SEGRE CLASSES OF MONOMIAL SCHEMES

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Abstract. We propose an explicit formula for the Segre classes of monomial subschemes of nonsingular varieties, such as schemes defined by monomial ideals in projective space. The Segre class is expressed as a formal integral on a region bounded by the corresponding Newton polyhedron. We prove this formula for monomial ideals in two variables and verify it for some families of examples in any number of variables.

1. Introduction

1.1. The *excess numbers* of a subscheme S of projective space $\mathbb{P}^n$ are roughly defined as the numbers of points of intersection in the complement of S of n general hypersurfaces of given degrees containing S. Many challenging open enumerative problems, such as the problem of computing characteristic numbers for families of plane curves, may be stated in terms of excess numbers. Recently, the problem of computing excess numbers has been raised in algebraic statistics and in applications to machine learning and ideal regression. See [Rod12] and references therein.

The excess numbers of a subscheme S may be computed from the push-forward of the *Segre class* $s(S, \mathbb{P}^n)$ to $\mathbb{P}^n$. Segre classes are defined for arbitrary closed embeddings of schemes, and in a sense carry all the intersection-theoretic information associated with the embedding ([Ful84], Chapters 4 and 6). Thus, they provide a general context which applies to the computation of excess numbers, and relates this problem with the well-developed tools of Fulton-MacPherson intersection theory. On the other hand, the computation of Segre classes is challenging, and the connection with excess numbers appears to have mostly been exploited in the reverse direction—providing algorithms for the computation of Segre classes starting from the explicit solution of enumerative problems by computer algebra systems such as Macaulay2 ([GS]). This strategy informs the author's implementation of an algorithm for Chern and Segre classes of subschemes of projective space in [Alu03], as well as more recent work on algorithmic computations of these classes ([DREPS11], [EJP13]).

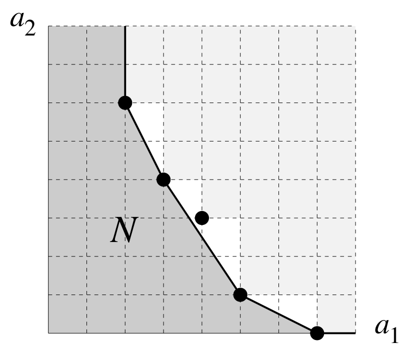
In this note we conjecture a general formula for the Segre class of a *monomial* subscheme, in terms of a corresponding Newton polyhedron. The monomial case is of independent interest, and in principle more general situations can be reduced to the monomial case by means of algebraic homotopies ([Rod12]). We prove the formula in the case of monomials in two variables in any nonsingular variety, and verify it for some nontrivial examples in arbitrarily many variables. The formula is expressed as a formal integral over the region bounded by a Newton polyhedron associated with the subscheme. This integral can be computed directly from a subdivision of the region into simplices.

1.2. We now state the proposed formula precisely. Let V be a nonsingular variety, and let $X_1, \dots, X_n$ be nonsingular hypersurfaces meeting with normal crossings in V. For $I = (i_1, \dots, i_n)$, we denote by X^I the hypersurface obtained by taking X_j with multiplicity i_j, and call this hypersurface a 'monomial' (supported on $X_1, \dots, X_n$). A *monomial subscheme* S of V is an intersection of monomials X^{I_k} supported on a fixed set of hypersurfaces. The exponents I_k determine a (possibly unbounded) region N in the orthant $\mathbb{R}^n_{\geq 0}$ in $\mathbb{R}^n$, namely, the complement of the convex hull of the union of the orthants with origins translated at I_k. We call this region the *Newton region* for the exponents I_k. Note that a lattice point is in the interior of the Newton region if and only if the corresponding monomial is *not* in the integral closure of I.

Example 1.1. For $n = 2$ and monomials

$$X^{(2,6)}, \quad X^{(3,4)}, \quad X^{(4,3)}, \quad X^{(5,1)}, \quad X^{(7,0)},$$

the Newton region N is as in the following picture:



The third monomial $X^{(4,3)}$ does not affect the Newton region, as it is contained in the convex hull of the other translated quadrants (cf. Remark 2.5). ⌟

Conjecture 1. *If ι is the inclusion morphism $S \hookrightarrow V$, then*

$$\iota_* s(S,V) = \int_N \frac{n! X_1 \cdots X_n \, da_1 \cdots da_n}{(1 + a_1 X_1 + \cdots + a_n X_n)^{n+1}} \quad . \tag{1}$$

The right-hand side is interpreted by evaluating the integral formally with $X_1, \dots, X_n$ as parameters; the result is a rational function in $X_1, \dots, X_n$, with a well-defined expansion as a power series in these variables. The claim in Conjecture 1 is that evaluating the terms of this series as intersection products of the corresponding divisor classes in V gives the push-forward $\iota_* s(S,V)$.

Example 1.2. Using Fubini's theorem to perform the integral for the monomials in Example 1.1 and taking $X_1 = X_2 = H$ (for example, the hyperplane class in projective space) gives

$$\begin{aligned}
&\int_0^2 \left(\int_0^\infty \frac{2\,H^2\,da_2}{(1+(a_1+a_2)H)^3} \right) da_1 + \int_2^3 \left(\int_0^{10-2a_1} \frac{2\,H^2\,da_2}{(1+(a_1+a_2)H)^3} \right) da_1 \\
&\quad + \int_3^5 \left(\int_0^{\frac{17}{2}-\frac{3a_1}{2}} \frac{2\,H^2\,da_2}{(1+(a_1+a_2)H)^3} \right) da_1 + \int_5^7 \left(\int_0^{\frac{7}{2}-\frac{a_1}{2}} \frac{2\,H^2\,da_2}{(1+(a_1+a_2)H)^3} \right) da_1 \\
&= \frac{2H}{1+2H} + \frac{10H^2(1+5H)}{(1+2H)(1+3H)(1+7H)(1+8H)} + \frac{2H^2(5+27H)}{(1+3H)(1+5H)(1+6H)(1+7H)} \\
&\qquad + \frac{2H^2}{(1+5H)(1+6H)(1+7H)} = \frac{2H(1+30H+168H^2)}{(1+6H)(1+7H)(1+8H)} \quad .
\end{aligned}$$

See Example 1.4 below for an alternative way to evaluate this integral. Expanding as a power series,

$$\frac{2H(1+30H+168H^2)}{(1+6H)(1+7H)(1+8H)} = 2H + 18H^2 - 334H^3 + 3714H^4 - 35278H^5 + \cdots$$

