HOW MANY HYPERSURFACES DOES IT TAKE TO CUT OUT A SEGRE CLASS?

PAOLO ALUFFI

Abstract. We prove an identity of Segre classes for zero-schemes of compatible sections of two vector bundles. Applications include bounds on the number of equations needed to cut out a scheme with the same Segre class as a given subscheme of (for example) a projective variety, and a 'Segre-Bertini' theorem controlling the behavior of Segre classes of singularity subschemes of hypersurfaces under general hyperplane sections.

These results interpolate between an observation of Samuel concerning multiplicities along components of a subscheme and facts concerning the integral closure of corresponding ideals. The Segre-Bertini theorem has applications to characteristic classes of singular varieties. The main results are motivated by the problem of computing Segre classes explicitly and applications of Segre classes to enumerative geometry.

1. Introduction

A result from P. Samuel's thesis states that, under mild hypotheses, in computing the multiplicity of a variety Y along a subscheme Z at an irreducible component V of Z we may replace the ideal determined by Z in the local ring $\mathscr{O}_{V,Y}$ by an ideal generated by $\operatorname{codim}_V Y$ elements (cf. [ZS60, Theorem 22]). In Fulton-MacPherson intersection theory, the same multiplicity may be defined by means of *Segre classes* ([Ful84, §4.3]); it is then natural to ask whether the number of equations needed to define a Segre class may be similarly bounded. This is one of the questions we answer in this note. We work over an algebraically closed field, and our schemes are embeddable in nonsingular varieties. We denote by $Z_{\rm red}$ the reduced scheme supported on Z, and by $s(Z,Y)$ the Segre class of Z in Y.

Theorem 1.1. *Let Y be a pure-dimensional scheme, and let $Z \subseteq Y$ be a closed subscheme. Let X_i, $i = 1, \dots$ be general elements of a linear system cutting out Z.*

(a) *Let $Z' := X_1 \cap \cdots \cap X_{\dim Y+1}$. Then ${Z'}_{red} = Z_{red}$, and $s(Z',Y) = s(Z,Y)$.*

(b) *Let $Z'' := X_1 \cap \cdots \cap X_{\dim Y}$. Then there exists an open neighborhood Y° of Z in Y such that $(Z'' \cap Y^\circ)_{red} = Z_{red}$, and $s(Z'' \cap Y^\circ, Y) = s(Z,Y)$.*

(The equality of supports allows us to identify the relevant Chow groups, as required in order to compare the Segre classes, cf. Remark 2.2.)

Thus, the Segre class of Z in Y can be 'cut out' by $\dim Y+1$ hypersurfaces, and by $\dim Y$ hypersurfaces in a neighborhood of Z. This fact is reminiscent of a well-known result of D. Eisenbud and G. Evans ([EE73]), stating that every subscheme Z of $\mathbb{P}^n$ may be cut out set-theoretically by n hypersurfaces, and of an observation by W. Fulton ([Ful84, Example 9.1.3]) pointing out that $n+1$ hypersurfaces suffice to cut out Z *scheme-theoretically* if Z is locally a complete intersection. As a particular case of Theorem 1.1, $n+1$ hypersurfaces suffice to cut out a subscheme $Z' \subseteq \mathbb{P}^n$ with the same Segre class in $\mathbb{P}^n$ as Z, without any requirement on Z. These hypersurfaces may be chosen to be general in a linear system cutting out Z, and n hypersurfaces suffice in a neighborhood of Z.

Theorem 1.1 may be further refined, as follows. Denote by $s(Z,Y)_k$ the k-dimensional component of the Segre class $s(Z,Y)$.

Theorem 1.1. (continued)

(b') *More generally, let $c \geq 0$ and let $Z_{(c)} := X_1 \cap \cdots \cap X_{\dim Y - c}$. Then there exists a closed subscheme S of dimension $\leq c$ in Y such that $\dim(S \cap Z) < c$, $(Z_{(c)} \smallsetminus S)_{red} = (Z \smallsetminus S)_{red}$, $s(Z_{(c)} \smallsetminus S, Y \smallsetminus S)_c = s(Z \smallsetminus S, Y \smallsetminus S)_c$, and $s(Z_{(c)}, Y)_k = s(Z, Y)_k$ for $k > c$.*

For $c = 0$, part (b') of Theorem 1.1 reduces to part (b): in this case S is a set of points disjoint from Z, and we can take $Y^\circ = Y \smallsetminus S$. Part ($a$) may also formally be seen as a particular case of (b'), by allowing $c = -1$ (and hence $S = \emptyset$).

If V is an irreducible component of Z, it follows from Theorem 1.1(b') with $c = \dim V$ that the coefficient of V in $s(Z, Y)$ equals the coefficient of V in $s(Z_{(\dim V)}, Y)$. As this coefficient equals the multiplicity of Y along Z at V ([Ful84, Example 4.3.4]), this recovers Samuel's result.

Theorem 1.1 is motivated by effective computations of Segre classes and of contributions of components to an intersection.

