mX_1} \frac{1}{1+\ell X_1+\frac{X_2}{m}} \quad .\end{aligned}$$

Thus, the right-hand side of (2) equals

$$1-\lim_{m\to\infty}\frac{1}{1+\ell X_1+\frac{X_2}{m}}=\frac{\ell X_1}{1+\ell X_1}$$

as expected.

For 'diagonal' ideals $I=(x_1^{\ell_1},\dots,x_n^{\ell_n})$, we have

$$\mathrm{lct}(I_{a_2\cdots a_n,\dots,a_1\cdots a_{n-1}})=\mathrm{lct}(x_1^{\ell_1 a_2\cdots a_n},\dots,x_n^{a_1\cdots a_{n-1}\ell_n})=\frac{1}{\ell_1 a_2\cdots a_n}+\cdots+\frac{1}{a_1\cdots a_{n-1}\ell_n}\quad;$$

the condition that this be $\geq m/a_1\cdots a_n$ is equivalent to

$$\frac{a_1}{\ell_1}+\cdots+\frac{a_n}{\ell_n}\geq m\quad.$$

For e.g., $n=2$, the content of Theorem 1.1 in this case is the identity

$$1+\lim_{m\to\infty}\frac{mX_1}{X_2^2}\left(\sum_{a_1=1}^{m\ell_1-1}\Psi^{(2)}\left(m\ell_2-\lfloor\frac{a_1\ell_2}{\ell_1}\rfloor+\frac{m+a_1X_1}{X_2}\right)+\sum_{a_1\geq m\ell_1}\Psi^{(2)}\left(1+\frac{m+a_1X_1}{X_2}\right)\right)$$
$$=\frac{\ell_1\ell_2X_1X_2}{(1+\ell_1X_1)(1+\ell_2X_2)}\quad.$$

3. Proof of Theorem 1.1

For positive integers $r_1,\dots,r_n$ and a homogeneous ideal I of $k[x_1,\dots,x_{M+1}]$ generated by polynomials in $x_1,\dots,x_n$, with $M+1\geq n$, we let $I_{r_1,\dots,r_n}$ denote the extension of I via the ring homomorphism $k[x_1,\dots,x_{M+1}]\to k[x_1,\dots,x_{M+1}]$ defined by $x_i\mapsto x_i^{r_i}, i=1\dots,n$. If I is a monomial ideal, let $N'\subset\mathbb{R}^n$ be the convex hull of the lattice points $(i_1,\dots,i_n)\in\mathbb{Z}^n$ such that $x_1^{i_1}\cdots x_n^{i_n}\in I$, and let N be the (closure of the) complement of N' in the positive orthant $\mathbb{R}^n_{\geq 0}$. We call N the 'Newton region' for I.

If I is monomial, the ideal $I_{r_1,\dots,r_n}$ is also monomial, and its Newton region is obtained by stretching N by a factor of r_1 in the x_1 direction, …, r_n in the x_n direction. We will denote by $N_{r_1,\dots,r_n}$ this stretched region.

Lemma 3.1. *Let I be a proper monomial ideal, and let N be as above. For $(a_1,\dots,a_n)\in\mathbb{Z}^n_{>1}$ and $m>0$,*

$$\left(\frac{a_1}{m},\dots,\frac{a_n}{m}\right)\in N\iff a_1\cdots a_n\,\mathrm{lct}(I_{a_2\cdots a_n,\dots,a_1\cdots a_{n-1}})\leq m\quad.$$

Proof. Let $a_1,\dots,a_n$ integers >1. Note that

$$\left(\frac{a_1}{m},\dots,\frac{a_n}{m}\right)\in N\iff\left(\frac{a_1}{m}a_2\cdots a_n,\dots,\frac{a_n}{m}a_1\cdots a_{n-1}\right)\in N_{a_2\cdots a_n,\dots,a_1\cdots a_{n-1}}$$
$$\iff\frac{a_1\cdots a_n}{m}(1,\dots,1)\in N_{a_2\cdots a_n,\dots,a_1\cdots a_{n-1}}\quad.$$

By Howald's result ([How01], Example 5) this is the case if and only if

$$\frac{a_1\cdots a_n}{m}\leq\frac{1}{\mathrm{lct}(I_{a_2\cdots a_n,\dots,a_1\cdots a_{n-1}})}\quad,$$

yielding the statement. □

Remark 3.2. The restriction to $a_i > 1$ in this statement is in order to ward off the 'annoying exception' raised in [How01], Example 5: the formula for the log canonical threshold used in the proof does not hold if the corresponding multiplier ideal is trivial. In any case, the difference between N and the region spanned by the n-tuples $(\frac{a_1}{m}, \dots, \frac{a_n}{m})$ satisfying the stated condition with $a_i > 0$ vanishes in the limit as $m \to \infty$, so we may (and will) adopt the condition for $(a_1, \dots, a_n) \in \mathbb{Z}^n_{>0}$ in the application to Theorem 1.1. ⌟

By Lemma 3.1, the limit in (1) equals

$$\lim_{m\to\infty} \frac{1}{m^n} \sum_{\substack{(a_1,\dots,a_n)\in\mathbb{Z}^n_{>0} \\ (\frac{a_1}{m},\dots,\frac{a_n}{m})\in N'}} \frac{n!\, X_1\cdots X_n}{(1+\frac{a_1}{m}X_1+\cdots+\frac{a_n}{m}X_n)^{n+1}} \quad .$$

This may be interpreted as a limit of Riemann sums for the integral

$$\int_{N'} \frac{n! X_1\cdots X_n\, da_1\cdots da_n}{(1+a_1X_1+\cdots+a_nX_n)^{n+1}} \quad .$$

Since the value of this integral on the positive orthant is 1, the right-hand side of (1) equals

$$\int_{N} \frac{n! X_1\cdots X_n\, da_1\cdots da_n}{(1+a_1X_1+\cdots+a_nX_n)^{n+1}} \quad .$$

This equals the Segre class $s(S, \mathbb{P}^M)$ once X_i is interpreted as the i-th coordinate hyperplane in $\mathbb{P}^M$, by Theorem 1.1 in [Alu]. ⌟

References

[Alu] Paolo Aluffi. Segre classes as integrals over polytopes. arXiv:1307.0830, to appear in J. Eur. Math. Soc. (JEMS).

[Apo13] Tom Apostol. Zeta and Related Functions. In *The Digital Library of Mathematical Functions* `http://dlmf.nist.gov/`. National Institute of Standards and Technology (NIST), Gaithersburg, U.S.A., 2013.

[Ful84] William Fulton. *Introduction to intersection theory in algebraic geometry*, volume 54 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1984.

[How01] J. A. Howald. Multiplier ideals of monomial ideals. *Trans. Amer. Math. Soc.*, 353(7):2665–2671, 2001.

[Mus12] Mircea Mustaţă. IMPANGA lecture notes on log canonical thresholds. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 407–442. Eur. Math. Soc., Zürich, 2012. Notes by Tomasz Szemberg.

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