According to Conjecture 1, the scheme S defined by the monomial ideal

$$I = (x_1^2x_2^6\,,\, x_1^3x_2^4\,,\, x_1^4x_2^3\,,\, x_1^5x_2\,,\, x_1^7)$$

in $\mathbb{P}^5$ has Segre class

$$\iota_* s(S,\mathbb{P}^5) = 2[\mathbb{P}^4] + 18[\mathbb{P}^3] - 334[\mathbb{P}^2] + 3714[\mathbb{P}^1] - 35278[\mathbb{P}^0] \quad .$$

This agrees with the output for I of the Macaulay2 procedure computing Segre classes given in [Alu03]:

```
Macaulay2, version 1.5
with packages: ConwayPolynomials, Elimination, IntegralClosure, LLLBases,
               PrimaryDecomposition, ReesAlgebra, TangentCone
i1 : load("CSM.m2");
i2 : QQ[x0,x1,x2,x3,x4,x5];
i3 : time segre ideal (x1^2*x2^6,x1^3*x2^4,x1^4*x2^3,x1^5*x2,x1^7)
                         5        4       3      2
Segre class : - 35278H  + 3714H  - 334H  + 18H  + 2H
     -- used 51.148 seconds
```

In terms of intersection numbers, the equivalence of the subscheme S defined by I in the intersection of hypersurfaces of degree $d_1,\dots,d_5$ is the coefficient of H^5 in

$$\left(\prod_{i=1}^{5}(1+d_iH)\right)\cdot(2H+18H^2-334H^3+3714H^4-35278H^5) \quad , \tag{2}$$

provided that the hypersurfaces cut out S in a neighborhood of S or, more generally, provided that the scheme cut out by the hypersurfaces near S has the same Segre class as S. (This is guaranteed if the hypersurfaces are general enough subject to the condition of containing S.) The excess number is the difference between this number and the Bézout number $d_1\cdots d_5$. ⌟

1.3. In this note we prove:

Theorem 1.3. *Conjecture 1 holds for $n \le 2$.*

The proof of Theorem 1.3 is a direct application of techniques in Fulton-MacPherson intersection theory. Concerning the validity of (1) for $n>2$, complete intersections of monomials $x_1^{m_1},\dots,x_n^{m_n}$ give a straightforward, but maybe too simple example (§3.2). We give more evidence for this formula in terms of a family of nontrivial examples for arbitrary n, namely the monomial ideals corresponding to the exponents

$$(0,1,\dots,1),\quad (1,0,1,\dots,1),\quad \dots,\quad (1,\dots,1,0) \quad .$$

For these examples we can compute independently the Segre class using the relation between Segre classes of singularity subschemes and Chern-Schwartz-MacPherson classes ([Alu99]), and we find (Proposition 3.4) that the expression we obtain does match the result of applying the formula given in Conjecture 1.

1.4. Excess numbers of monomial ideals admit an expression in terms of mixed volumes of polytopes, via Bernstein's theorem; the example of the monomial ideal $(x_1^{p_1},\dots,x_k^{p_k})$ is worked out explicitly in [Rod12]. Thus, the expression obtained in (2) may be interpreted as a computation of the mixed volumes of certain polytopes in terms of the integral in (1), for the monomial subscheme of Example 1.1. Conversely, Bernstein's theorem may offer a path to the proof of the conjecture stated above for $n>2$, at least if the classes X_i are ample enough. We do not pursue this approach here; Bernstein's theorem is not used in our proof of Theorem 1.3.

A precise relation between Segre classes, volumes of convex bodies, and integrals such as those appearing in Conjecture 1 would be very valuable. Formula (1) (if verified) suggests that the Segre class of the scheme defined by an ideal I may be computed as a suitable integral over a region in $\mathbb{R}^n_{\geq 0}$ associated with I. A naive guess is that the convex bodies appearing in the work of Lazarsfeld and Mustaţă ([LM09]) and Kaveh and Khovanskii ([KK12]) would play a key role in such a result.

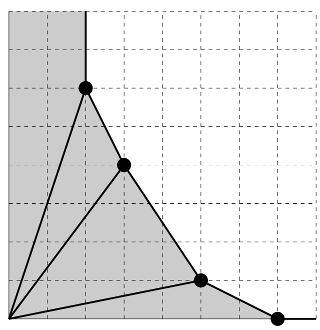
As mentioned above, current algorithms for Segre classes essentially reduce the computation to enumerative problems, which are then solved by methods in computer algebra. This limits substantially the scope of these algorithms, and runs against one of the main applications of Segre classes: in principle one would want to compute Segre classes in order to solve hard enumerative problems, not the other way around. Formulas such as (1) do not rely on enumerative information, so they have the potential for broader applications.

1.5. We end this introduction by noting that the integrals appearing in (1) have the following property: if T is an n-dimensional simplex, then

$$\int_T \frac{n! X_1 \cdots X_n \, da_1 \cdots da_n}{(1 + a_1 X_1 + \cdots + a_n X_n)^{n+1}} = \frac{n! \operatorname{Vol}(T) \, X_1 \cdots X_n}{\prod_{(a_1,\dots,a_n)} (1 + a_1 X_1 + \cdots a_n X_n)} \tag{3}$$

where the product ranges over the vertices of T. An analogous expression may be given for unbounded regions dominating a simplex in lower dimension; see Proposition 3.1. Thus, one may compute the integral in (1) by splitting the region N in any way into (possibly unbounded) simplices and applying (3).

Example 1.4. The computation carried out in Example 1.2 may be performed by splitting the region N as a union of triangles and one unbounded region as follows:



Each triangle contribues a rational function according to (3):

$$\frac{2H}{(1+8H)} + \frac{10H^2}{(1+8H)(1+7H)} + \frac{17H^2}{(1+7H)(1+6H)} + \frac{7H^2}{(1+6H)(1+7H)}$$
$$= \frac{2H(1+30H+168H^2)}{(1+6H)(1+7H)(1+8H)}$$

(the first term accounts for the unbounded region). This reproduces the result obtained in Example 1.2. ⌟

This approach may in fact be taken as an alternative interpretation of the meaning of the right-hand side of (1). What is possibly surprising from this point of view, and is transparent from the interpretation as an integral, is that the result *does not depend on the chosen decomposition.*

Theorem 1.3 is proven in §2. Generalizations of (3) and examples giving evidence for the validity of (1) for monomial ideals in arbitrarily many variables are discussed in §3.