Example 1.2. Consider the scheme $Z \subseteq \mathbb{P}^3$ defined by the ideal

$$(z^2, yz, xz, y^2 w - x^2(x + w)) \quad .$$

(This is the flat limit of a family of twisted cubics, cf. [Har77, Example 9.8.4].) Note that Z may be cut out by cubics. Standard techniques give $s(Z, \mathbb{P}^3) = [Z] - 10[pt]$. If X_1, X_2, X_3 are general cubic hypersurfaces containing Z, then by [Ful84, Proposition 9.1.1] the contribution of Z to the intersection X_1, X_2, X_3 is

$$\int c(\mathscr{O}(3H))^3 \cap s(Z^-, \mathbb{P}^3) \tag{1}$$

where H denotes the hyperplane class and Z^- is the component of $X_1 \cap X_2 \cap X_3$ supported on Z. (So $Z^- = Z'' \cap Y^\circ$ with the notation of Theorem 1.1(b).) A Macaulay2 ([GS]) computation shows that the schemes Z and Z^- have the same support but are not equal. In fact, the scheme Z^- depends on the choice of X_1, X_2, X_3, so it seems *a priori* difficult to perform the computation of (1), barring an exhaustive analysis of the specific choice of these hypersurfaces. However, by Theorem 1.1(b) we must have

$$s(Z^-, \mathbb{P}^3) = s(Z, \mathbb{P}^3) = [Z] - 10[pt] \quad ,$$

and it follows that the contribution computed by (1) is $\int (1 + 9H) \cap ([Z] - 10[pt]) = 17$. Taking this off the Bézout number 27 for the intersection of three cubics, it follows that X_1, X_2, X_3 meet at 10 points outside of Z (if the ground field is algebraically closed of characteristic 0). ⌟

The point of this example is that even though the relevant component Z^- of the intersection of the hypersurfaces is *not* equal to Z, we may carry out the computation of the corresponding contribution to the intersection as if it were. This is a common issue in applications of Segre classes to enumerative geometry, where Z may have a compelling scheme-theoretic description, but the scheme Z^- corresponding to a choice of hypersurfaces realizing general constraints may retain features due to the specific chosen hypersurfaces. Example 1.2 illustrates the fact that the hypothesis that the expected number of divisors in the linear system cut out the base scheme in a neighborhood (cf. [Ful84, Example 4.4.1]) may be weakened: in general this hypothesis should not be expected to be verified, but the corresponding formulas remain true if the divisors are general. We should also point out that, as a rule, judicious use of [Ful84, Proposition 4.4] suffices to address this issue; in fact, Theorem 1.1(b) is little more than a recasting of this result, presented here in an attempt to streamline its utilization.

The same complication arises in some approaches to the algorithmic computation of Segre classes. For example, the computation of the Segre class of a subscheme in $\mathbb{P}^n$ is reduced in [EJP13] to residual intersection computations, and these are controlled by the Segre class of an *a priori* different subscheme; again, Example 1.2 provides an example. Theorem 3.2 in [EJP13] implicitly includes a proof of Theorem 1.1(b) in the particular case $Y = \mathbb{P}^n$, resolving this issue in this case for the numerical degrees of the classes (i.e., their push-forward to projective space). Theorem 2 in [MQ13] does the same in the toric setting. Theorem 1.1 has no restrictions on the ambient scheme Y, and gives the result in the Chow group of Z rather than pushing-forward to Y. Also, Theorem 1.1(b') can in principle lead to an improvement in the speed of such algorithms when only terms of a fixed dimension in the Segre class are needed.

We prove Theorem 1.1 in §3 as an application of a more general observation presented in §2, to the effect that the Segre class of the zero-scheme of a section of a vector bundle $\mathscr{E}$ is preserved in a range of dimensions by taking suitable quotients of $\mathscr{E}$. See §2 for a precise statement. For the '$c = 0$ case', corresponding to part (b) of Theorem 1.1, the commutative algebra counterpart of this observation is the statement that such quotients do not change the integral closure of the ideal sheaf determined by the section. The argument given in §2 to prove the Segre class identity may be used to draw this conclusion (Remark 2.5); but our proof of the Segre class identity bypasses the commutative algebra, and hence seems more direct in the context of this paper.

In §4 we give a second application of the same tool, proving a 'Bertini' type statement for Segre classes; here we require the characteristic of the field to be 0. Let $Y \subseteq \mathbb{P}^n$ now be a nonsingular variety, and let X be a (possibly singular) hypersurface in Y. Let H be a general hyperplane. By the Bertini theorem, the singular locus of $H \cap X$ is set-theoretically equal to the intersection of H with the singular locus of X:

$$(\mathrm{Sing}(H \cap X))_{\mathrm{red}} = (H \cap \mathrm{Sing}(X))_{\mathrm{red}}$$

where $\mathrm{Sing}(-)$ denotes the singularity subscheme. While this equality is not true at the level of schemes, we prove that it does lift to an equality of Segre classes.

Theorem 1.3. *Let H be a general hyperplane, and let W be a pure-dimensional subscheme of $H \cap Y$. Then $s(\mathrm{Sing}(H \cap X) \cap W, W) = s(H \cap \mathrm{Sing}(X) \cap W, W)$.*

Again this statement could be proven from commutative algebra considerations. The argument given here follows easily from the tool presented in §2, and seems more direct. Theorem 1.3 may be used to prove that certain characteristic classes associated with X behave as expected with respect to general hyperplane sections. There are other approaches to such questions; see for example [PP95, Lemma 1.2].

Acknowledgments. The author's research was supported in part by the Simons foundation and by NSA grants H98230-15-1-0027 and H98230-16-1-0016. The author is grateful to Caltech for hospitality while this work was carried out. The author especially thanks Prof. Matilde Marcolli for suggesting the title of this paper. The author also thanks Sam Huckaba for insightful conversations on the commutative algebraic aspects of the questions studied in this paper.

2. The main tool

In this note Y denotes a separated pure-dimensional scheme of finite type over an algebraically closed field, embeddable in a nonsingular scheme. We are interested in the Segre classes $s(Z, Y)$ of subschemes Z of Y. By [Ful84, Lemma 4.2] the Segre class of Z in Y is a linear combination of the Segre classes in the irreducible components of Y; so we may and will assume that Y is a variety. Also, $s(Y, Y) = [Y]$; so we will assume the subschemes we consider are properly contained in Y.

Every subscheme Z of Y may be realized as the zero-scheme of a section of a vector bundle $\mathscr{E}$ on Y, and this yields an embedding of the blow-up $B\ell_Z Y$ of Y along Z as a subscheme of $\mathbb{P}(\mathscr{E})$ ([Ful84, B.8.2]). We consider the following situation:

- $\mathscr{E}$, $\mathscr{F}$ are vector bundles on Y;
- $s_{\mathscr{E}}$, $s_{\mathscr{F}}$ are sections of $\mathscr{E}$, $\mathscr{F}$, with zero-schemes Z, Z', respectively;
- We have an epimorphism $p : \mathscr{E} \to \mathscr{F}$ with kernel $\mathscr{K} = \ker p$, such that the diagram

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\;p\;} & \mathscr{F} \\ {\scriptstyle s_{\mathscr{E}}}\nwarrow & & \nearrow{\scriptstyle s_{\mathscr{F}}} \\ & Y & \end{array}$$

  is commutative;

Clearly $Z \subseteq Z'$, in the sense that the ideal sheaf $\mathscr{I}_{Z,Y}$ of Z in Y contains $\mathscr{I}_{Z',Y}$.

As recalled above, the blow-up $B\ell_Z Y$ may be embedded in the projectivization $\mathbb{P}(\mathscr{E})$. The image of this embedding is the variety $\widetilde{Y}$ obtained as the closure of the image in $\mathbb{P}(\mathscr{E})$ of the rational section $Y \dashrightarrow \mathbb{P}(\mathscr{E})$ induced by $s_{\mathscr{E}}$. We will denote by $E \subseteq \widetilde{Y}$ the exceptional divisor. We adopt the convention that the empty set has negative dimension.

Theorem 2.1. *Let $c = \dim(\mathbb{P}(\mathscr{K}) \cap \widetilde{Y})$ and assume that no component of $\mathbb{P}(\mathscr{K}) \cap \widetilde{Y}$ is contained in E. Then there exists a subscheme $S \subseteq Y$ of dimension $\leq c$ such that* $\dim(S \cap Z) < c$ *and*

(i) $(Z' \smallsetminus S)_{red} = (Z \smallsetminus S)_{red}$;
(ii) $s(Z' \smallsetminus S, Y \smallsetminus S)_c = s(Z \smallsetminus S, Y \smallsetminus S)_c$;
(iii) $s(Z', Y)_k = s(Z, Y)_k$ *for* $k > c$.

Remark 2.2. The equality of supports in (*i*) allows us to identify the Chow groups as needed for (*ii*) and (*iii*). Indeed, if (*i*) holds, then $A_c(Z' \smallsetminus S) = A_c(Z \smallsetminus S)$ ([Ful84, Example 1.3.1]); and recall ([Ful84, Proposition 1.8]) that for all $k \geq 0$ there is an exact sequence

$$A_k(Z \cap S) \longrightarrow A_k(Z) \longrightarrow A_k(Z \smallsetminus S) \longrightarrow 0 \quad ;$$

if $\dim(S) \leq c$ and (*i*) holds, then it follows that for $k > c$ we have canonical isomorphisms

$$A_k(Z) \cong A_k(Z \smallsetminus S) \cong A_k((Z \smallsetminus S)_{\text{red}}) \cong A_k((Z' \smallsetminus S)_{\text{red}}) \cong A_k(Z' \smallsetminus S) \cong A_k(Z') \quad .$$