1.6. **Acknowledgments.** The author thanks the referee for constructive comments. The author's research is partially supported by a Simons collaboration grant.

2. Proof of Theorem 1.3

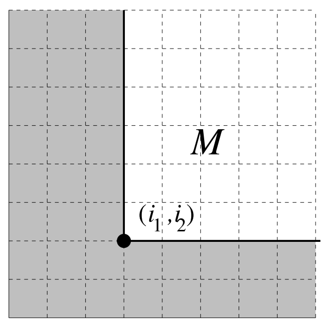
2.1. **Principal monomial ideals.** We maintain the notation used in the introduction. Notice that

$$\int_{\mathbb{R}^n_{\geq 0}} \frac{n!\, X_1 \cdots X_n \, da_1 \cdots da_n}{(1 + a_1 X_1 + \cdots + a_n X_n)^{n+1}} = 1 \quad ;$$

this follows from a simple induction. More generally, if $i_j \geq 0$ and M denotes the region defined by the inequalities $a_1 \geq i_1, \ldots, a_n \geq i_n$, then

$$\int_M \frac{n!\, X_1 \cdots X_n \, da_1 \cdots da_n}{(1 + a_1 X_1 + \cdots + a_n X_n)^{n+1}} = \frac{1}{1 + i_1 X_1 + \cdots + i_n X_n} \quad .$$

(Again, this is a simple induction; see Proposition 3.1 for generalizations.)



Now consider the subscheme $S \overset{\iota}{\hookrightarrow} V$ defined by a principal ideal (X^I) generated by a single monomial, with $I = (i_1, \ldots, i_n)$. Then S is a Cartier divisor, with normal bundle

$$N_S V \cong \mathscr{O}(i_1 X_1 + \cdots + i_n X_n) \quad ,$$

and therefore

$$\iota_* s(S, V) = c(N_S V)^{-1} \cap [S] = \frac{i_1 X_1 + \cdots + i_n X_n}{1 + i_1 X_1 + \cdots + i_n X_n} \quad .$$

The integral from equation 1 for this example is

$$\int_{\mathbb{R}^n_{\geq 0} \smallsetminus M} \frac{n!\, X_1 \cdots X_n \, da_1 \cdots da_n}{(1 + a_1 X_1 + \cdots + a_n X_n)^{n+1}} = 1 - \frac{1}{1 + i_1 X_1 + \cdots + i_n X_n} = \frac{i_1 X_1 + \cdots + i_n X_n}{1 + i_1 X_1 + \cdots + i_n X_n}$$

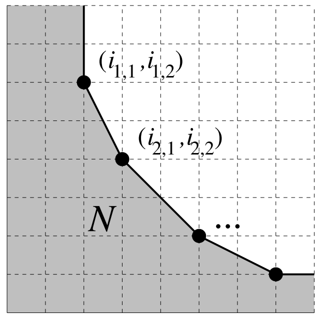
verifying Conjecture 1 for principal monomial ideals.

In particular, this verifies Conjecture 1 for $n = 1$.

2.2. **The case $n = 2$.** Monomial ideals in two variables can be principalized by a sequence of blow ups along codimension 2 loci. (In fact, *every* monomial ideal may be principalized by blowing up codimension 2 loci, cf. [Gow05].) A principalization algorithm may be described as follows: if S is a monomial ideal supported on X_1, X_2 in V, let $\pi : \widetilde{V} \to V$ be the blow-up of V along $X_1 \cap X_2$ (which is nonsingular by hypothesis), let E be the exceptional divisor, and $\widetilde{X}_1, \widetilde{X}_2$ the proper transforms of X_1, X_2. (Note that $\widetilde{X}_1$, $\widetilde{X}_2$ are disjoint.) Up to a principal component supported on E, $\pi^{-1}(S)$ is again a union of disjoint monomial subschemes S_1, S_2. Iterating this process on S_1, S_2 leads to a sequence of blow-ups principalizing S.

Therefore, the $n = 2$ case of Conjecture 1 may be proven by showing that the validity of the conjecture for S_1, S_2 implies that (1) holds for S: indeed, induction on the number of blow-ups needed for a principalization reduces the conjecture to the principal case, which has been verified in §2.1. We carry out this strategy in the rest of this section.

Let S be defined by monomials X^{I_k}, with $I_k = (i_{k,1}, i_{k,2})$, $k = 1, \dots, r$. The Newton region N is the complement of the convex hull of the union of the $(i_{k,1}, i_{k,2})$ translations of the positive quadrants:



For $n = 2$, (1) states that

$$\iota_* s(S, V) = \int_N \frac{2\, X_1 X_2\, da_1 da_2}{(1 + a_1 X_1 + a_2 X_2)^3} \quad ,$$

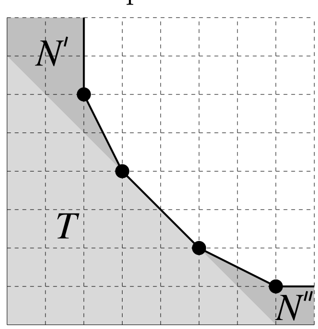
where $\iota : S \hookrightarrow V$ is the embedding. With $\pi : \widetilde{V} \to V$ as above, let j be the inclusion of $\pi^{-1}(S)$ in $\widetilde{V}$.

$$\begin{tikzcd} \pi^{-1}(S) \arrow[r, hook, "j"] \arrow[d] & \widetilde{V} \arrow[d, "\pi"] \\ S \arrow[r, hook, "\iota"] & V \end{tikzcd}$$

By the birational invariance of Segre classes (Proposition 4.2 (a) in [Ful84]),

$$\iota_* s(S, V) = \pi_* j_* s(\pi^{-1}(S), \widetilde{V}) \quad .$$

The scheme $\pi^{-1}(S)$ contains a copy of the exceptional divisor E with multiplicity m, where m is the minimum of $i_{k,1} + i_{k,2}$ for $k = 1, \dots, r$. The region N is split into three areas: the triangle T with vertices $(0,0), (0,m), (m,0)$, and the two (possibly empty) top-left (N') and bottom-right (N'') components of the complement of this triangle:



The residual of mE in $\pi^{-1}(S)$ consists of two disjoint subschemes S_1, S_2 of $\widetilde{V}$. These are monomial subschemes, supported respectively on $\widetilde{X}_1, E$ and $E, \widetilde{X}_2$.