The claimed equality $s(Z, Y)_k = s(Z', Y)_k$ holds in this group. ⌟

Example 2.3. It is important to note that $Z \neq Z'$ in general, even if $\mathbb{P}(\mathscr{K}) \cap \widetilde{Y} = \emptyset$. For example, let $Y = \mathbb{A}^2$ with coordinates x, y, and let $\mathscr{E} = \mathscr{O}^{\oplus 3}$, projecting to the first two factors $\mathscr{F} = \mathscr{O}^{\oplus 2}$. We have $\mathscr{K} \cong \mathscr{O}$, identified with the third factor of $\mathscr{E}$. Define $s_{\mathscr{E}}$ by

$$(x, y) \mapsto (x^2, y^2, xy) \quad .$$

It is straightforward to verify that $\widetilde{Y}$ is given by the ideal

$$(sy - ux, tx - uy, st - u^2)$$

in $\mathbb{P}(\mathscr{E}) \cong \mathbb{A}^2 \times \mathbb{P}^2$, where $(s : t : u)$ are homogeneous components in the $\mathbb{P}^2$ factor. The projectivization $\mathbb{P}(\mathscr{K})$ consists of $\mathbb{A}^2 \times \{(0 : 0 : 1)\}$, hence it has empty intersection with $\widetilde{Y}$. On the other hand, the ideals of Z, Z' are (x^2, y^2, xy), (x^2, y^2), respectively; so $Z \neq Z'$. ⌟

Proof of Theorem 2.1. Identify $\widetilde{Y}$ with $B\ell_Z Y$; the blow-up map is the projection $\pi : \widetilde{Y} \to Y$, and $E = \pi^{-1}(Z)$ is the exceptional divisor. Since $Z \subseteq Z'$, we have $E \subseteq \pi^{-1}(Z')$. We will verify that the residual scheme to E in $\pi^{-1}(Z')$ is supported on $S' := \mathbb{P}(\mathscr{K}) \cap \widetilde{Y}$. We claim that the statement of the theorem follows from this, by setting $S = \pi(S')$. Indeed, since no component of S' is contained in E, we have $\dim(Z \cap S) < c$. Since (set-theoretically) $Z' = \pi(\pi^{-1}(Z')) = \pi(E \cup S') = Z \cup S$, the equality (i) of supports holds. Further, we will have

$$s(Z', Y)_k \overset{(1)}{=} \pi_* s(\pi^{-1}(Z'), \widetilde{Y})_k \overset{(2)}{=} \pi_* s(E, \widetilde{Y})_k \overset{(3)}{=} s(Z, Y)_k$$

for $k > c$, where equalities (1) and (3) hold by the birational invariance of Segre classes ([Ful84, Proposition 4.2(a)]), and (2) follows from the residual formula for Segre classes ([Ful84, Proposition 9.2]). This implies (iii). The residual formula also shows that $s(E, \widetilde{Y})_c$ and $s(\pi^{-1}(Z), \widetilde{Y})_c$ differ by a class supported on S'. It follows that $s(E \smallsetminus S', \widetilde{Y} \smallsetminus S')_c = s(\pi^{-1}(Z) \smallsetminus S', \widetilde{Y} \smallsetminus S')_c$, and (ii) follows, again by the birational invariance of Segre classes.

Thus, in order to prove the theorem it suffices to show that the residual scheme to E in $\pi^{-1}(Z')$ equals $S' := \mathbb{P}(\mathscr{K}) \cap \widetilde{Y}$. To compute this residual scheme we may work locally, hence assume that $\mathscr{E}$, $\mathscr{K}$ and $\mathscr{F}$ are trivial and that $p : \mathscr{E} = \mathscr{O}^{\oplus N} \to \mathscr{F} = \mathscr{O}^{\oplus M}$ is the projection onto the first M factors. Write the section $s_{\mathscr{E}}$ in components as $s_{\mathscr{E}} = (s_1, \dots, s_N)$; the induced rational section $Y \dashrightarrow \mathbb{P}(\mathscr{E}) = Y \times \mathbb{P}^{N-1}$ is $(s_1 : \cdots : s_N)$, and this lifts to the embedding

$$\rho : \widetilde{Y} \to Y \times \mathbb{P}^{N-1} \quad .$$

The exceptional divisor E is given by a section e of $\mathscr{O}(E)$. Locally, E is defined by the ideal $(e) = (\pi^{-1}s_1, \dots, \pi^{-1}s_N)$. We can factor

$$\pi^{-1}(s_i) = \hat{s}_i e$$

for $i = 1, \dots, N$; then $\hat{s}_1, \dots, \hat{s}_N$ locally generate the unit ideal (1), and the embedding $\rho : \widetilde{Y} \to Y \times \mathbb{P}^{N-1}$ is given by $\tilde{y} \mapsto (\tilde{y}, (\hat{s}_1(\tilde{y}) : \cdots : \hat{s}_N(\tilde{y})))$.

With the above notation, the ideal for $S' = \mathbb{P}(\mathscr{K}) \cap \widetilde{Y}$ in $\widetilde{Y}$ is $\mathscr{I}_{S', \widetilde{Y}} = (\hat{s}_1, \dots, \hat{s}_M)$.

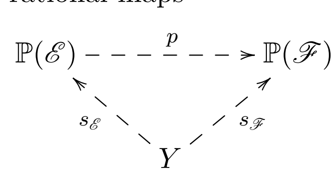
Now Z' is defined by the ideal $(s_1, \dots, s_M)$ in Y. Therefore $\pi^{-1}(Z')$ has ideal

$$(\hat{s}_1 e, \dots, \hat{s}_M e) = (\hat{s}_1, \dots, \hat{s}_M) \cdot (e) = \mathscr{I}_{S', \widetilde{Y}} \cdot \mathscr{I}_{E, \widetilde{Y}} \quad .$$

This verifies that the residual scheme to E in $\widetilde{Y}$ is $S' = \mathbb{P}(\mathscr{K}) \cap \widetilde{Y}$ and concludes the proof. □

Remark 2.4. Suppose $c = \dim V$, where V is an irreducible component of Z. By Theorem 2.1(i), V is also an irreducible component of Z', and by Theorem 2.1(ii), V appears with the same multiplicity in $s(Z, Y)$ and in $s(Z', Y)$. Thus the multiplicity of Y along Z and along Z' at V coincide. This will recover the result by Samuel recalled at the beginning of §1. ⌟

Remark 2.5. If $c = 0$ in Theorem 2.1, then $S' = \mathbb{P}(\mathscr{K}) \cap \widetilde{Y}$ consists of a finite set disjoint from E; replacing Y by the complement of $S = \pi(S')$, we may assume $\mathbb{P}(\mathscr{K}) \cap \widetilde{Y} = \emptyset$. The resulting situation has a compelling interpretation in terms of commutative algebra. We have a commutative diagram of rational maps



and the projection induces a birational morphism between the closures of the images of $s_{\mathscr{E}}$ and $s_{\mathscr{F}}$, i.e.,

$$p|_Y : \widetilde{Y} = B\ell_Z Y \dashrightarrow B\ell_{Z'} Y \quad .$$

If $\mathbb{P}(\mathscr{K}) \cap \widetilde{Y} = \emptyset$, this morphism is regular and finite, because $\widetilde{Y}$ is disjoint from the center of the projection. This implies that the ideal of Z' is a reduction of the ideal of Z at every point of Z, cf. [Vas05, Proposition 1.44]. Therefore, we can conclude that if $c = 0$ in Theorem 2.1, then the ideals of Z and Z' have the same integral closure at every point of Z. Theorem 2.1 may be seen as a Segre class version of this observation, extended to all $c \geq 0$. ⌟

3. Proof of Theorem 1.1

As above, Y is a pure-dimensional scheme and $Z \subseteq Y$ is a closed subscheme, and we can in fact assume that Y is a variety and $Z \subsetneq Y$ (cf. the beginning of §2).

Let $\mathscr{A}$ be a line bundle on Y, and let $L \subseteq \mathbb{P}H^0(\mathscr{A}, Y)$ be a linear system of which Z is the base scheme. For example, if $Y \subseteq \mathbb{P}^n$, then L can be the restriction of the linear system of degree-d hypersurfaces containing Z, provided d is large enough; in fact, the maximum degree in a set of generators of any ideal defining Z scheme-theoretically in $\mathbb{P}^n$ will do. If $X_1, \dots, X_N$ are general elements of L and $N \gg 0$, then $X_1, \dots, X_N$ generate L, and $Z = X_1 \cap \cdots \cap X_N$ scheme-theoretically. Equivalently, Z may be realized as the zero-scheme of the section $s_{\mathscr{E}}$ of $\mathscr{E} := \mathscr{A}^{\oplus N}$ given by $(s_1, \dots, s_N)$, where s_i is a section of $\mathscr{A}$ defining X_i. As in §2, we consider the closure $\widetilde{Y}$ of the image of the rational section $(s_1 : \cdots : s_N)$ of $\mathbb{P}(\mathscr{E})$ determined by $s_{\mathscr{E}}$. For $c \geq 0$, we let $M = \dim Y - c$, and we let $Z_{(c)} = X_1 \cap \cdots \cap X_{\dim Y - c}$ be the zero-scheme of the section $s_{\mathscr{F}}$ of $\mathscr{F} := \mathscr{A}^{\oplus M}$ given by $(s_1, \dots, s_M)$.

We are then in the situation of §2. Let $\mathscr{K}$ be the kernel of the projection $p : \mathscr{E} \to \mathscr{F}$. By Theorem 2.1, in order to prove Theorem 1.1 it suffices to show that $\mathbb{P}(\mathscr{K}) \cap \widetilde{Y}$ has pure dimension c (if nonempty) and $\mathbb{P}(\mathscr{K}) \cap E$ has dimension $< c$. We have

$$\mathbb{P}(\mathscr{E}) = \mathbb{P}(\mathscr{A}^{\oplus N}) \cong \mathbb{P}(\mathscr{O}_Y^{\oplus N}) = Y \times \mathbb{P}^{N-1} \quad ;$$

$\mathbb{P}(\mathscr{F}) \cong Y \times \mathbb{P}^{M-1}$ under the same identification, and $\mathbb{P}(\mathscr{K})$ is defined by the intersection of M hypersurfaces $Y \times H^{(1)}, \dots, Y \times H^{(M)}$, where $H^{(i)}$ are general hyperplanes. Now the linear system cut out on $\widetilde{Y}$ by the hypersurfaces $Y \times H$, with H a hyperplane, is base-point free. It follows that every component of $\mathbb{P}(\mathscr{K}) \cap \widetilde{Y}$ has codimension $M = \dim Y - c$ in $\widetilde{Y}$, hence dimension c. The dimension of every component of $\mathbb{P}(\mathscr{K}) \cap E$ is $c - 1$ by the same token, concluding the proof. □

If V is an irreducible component of Z, the $c = \dim V$ case of Theorem 1.1(b') recovers Samuel's result on multiplicities, cf. Remark 2.4.