Claim 2.1. *Exponents for S_1, resp. S_2 are*

$$(i_{k,1}, i_{k,1} + i_{k,2} - m), \quad k = 1, \dots, r, \quad \textit{resp.} \quad (i_{k,1} + i_{k,2} - m, i_{k,2}), \quad k = 1, \dots, r \quad .$$

Proof. In a neighborhood of S_1, $\pi^{-1}(S)$ is the intersection of divisors denoted additively as

$$i_{k,1}\pi^{-1}(X_1)+i_{k,2}\pi^{-1}(X_2)=i_{k,1}(\widetilde{X}_1+E)+i_{k,2}E=i_{k,1}(\widetilde{X}_1)+(i_{k,1}+i_{k,2})E$$

(since S_1 is disjoint from $\widetilde{X}_2$, $\pi^{-1}(X_2)$ agrees with E near S_1). The monomial scheme S_1 is obtained as the residual of mE in this intersection, with corresponding exponents as stated. The analysis is identical near S_2. □

By residual intersection ([Ful84], Proposition 9.2, in the form given in [Alu94], Proposition 3), $s(\pi^{-1}(S),\widetilde{V})$ equals

$$\frac{mE}{1+mE}+\frac{1}{1+mE}\cap(s(S_1,\widetilde{V})\otimes\mathscr{O}(mE))+\frac{1}{1+mE}\cap(s(S_2,\widetilde{V})\otimes\mathscr{O}(mE)) \quad .$$

(Here and in what follows we use freely notation as in [Alu94], §2.) Therefore, $i_*s(S,V)$ is naturally the sum of three terms: the push-forwards to V of

$$\text{(i)}\ \frac{mE}{1+mE},\quad \text{(ii)}\ \frac{1}{1+mE}\cap(s(S_1,\widetilde{V})\otimes\mathscr{O}(mE)),\quad \text{(iii)}\ \frac{1}{1+mE}\cap(s(S_2,\widetilde{V})\otimes\mathscr{O}(mE)) \quad .$$

The following claim will conclude the proof of Theorem 1.3.

Claim 2.2. *The terms (i), resp. (ii), (iii) push-forward to the values of the integral on the subregions T, resp. N', N'' of N determined above.*

Proof. (i): By the birational invariance of Segre classes, the push-forward of $E/(1+E)$ is the Segre class of the center of the blow-up. Therefore, the push-forward of $mE/(1+mE)$ is the m-th Adams of this Segre class. Since the center is the complete intersection of X_1 and X_2, this is given by

$$\frac{mE}{1+mE}\mapsto\frac{m^2X_1\cdot X_2}{(1+mX_1)(1+mX_2)} \quad .$$

(The Segre class of a complete intersection equals the inverse Chern class of its normal bundle.) The claim is that this expression equals the integral

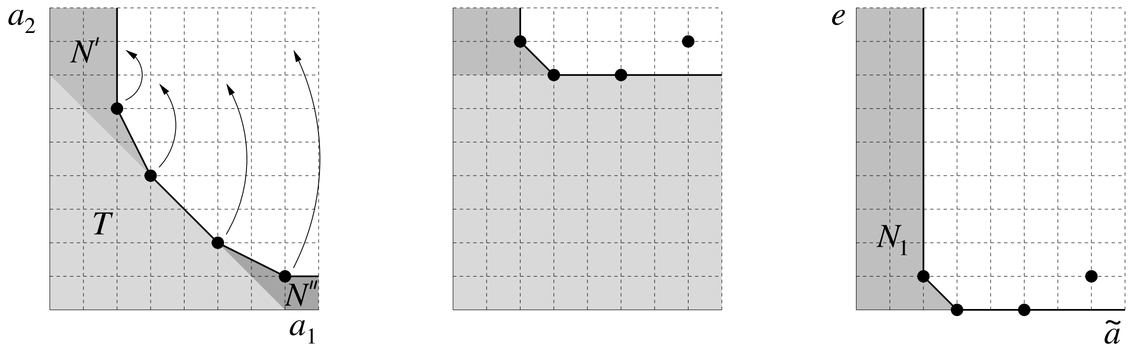
$$\int_T\frac{2X_1X_2\,da_1\,da_2}{(1+a_1X_1+a_2X_2)^3}$$

where T is the triangle with vertices $(0,0)$, $(0,m)$, $(m,0)$. The verification of this fact is a trivial calculus exercise (see Proposition 3.1 for a generalization).

(ii): Term (ii) is

$$\frac{1}{1+mE}\cap(s(S_1,\widetilde{V})\otimes\mathscr{O}(mE)) \quad ,$$

where S_1 is monomial on $\widetilde{X}_1$, E with exponents $(i_{k,1},i_{k,1}+i_{k,2}-m)$ as observed in Claim 2.1. Each vertex of the Newton polyhedron for S determines a vertex for S_1 by the transformation $(a_1,a_2)\mapsto(\tilde{a},e)=(a_1,a_1+a_2-m)$.



By induction on the number of blow-ups needed for a principalization, Conjecture 1 holds for S_1. Thus

$$\iota_{1*}s(S_1,\widetilde{V}) = \int_{N_1} \frac{2\widetilde{X}_1 E\, d\tilde{a}\, de}{(1+\tilde{a}\widetilde{X}_1+eE)^3} \tag{4}$$

where $\iota_1 : S_1 \hookrightarrow \widetilde{V}$ is the embedding, and N_1 is the Newton region for S_1. Note that N_1 maps onto region N' via the transformation $(\tilde{a}, e) \mapsto (a_1, a_2) = (\tilde{a}, e - \tilde{a} + m)$.