As a consequence of the considerations in Remark 2.5, we see that for $c = 0$ this argument in fact proves that the local equations of $\dim Y$ general elements of L generate a reduction of the ideal $\mathscr{I}_{Z,Y}$ of Z in Y at every point $p \in Z$. For example, if $Y \subseteq \mathbb{P}^n$, $\dim Y$ general homogeneous polynomials of degree d containing Z (where $d \geq$ maximum degree of a polynomial in any set of generators of any ideal for Z in $\mathbb{P}^n$) generate a reduction of $\mathscr{I}_{Z,Y}$ at every point of Z. In the local case, it is known that a reduction may in fact be generated by ℓ generators, where ℓ equals the *analytic spread* of the ideal ([HS06, Proposition 8.3.7]). In our context, this equals 1 plus the dimension of the fiber of the exceptional divisor E at the given point. Theorem 1.1(b) amounts to the consequence for Segre classes of a global version of this result.

4. A Segre-Bertini theorem

We now move on to the proof of Theorem 1.3. In this section we assume that the characteristic of the ground field is 0.

We consider a nonsingular variety $Y \subseteq \mathbb{P}^n$ and let $X \subseteq Y$ be a hypersurface. We denote by $Z = \mathrm{Sing}(X)$ the *singularity subscheme* of X, i.e., the subscheme of Y locally defined by an equation for X and by its partial derivatives. The Segre classes of Z in X and in Y play an important role in the theory of characteristic classes for singular varieties: there are formulas relating directly the class $s(Z, X)$ with the *Chern-Mather* class of X and the class $s(Z, Y)$ with the *Chern-Schwartz-MacPherson* class of X (see e.g., [AB03, Proposition 2.2]).

By the ordinary Bertini theorem $\underline{Y} := H \cap Y$ is nonsingular for a general hyperplane H, and the singularity subscheme of $\underline{X} := H \cap X$ is supported on $\underline{\mathrm{Sing}(X)} := H \cap Z$. It is clear that $\underline{\mathrm{Sing}(X)} \subseteq \mathrm{Sing}(\underline{X})$, but these two subschemes may be different, even for general H. According to Theorem 1.3, this difference does not affect their Segre classes: we will prove that if W is any pure-dimensional subscheme of $\underline{Y}$, then

$$s(\mathrm{Sing}(\underline{X}) \cap W, W) = s(\underline{\mathrm{Sing}(X)} \cap W, W) \quad . \tag{2}$$

For example, this holds for $W = \underline{X}$ and $W = \underline{Y}$, the cases most relevant to characteristic classes as mentioned above. Also, since $\underline{Y}$ is nonsingular, the same equality follows for any nonsingular variety $W \subseteq \mathbb{P}^n$ containing $\mathrm{Sing}(\underline{X})$ (by [Ful84, Example 4.2.6(a)]).

The main value of identities such as (2) is that they yield tools for the effective computation of Segre classes. For example, for $W = \underline{Y}$, (2) implies that

$$s(\mathrm{Sing}(\underline{X}), \underline{Y}) = H \cdot s(\mathrm{Sing}(X), Y)$$

by essentially the same argument used in the proof of [Alu12, Claim 3.2], and this implies an 'adjunction formula' for Chern-Schwartz-MacPherson classes (cf. [Alu13, Proposition 2.6]).

Like Theorem 1.1, Theorem 1.3 (i.e., (2)) follows from Theorem 2.1: it suffices to realize the two schemes as zero-schemes of compatible sections of vector bundles under a projection from a suitable center. The main technical point of the proof is the existence of such a center.

Proof. As in the proof of Theorem 1.1 we may assume that W is a variety. Let $\mathscr{L} = \mathscr{O}(X)$; so X is defined by a section F of $\mathscr{L}$ on Y. This section lifts to a section s_F of the bundle of principal parts $\mathscr{P}^1_Y\mathscr{L}$, and the subscheme $\mathrm{Sing}(X)$ is the zero-scheme of this section. Therefore, $\underline{\mathrm{Sing}(X)}$ is the zero-scheme of the restriction of s_F to $\underline{Y}$, a section of $(\mathscr{P}^1_Y\mathscr{L})|_{\underline{Y}}$. By the same token, $\mathrm{Sing}(\underline{X})$ is the zero-scheme of the section $s_{\underline{F}}$ of $\mathscr{P}^1_{\underline{Y}}\mathscr{L}$ determined by the restriction $\underline{F}$ of F to $\underline{Y}$ (where for brevity we denote by $\mathscr{L}$ the restriction $\mathscr{L}|_{\underline{Y}}$). These sections are compatible with the natural surjective morphism of vector bundles $(\mathscr{P}^1_Y\mathscr{L})|_{\underline{Y}} \to \mathscr{P}^1_{\underline{Y}}\mathscr{L}$: the diagram