Claim 2.3.

$$\iota_{1*}\left(\frac{1}{1+mE} \cap \left(s(S_1,\widetilde{V}) \otimes \mathscr{O}(mE)\right)\right) = \int_{N_1} \frac{2\widetilde{X}_1 E\, d\tilde{a}\, de}{(1+\tilde{a}\widetilde{X}_1+(e+m)E)^3}$$

Proof. Applying (4), the projection formula, and the fact that the line bundle $\mathscr{O}(mE)$ is constant with respect to the integration variables $\tilde{a}$ and e, we see that the left-hand side equals

$$\begin{aligned}
\frac{1}{1+mE}&\cap\left(\left(\int_{N_1} \frac{2\widetilde{X}_1 E\, d\tilde{a}\, de}{(1+\tilde{a}\widetilde{X}_1+eE)^3}\right) \otimes \mathscr{O}(mE)\right)\\
&= \int_{N_1} \frac{1}{1+mE}\left(\frac{2\widetilde{X}_1 E}{(1+\tilde{a}\widetilde{X}_1+eE)^3} \otimes \mathscr{O}(mE)\right) d\tilde{a}\, de\\
&= \int_{N_1} \frac{1}{1+mE}\left(\frac{(1+mE)\, 2\widetilde{X}_1 E}{(1+\tilde{a}\widetilde{X}_1+(e+m)E)^3}\right) d\tilde{a}\, de\\
&= \int_{N_1} \frac{2\widetilde{X}_1 E\, d\tilde{a}\, de}{(1+\tilde{a}\widetilde{X}_1+(e+m)E)^3}
\end{aligned}$$

as stated. (We have used here the formal properties of the $\otimes$ operation, in particular Proposition 1 of [Alu94].) $\square$

Thus, we are reduced to proving the following.

Claim 2.4.

$$\pi_*\int_{N_1} \frac{2\widetilde{X}_1 E\, d\tilde{a}\, de}{(1+\tilde{a}\widetilde{X}_1+(e+m)E)^3} = \int_{N'} \frac{2\, X_1 X_2\, da_1 da_2}{(1+a_1X_1+a_2X_2)^3} \quad . \tag{5}$$

Proof. As observed above, N_1 maps onto N' via $(a_1, a_2) \mapsto (\tilde{a}, e) = (a_1, a_1 + a_2 - m)$. This transformation has Jacobian 1, therefore (5) follows from the equality

$$\pi_*\left(\frac{\widetilde{X}_1 \cdot E}{(1+\tilde{a}\widetilde{X}_1+(e+m)E)^3}\right) = \frac{X_1 \cdot X_2}{(1+a_1X_1+a_2X_2)^3} \quad . \tag{6}$$

In turn, (6) follows from the projection formula. Indeed,

$$\begin{aligned}\pi^*(a_1X_1+a_2X_2) &= a_1(\widetilde{X}_1+E)+a_2(\widetilde{X}_2+E) = \tilde{a}(\widetilde{X}_1+E)+(e-\tilde{a}+m)(\widetilde{X}_2+E)\\ &= \tilde{a}\widetilde{X}_1+(e+m)E+(e-\tilde{a}+m)\widetilde{X}_2 \quad ,\end{aligned}$$

so that $\dfrac{\widetilde{X}_1\cdot E}{(1+\tilde{a}\widetilde{X}_1+(e+m)E)^3}$ equals

$$\frac{\widetilde{X}_1\cdot E}{(\pi^*(1+a_1X_1+a_2X_2)-(e-\tilde{a}+m)\widetilde{X}_2)^3}=\frac{\widetilde{X}_1\cdot E}{(\pi^*(1+a_1X_1+a_2X_2))^3}$$

as $\widetilde{X}_1\cdot\widetilde{X}_2=0$. Hence

$$\pi_*\left(\frac{\widetilde{X}_1\cdot E}{(1+\tilde{a}\widetilde{X}_1+(e+m)E)^3}\right)=\pi_*\left(\frac{\widetilde{X}_1\cdot E}{(\pi^*(1+a_1X_1+a_2X_2))^3}\right)=\frac{\pi_*(\widetilde{X}_1\cdot E)}{(1+a_1X_1+a_2X_2)^3} \quad ,$$

and $\pi_*(\widetilde{X}_1\cdot E)=X_1\cdot X_2$, concluding the proof. □

(iii) follows by symmetry. □

Remark 2.5. As a consequence of Theorem 1.3, monomial generators which do not affect the Newton region (i.e., which are in the convex hull of the translated quadrants determined by the other generators) do not affect the Segre class.

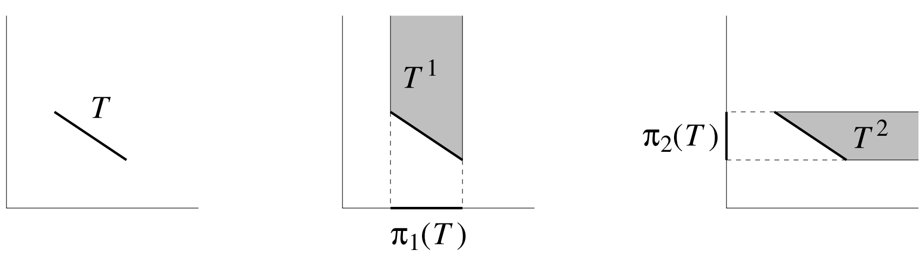
This fact is not surprising, and holds for arbitrary n. Indeed, such generators do not affect the integral closure of the ideal, and the Segre class only depends on the integral closure, cf. the proof of Lemma 1.2 in [Alu95]. ⌟

3. Examples for arbitrary n

3.1. **Calculus.** The following observation simplifies the computation of the integral in Conjecture 1. We work in $\mathbb{R}^n$, with coordinates $(a_1,\dots,a_n)$. Let $\mathbf{e}_1=(1,0,\dots,0),\dots,\mathbf{e}_n=(0,\dots,0,1)$. We also consider indeterminates X_i, and for $\mathbf{a}=(a_1,\dots,a_n)\in\mathbb{R}^n$ we write $a_1X_1+\cdots+a_nX_n=\mathbf{a}\cdot\mathbf{X}$. The integral in (1) is

$$\int_N \frac{n!\,X_1\cdots X_n\,da_1\cdots da_n}{(1+\mathbf{a}\cdot\mathbf{X})^{n+1}} \quad .$$

Let T be a k-dimensional simplex in $\mathbb{R}^n$, with vertices $\mathbf{v}_0,\dots,\mathbf{v}_k$. For $J=\{j_1,\dots,j_k\}$, with $1\le j_i<\cdots<j_k\le n$, let $\pi_J:\mathbb{R}^n\to\mathbb{R}^k$ be the projection to the span of $\mathbf{e}_{j_1},\dots,\mathbf{e}_{j_k}$. We denote by T^J the region $\{\mathbf{a}+\sum_{i\notin J}\lambda_i\mathbf{e}_i\,|\,\mathbf{a}\in T,\lambda_i\ge 0\}$.



Proposition 3.1. *With notation as above,*

$$\int_{T^J}\frac{n!\,X_1\cdots X_n\,da_1\cdots da_n}{(1+\mathbf{a}\cdot\mathbf{X})^{n+1}}=\frac{k!\,\mathrm{Vol}(\pi_J(T))\,X_{j_1}\cdots X_{j_k}}{(1+\mathbf{v}_0\cdot\mathbf{X})\cdots(1+\mathbf{v}_k\cdot\mathbf{X})} \quad .$$

Proof. For simplicity of notation, assume $J = \{1, \dots, k\}$. If $\dim \pi_J(T) < k = \dim T$, then the integral is 0. Thus, we may assume that $\pi_J(T)$ is a k-simplex, with vertices $\mathbf{v}'_i = \pi_J(\mathbf{v}_i)$. Parametrize $\pi_J(T)$ by the simplex Σ defined by

$$\{(t_1, \dots, t_k) \,|\, t_i \geq 0, t_1 + \dots + t_k \leq 1\}$$

in $\mathbb{R}^k_{\geq 0}$, via

$$(t_1, \dots, t_k) \mapsto \mathbf{v}'(\mathbf{t}) := \mathbf{v}'_0 + t_1(\mathbf{v}'_1 - \mathbf{v}'_0) + \dots + t_k(\mathbf{v}'_k - \mathbf{v}'_0) \quad .$$

The Jacobian of this parametrization is $\mathrm{Vol}(\pi_J(T))$. There are linear functions $\ell_j(\mathbf{t})$ such that

$$(t_1, \dots, t_k) \mapsto \mathbf{v}(\mathbf{t}) := \mathbf{v}'(\mathbf{t}) + \sum_{j=k+1}^{n} \ell_j(\mathbf{t})\, \mathbf{e}_j$$

parametrizes T. With this notation,

$$\mathbf{v}(0, \dots, 0) = \mathbf{v}_0, \quad \mathbf{v}(1, 0, \dots, 0) = \mathbf{v}_1, \quad \cdots, \quad \mathbf{v}(0, \dots, 0, 1) = \mathbf{v}_k,$$

and the integral to be evaluated is

$$\int_\Sigma \int_{\ell_{k+1}(\mathbf{t})}^{\infty} \cdots \int_{\ell_n(\mathbf{t})}^{\infty} \frac{\mathrm{Vol}(\pi_J(T))\, X_1 \cdots X_n\, n!\, da_n \cdots da_{k+1}\, dt_k \cdots dt_1}{(1 + \mathbf{v}'(\mathbf{t}) \cdot (X_1, \dots, X_k) + a_{k+1}X_{k+1} + \dots + a_n X_n)^{n+1}} \quad .$$

Performing the unbound integrations shows that this equals

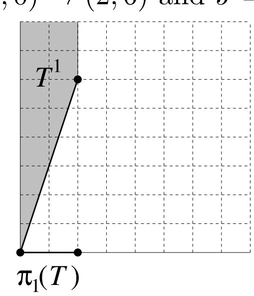
$$\int_\Sigma \frac{\mathrm{Vol}(\pi_J(T))\, X_1 \cdots X_k\, k!\, dt_k \cdots dt_1}{(1 + \mathbf{v}'(\mathbf{t}) \cdot (X_1, \dots, X_k) + \ell_{k+1}(\mathbf{t})X_{k+1} + \dots + \ell_n(\mathbf{t})X_n)^{n+1}}$$
$$= \mathrm{Vol}(\pi_J(T))\, X_1 \cdots X_k \int_\Sigma \frac{k!\, dt_k \cdots dt_1}{(1 + \mathbf{v}(\mathbf{t}) \cdot \mathbf{X})^{k+1}}$$

Thus, the statement of Proposition 3.1 is reduced to the following fact:

$$\int_\Sigma \frac{k!\, dt_k \cdots dt_1}{(M + L_1 t_1 + \dots + L_k t_k)^{k+1}} = \frac{1}{M(M + L_1) \cdots (M + L_k)} \quad ,$$

with $M, L_1, \dots, L_k$ independent of $\mathbf{t}$. This equality is immediately verified by induction. □

Example 3.2. The integral over the unbounded region in Example 1.4 is evaluated by Proposition 3.1 with $T =$ the segment $(0, 0) \to (2, 6)$ and $J = \{1\}$:

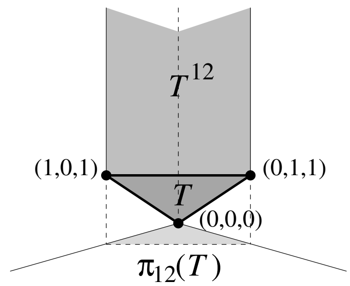


$$\int_{T^1} \frac{2!\, X_1 X_2\, da_1 da_2}{(1 + \mathbf{a} \cdot \mathbf{X})^3} = \frac{1!\, 2\, X_1}{(1 + 0X_1 + 0X_2)(1 + 2X_1 + 6X_2)} \quad .$$

With $X_1 = X_2 = H$, this gives the term $\dfrac{2H}{1 + 8H}$ used in Example 1.4.

The integral over the triangle T with vertices $(0, 0)$, $(0, m)$, $(m, 0)$ is evaluated by taking $J = \{1, 2\}$, giving the expression used in the proof of Claim 2.2 (i).

For a more interesting example, consider the 'infinite column' T^{12} determined by the triangle in $\mathbb{R}^3$ with vertices $(0, 0, 0)$, $(1, 0, 1)$, $(0, 1, 1)$ by taking $J = \{1, 2\}$:



The area of $\pi_{12}(T)$ is $\frac{1}{2}$, so Proposition 3.1 evaluates the integral over T^{12} as

$$\frac{X_1 \cdot X_2}{(1+X_1+X_3)(1+X_2+X_3)} \quad .$$

This will be used below in Example 3.3. ⌟

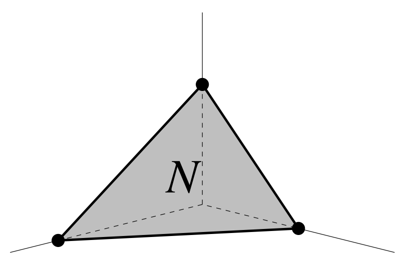
3.2. **Complete intersections.** As we are assuming that $X_1, \dots, X_n$ are nonsingular hypersurfaces meeting with normal crossings, the monomial subscheme corresponding to exponents

$$(m_1, 0, \dots, 0), \quad (0, m_2, 0, \dots, 0), \quad \cdots, \quad (0, \dots, 0, m_n)$$

is a complete intersection, with normal bundle $\mathscr{O}(m_1X_1) \oplus \cdots \oplus \mathscr{O}(m_nX_n)$. Therefore

$$\iota_* s(S, V) = \frac{[S]}{\prod_i c(\mathscr{O}(m_iX_i)} = \prod_i \frac{m_iX_i}{(1+m_iX_i)} \quad .$$