$$\begin{array}{ccc} (\mathscr{P}^1_Y\mathscr{L})|_{\underline{Y}} & \xrightarrow{\quad p \quad} & \mathscr{P}^1_{\underline{Y}}\mathscr{L} \\ & {}_{(s_F)|_{\underline{Y}}}\nwarrow \quad \nearrow_{s_{\underline{F}}} & \\ & \underline{Y} & \end{array}$$

is commutative. We are therefore in the situation studied in §2, and in order to complete the proof we only need to verify that, for a general H, the projectivization of the kernel of p is disjoint from the closure of the image of $s_F(W)$ in $\mathbb{P}((\mathscr{P}^1_Y\mathscr{L})|_W)$. *A fortiori,* it suffices to show that this is the case for $W = \underline{Y}$. This will verify the hypothesis of Theorem 2.1 with $c = -1$, hence prove the equality of Segre class in all dimensions.

Recall that, by [Gro67, 16.4.20], there is an exact sequence

$$0 \longrightarrow \mathscr{L} \otimes \mathscr{O}(-1)|_{\underline{Y}} \longrightarrow (\mathscr{P}^1_Y \mathscr{L})|_{\underline{Y}} \xrightarrow{p} \mathscr{P}^1_{\underline{Y}} \mathscr{L} \longrightarrow 0$$

extending the standard exact sequence of differentials from [Har77, Proposition II.8.12],

$$0 \longrightarrow \mathscr{O}(-1)|_{\underline{Y}} \longrightarrow \Omega_Y|_{\underline{Y}} \longrightarrow \Omega_{\underline{Y}} \longrightarrow 0$$

tensored by $\mathscr{L}$. (This sequence is exact on the left since Y and $\underline{Y}$ are nonsingular, [Har77, Proposition II.8.17].) Therefore, the kernel $\mathscr{K}$ of p is the image of $\mathscr{L} \otimes \mathscr{O}(-1)|_{\underline{Y}}$ in $(\mathscr{P}^1_Y \mathscr{L})|_{\underline{Y}}$, and $p : \mathbb{P}((\mathscr{P}^1_Y \mathscr{L})|_{\underline{Y}}) \dashrightarrow \mathbb{P}(\mathscr{P}^1_{\underline{Y}} \mathscr{L})$ is the projection with center at the section $\mathbb{P}(\mathscr{K})$. By Theorem 2.1, in order to prove Theorem 1.3 it suffices to prove that, for a general choice of H, $s_H = \mathbb{P}(\mathscr{L} \otimes \mathscr{O}(-1)|_{\underline{Y}})$ is disjoint from $B\ell_{\underline{\mathrm{Sing}(X)}} \underline{Y}$ in $\mathbb{P}((\mathscr{P}^1_Y \mathscr{L})|_{\underline{Y}})$.

To study this question, it is helpful to work over a trivializing open set. (A local trivialization for the bundle of principal parts is discussed in e.g., [Per95, §A.4].) Let $U \subseteq Y$ be a dense open set such that

$$\mathbb{P}(\mathscr{P}^1_U \mathscr{L}) \cong U \times \mathbb{P}^m \tag{3}$$

with $m = \operatorname{rk} \mathbb{P}(\mathscr{P}^1_U \mathscr{L}) = \dim Y$. We may choose U so that the projection from a fixed subspace $P := \mathbb{P}^{n-m-1}$ is an isomorphism on each embedded tangent space to Y at $y \in U$; we then have a natural identification of the fiber $\mathbb{P}^m$ in (3) with the subspace of the dual space $\mathbb{P}^{n\vee}$ consisting of hyperplanes containing P.

We will denote by (y, H) the point of $\mathscr{P}^1_U \mathscr{L}$ determined by the choice of a point $y \in U$ and a hyperplane $H \supseteq P$.

With this notation, $(y, H) \in \mathbb{P}(\Omega^1_U \otimes \mathscr{L}) \subseteq \mathbb{P}(\mathscr{P}^1_U \mathscr{L})$ if and only if $y \in H$. Further, each $H \supseteq P$ determines a section of $\mathbb{P}^1(\mathscr{P}^1_U \mathscr{L})$, given by $y \mapsto (y, H)$ for $y \in U$, and hence a section of $\mathbb{P}((\Omega^1_U \otimes \mathscr{L})|_{\underline{U}})$ for $\underline{U} = H \cap U \subseteq \underline{Y}$. It is straightforward to verify that this section agrees with the restriction to U of the section s_H determined by H as explained above.