The corresponding Newton region is a simplex with vertices on the coordinate axes:



and according to Proposition 3.1

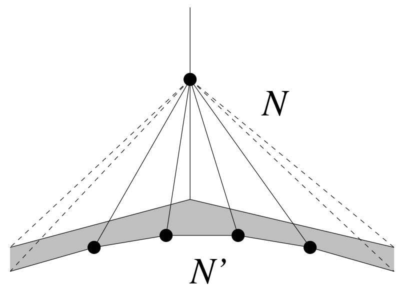
$$\begin{aligned}\int_N \frac{n! X_1 \cdots X_n \, da_1 \cdots da_n}{(1+a_1X_1+\cdots+a_nX_n)^{n+1}} &= \frac{n! \operatorname{Vol}(N)\, X_1 \cdots X_n}{(1+m_1X_1)\cdots(1+m_nX_n)} \\ &= \frac{m_1 \cdots m_n \, X_1 \cdots X_n}{(1+m_1X_1)\cdots(1+m_nX_n)} \quad .\end{aligned}$$

This verifies Conjecture 1 for these complete intersection.

A somewhat harder calculus exercise verifies Conjecture 1 for arbitrary complete intersections of monomials. As an example of what is involved in this verification, consider a monomial subscheme $S' \overset{\iota'}{\hookrightarrow} V$ supported on $X_1, \dots, X_{n-1}$, and let S be the intersection of S' with the m-multiple mX_n. Standard facts about Segre classes imply that

$$\iota'_* s(S', V) = \frac{mX_n}{1+mX_n} \cap \iota_* s(S, V) \quad . \tag{7}$$

To see that Conjecture 1 is compatible with this formula, observe that the Newton region N for S is a cone over the Newton region N' for S' with vertex at $(0, \cdots, 0, m)$.



The equality (7) amounts to

$$\int_N \frac{n!\, X_1\cdots X_n\, da_1\cdots da_n}{(1+a_1X_1+\cdots+a_nX_n)^{n+1}} = \frac{mX_n}{1+mX_n}\cap\int_{N'}\frac{(n-1)!\, X_1\cdots X_{n-1}\, da_1\cdots da_{n-1}}{(1+a_1X_1+\cdots+a_{n-1}X_{n-1})^n}$$

and in the cone situation this follows from

$$\int_0^1 \frac{n!\, t^{n-1}dt}{(A+Bt)^{n+1}} = \frac{(n-1)!}{A(A+B)^n}$$

with suitable positions for A and B, as the reader may check. A similar (but harder) computation verifies the corresponding formula whenever S' is a monomial scheme in $X_1,\dots,X_k$ and the lone vertex is a single monomial in $X_{k+1},\dots,X_n$. Repeated application of this more general formula implies that (1) holds for any complete intersection of monomials.

3.3. **Singularity subschemes.** As a less straightforward family of examples verifying Conjecture 1, we consider the monomial subschemes on $X_1,\dots,X_n$ with exponents

$$\mathbf{f}_1 := (0,1,\dots,1),\quad \mathbf{f}_2 := (1,0,1,\dots,1),\quad \cdots,\quad \mathbf{f}_n := (1,\dots,1,0)\quad . \tag{8}$$

These subschemes are very far from being complete intersections (for $n>1$), and computing their Segre class requires some nontrivial work, which depends on features of these schemes which do not hold for arbitrary monomial schemes. *Ad-hoc* alternative methods are occasionally available, as in the following example.

Example 3.3. For $n=3$ in $\mathbb{P}^3$, with coordinates x,y,z,w, the exponents $\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3$ define the monomial subscheme S with ideal

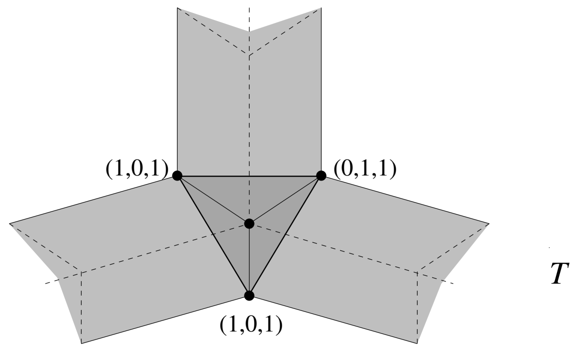
$$(xy, xz, yz)\quad .$$

This scheme is reduced and consists of three coordinate axes, so its Segre class must be $\iota_*s(S,\mathbb{P}^3)=3[\mathbb{P}^1]+a[\mathbb{P}^0]$ for some integer a. On the other hand, S is the scheme-theoretic intersection of three quadrics Q_1, Q_2, Q_3 (each consisting of two coordinate planes). By Fulton-MacPherson intersection theory and Bézout's theorem, and denoting by H the hyperplane class,

$$8=\int Q_1\cdot Q_2\cdot Q_3=\int c(\mathscr{O}(2))\cap s(S,\mathbb{P}^3)=\int(1+2H)^3\cap(3[\mathbb{P}^1]+a[\mathbb{P}^0])=18+a\quad .$$

It follows that $a=-10$, so $\iota_*s(S,\mathbb{P}^3)=3[\mathbb{P}^1]-10[\mathbb{P}^0]$.

The Newton region for S may be decomposed into 3 infinite columns and one tetrahedron with vertices $(0,0,0),(1,1,0),(1,0,1),(0,1,1)$.