We have to prove that, for a general hyperplane H, s_H is disjoint from $B\ell_{\underline{\mathrm{Sing}(X)}} \underline{Y}$ in $\mathbb{P}((\mathscr{P}^1_Y \mathscr{L})|_{\underline{Y}})$. It suffices to prove that there is *one* such hyperplane. Arguing by contradiction, assume that for all H there exists a point $y \in Y$ such that s_H and $B\ell_{\underline{\mathrm{Sing}(X)}} \underline{Y}$ meet over y. After a choice of U and P as above, we may represent points of $\mathbb{P}(\mathscr{P}^1_U \mathscr{L})$ by pairs (y, H) with $y \in U$ and $H \supseteq P$. By our assumption, we would have that for a general H containing P there exists $y \in \underline{U}$ such that

$$(y, H) \in B\ell_{\underline{\mathrm{Sing}(X)}} \underline{Y} \subseteq B\ell_{\mathrm{Sing}(X)} Y \quad . \tag{4}$$

The set of such $(y, H) \in B\ell_{\mathrm{Sing}(X)} Y$ is m-dimensional, since it dominates the fiber $\mathbb{P}^m$ in the trivialization (3). Since $\dim B\ell_{\mathrm{Sing}(X)} Y = m$, it follows that *every* $(y, H) \in B\ell_{\mathrm{Sing}(X)} Y$ is of this type. But $(y, H) \in \mathbb{P}(\Omega^1_U \otimes \mathscr{L})$ since $y \in \underline{U} \subseteq H$. Thus, it would follow that

$$B\ell_{\mathrm{Sing}(X)} Y \subseteq \mathbb{P}(\Omega^1_Y \otimes \mathscr{L}) \quad .$$

However, this is clearly not the case: the fiber of $B\ell_{\mathrm{Sing}(X)} Y$ over a point $y \notin X$ equals $s_F(y)$, which is not an element of the fiber of $\mathbb{P}(\Omega^1_Y \otimes \mathscr{L})$ since $F(y) \neq 0$.

This contradiction concludes the proof of Theorem 1.3. □

As in Remark 2.5, we can also observe that this argument proves that for a general hyperplane H there is a regular finite map

$$B\ell_{\underline{\mathrm{Sing}(X)}} Y \to B\ell_{\mathrm{Sing}(\underline{X})} Y \tag{5}$$

(restricting to a finite regular map $B\ell_{\underline{\mathrm{Sing}(X)}\cap W}W \to B\ell_{\mathrm{Sing}(\underline{X})\cap W}W$ for all $W \subseteq Y$). It follows that the ideal of $\underline{\mathrm{Sing}(X)}$ is integral over the ideal of $\mathrm{Sing}(\underline{X})$ for a general H. This (re)proves a particular case of Teissier's 'idealistic Bertini theorem', [Tei77, §2.8]. In fact, the idealistic Bertini can conversely be used to prove Theorem 1.3.

References

[AB03] Paolo Aluffi and Jean-Paul Brasselet. Interpolation of characteristic classes of singular hypersurfaces. *Adv. Math.*, 180(2):692–704, 2003.

[Alu12] Paolo Aluffi. Chern classes of graph hypersurfaces and deletion-contraction relations. *Mosc. Math. J.*, 12(4):671–700, 883, 2012.

[Alu13] Paolo Aluffi. Euler characteristics of general linear sections and polynomial Chern classes. *Rend. Circ. Mat. Palermo (2)*, 62(1):3–26, 2013.

[EE73] David Eisenbud and E. Graham Evans, Jr. Every algebraic set in n-space is the intersection of n hypersurfaces. *Invent. Math.*, 19:107–112, 1973.

[EJP13] David Eklund, Christine Jost, and Chris Peterson. A method to compute Segre classes of subschemes of projective space. *J. Algebra Appl.*, 12(2), 2013.

[Ful84] William Fulton. *Intersection theory.* Springer-Verlag, Berlin, 1984.

[Gro67] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[Har77] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977.

[HS06] Craig Huneke and Irena Swanson. *Integral closure of ideals, rings, and modules*, volume 336 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2006.

[MQ13] Torgunn Karoline Moe and Nikolay Qviller. Segre classes on smooth projective toric varieties. *Math. Z.*, 275(1-2):529–548, 2013.

[Per95] David Perkinson. Curves in Grassmannians. *Trans. Amer. Math. Soc.*, 347(9):3179–3246, 1995.

[PP95] Adam Parusiński and Piotr Pragacz. Chern-Schwartz-MacPherson classes and the Euler characteristic of degeneracy loci and special divisors. *J. Amer. Math. Soc.*, 8(4):793–817, 1995.

[Tei77] Bernard Teissier. The hunting of invariants in the geometry of discriminants. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pages 565–678. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[Vas05] Wolmer Vasconcelos. *Integral closure.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[ZS60] Oscar Zariski and Pierre Samuel. *Commutative algebra. Vol. II.* Van Nostrand Co., Inc., Princeton-Toronto-London-New York, 1960.

Mathematics Department, Florida State University, Tallahassee FL 32306, U.S.A.
E-mail address: aluffi@math.fsu.edu