The contribution of a column to the integral in (1) was computed in Example 3.2, and equals (setting $X_1 = X_2 = H$, the hyperplane class)

$$\frac{H^2}{(1+2H)^2} \quad .$$

The tetrahedron has volume $\frac{1}{3}$, hence it contributes (using Proposition 3.1, and again setting $X_1 = X_2 = X_3 = H$)

$$\frac{3!\,\frac{1}{3}\,X_1X_2X_3}{(1+X_1+X_2)(1+X_1+X_3)(1+X_2+X_3)} = \frac{2H^3}{(1+2H)^3} \quad .$$

Therefore, according to Conjecture 1, the Segre class equals

$$\int_N \frac{n!\,X_1X_2X_3\,da_1da_2da_3}{(1+a_1X_1+a_2X_2+a_3X_3)^4} = \frac{2H^3}{(1+2H)^3} + 3\frac{H^2}{(1+2H)^2} = \frac{3H^2+8H^3}{(1+2H)^3} = 3H^2 - 10H^3 + \cdots$$

in agreement with the direct computation. ⌟

For any n, the subscheme S defined by the exponents (8) is the *singularity subscheme* of the union X of the hypersurfaces $X_1, \dots, X_n$, i.e., the subscheme of X locally defined by the partials of an equation for X in V. (For example, (xy, xz, yz) is the ideal generated by the partials of xyz.) This is what gives us independent access to the Segre classes for these subschemes, and allows us to verify Conjecture 1 in these cases.

Proposition 3.4. *For all $n > 0$, Conjecture 1 holds for the monomial subschemes defined by the exponents $\mathbf{f}_1, \dots, \mathbf{f}_n$ listed in* (8)*.*

Proof. Let $F = \{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be the set of exponents. The Newton region N may be described as follows. For any $J \subseteq \{1, \dots, n\}$, let Σ_J be the simplex with vertices at the origin and at the points $\mathbf{f}_j$ with $j \in J$, and consider the subsets Σ_J^J, with notation as in §3.1. The reader can verify that the Newton region N is then the union of the sets Σ_J^J with $|J| \geq 2$. (For example, the region N for $n = 3$ decomposes as the union of the three columns Σ_{12}^{12}, Σ_{13}^{13}, Σ_{23}^{23}, and the 3-simplex Σ_{123}^{123}. Cf. Example 3.3.)

The volume of the projection $\pi_J(\Sigma_J)$ is easily found to be $(|J|-1)/|J|!$. By Proposition 3.1,

$$\int_N \frac{n!\,X_1\cdots X_n\,da_1\cdots da_n}{(1+a_1X_1+\cdots+a_nX_n)^{n+1}} = \sum_{|J|\geq 2} \frac{(|J|-1)\prod_{j\in J} X_j}{\prod_{j\in J}(1+\mathbf{f}_j\cdot\mathbf{X})} \quad .$$

We have to verify that this equals the class $\iota_* s(S, V)$. This Segre class is computed in [Alu99], §2.2 (p. 4002):

$$\iota_* s(S,V) = \left(\left(1 - \frac{c(\mathscr{L}^\vee)}{c(\mathscr{L}_1^\vee)\cdots c(\mathscr{L}_n^\vee)}\right) \cap [V]\right) \otimes \mathscr{L} = \left(1 - \frac{c(\mathscr{L})^{n-1}}{c(\mathscr{L}\otimes\mathscr{L}_1^\vee)\cdots c(\mathscr{L}\otimes\mathscr{L}_n^\vee)}\right) \cap [V]$$

where $\mathscr{L}_i = \mathscr{O}(X_i)$, $\mathscr{L} = \mathscr{O}(X_1 + \cdots + X_n)$. (This is an instance of the relation between the Chern-Schwartz-MacPherson class of a hypersurface and the Segre class of its singularity subscheme, in the particular case of divisors with normal crossings.) Thus, we have to verify that

$$\sum_{|J|\ge 2} \frac{(|J|-1)\prod_{j\in J} X_j}{\prod_{j\in J}(1+\mathbf{f}_j\cdot\mathbf{X})} = 1 - \frac{(1+X_1+\cdots+X_n)^{n-1}}{(1+\mathbf{f}_1\cdot\mathbf{X})\cdots(1+\mathbf{f}_n\cdot\mathbf{X})} \quad . \tag{9}$$

This is in fact an identity of rational functions in indeterminates $X_1,\dots,X_n$. To prove it, interpret the left-hand side as the value at $t=1$ of

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{t}\sum_{|J|\ge 2}\prod_{j\in J}\frac{X_j\,t}{1+\mathbf{f}_j\cdot\mathbf{X}}\right) &= \frac{d}{dt}\left(\frac{1}{t}\left(\prod_{j=1}^{n}\left(1+\frac{X_j\,t}{1+\mathbf{f}_j\cdot\mathbf{X}}\right)-1\right)\right)\\ &= \frac{1}{\prod_j(1+\mathbf{f}_j\cdot\mathbf{X})}\cdot\frac{d}{dt}\left(\frac{1}{t}\prod_j(1+\mathbf{f}_j\cdot\mathbf{X}+X_j\,t)-\frac{1}{t}\prod_j(1+\mathbf{f}_j\cdot X)\right)\end{aligned}$$

As

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{t}\prod_j(1+\mathbf{f}_j\cdot\mathbf{X}+X_j\,t)-\frac{1}{t}\prod_j(1+\mathbf{f}_j\cdot X)\right)\\ &= -\frac{1}{t^2}\prod_j(1+\mathbf{f}_j\cdot\mathbf{X}+X_j\,t)+\frac{1}{t}\prod_j(1+\mathbf{f}_j\cdot\mathbf{X}+X_j\,t)\sum\frac{X_j}{1+\mathbf{f}_j\cdot\mathbf{X}+X_j\,t}+\frac{1}{t^2}\prod_j(1+\mathbf{f}_j\cdot\mathbf{X})\end{aligned}$$

evaluating at $t=1$ gives

$$\begin{aligned}&-(1+X_1+\cdots+X_n)^n+(1+X_1+\cdots+X_n)^{n-1}(X_1+\cdots+X_n)+\prod_j(1+\mathbf{f}_j\cdot\mathbf{X})\\ &\qquad = -(1+X_1+\cdots+X_n)^{n-1}+\prod_j(1+\mathbf{f}_j\cdot\mathbf{X}) \quad .\end{aligned}$$

This shows that the left-hand side of (9) equals

$$\frac{1}{\prod_j(1+\mathbf{f}_j\cdot\mathbf{X})}\cdot\left(\prod_j(1+\mathbf{f}_j\cdot\mathbf{X})-(1+X_1+\cdots+X_n)^{n-1}\right) \quad ,$$

that is, the right-hand side, and concludes the proof of Proposition 3.4. □

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