TENSORED SEGRE CLASSES

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ABSTRACT. We study a class obtained from the Segre class $s(Z, Y)$ of an embedding of schemes by incorporating the datum of a line bundle on Z. This class satisfies basic properties analogous to the ordinary Segre class, but leads to remarkably simple formulas in standard intersection-theoretic situations such as excess or residual intersections. We prove a formula for the behavior of this class under linear joins, and use this formula to prove that a 'Segre zeta function' associated with ideals generated by forms of the same degree is a rational function.

1. Introduction

Segre classes of subschemes are fundamental ingredients in Fulton-MacPherson intersection theory: the very definition of intersection product may be given as a component of a class obtained by capping a Segre class by the Chern class of a bundle determined by the data ([Ful84, Proposition 6.1(a)]). Segre classes also have applications in the theory of characteristic classes of singular varieties: both the *Chern-Mather* and the *Chern-Schwartz-MacPherson* class of a hypersurface of a nonsingular variety may be written in terms of Segre classes determined by the singularity subscheme of the hypersurface ([AB03, Proposition 2.2]). Precisely because they carry so much information, Segre classes are as a rule very challenging to compute, and their manipulation often leads to overly complex formulas. The main goal of this note is to study a variation on the definition of Segre class which produces a class with essentially the same amount of information, but enjoying features that may simplify its computation and often lead to much simpler expressions. For example, standard applications to enumerative geometry may be streamlined by the use of this 'tensored' class; and we will use this notion to give an efficient proof of the rationality of a *Segre zeta function* of a homogeneous ideal in a polynomial ring, subject to the condition that the generators of the ideal all have the same degree. Concrete applications of the tensored Segre class to intersection-theoretic computations are given in [Alu15], where several of its properties are stated without proof. The proofs of those properties may be found (among others) in this note.

We work over an algebraically closed field k. Throughout this note, Y will denote an algebraic variety over k, and Z will be a closed subscheme of Y. Segre classes of subschemes are defined as Segre classes of related *cones.* Recall the definition, from [Ful84, Chapter 4]: for a closed subscheme Z of Y, with normal cone $C_Z Y$, the Segre class of Z in Y is the class

$$s(Z, Y) := q_* \left(\sum_{i \geq 0} c_1(\mathscr{O}_{\mathbb{P}}(1))^i \cap [\mathbb{P}] \right) \in A_* Z \tag{1.1}$$

where $\mathbb{P} = \mathbb{P}(C_Z(Y \times \mathbb{A}^1))$ is the projectivization of the normal cone of $Z \cong Z \times \{0\}$ in $Y \times \mathbb{A}^1$, $q : \mathbb{P} \to Z$ is the projection, and $\mathscr{O}_{\mathbb{P}}(1)$ is the tautological line bundle on $\mathbb{P}$. (The extra $\mathbb{A}^1$ factor takes care of the possibility that Z may equal Y.)

One motivation for the introduction of the class studied here is the observation that there are two ingredients to the definition recalled above: the projective cone $\mathbb{P}$ and the tautological line bundle $\mathscr{O}_{\mathbb{P}}(1)$. The scheme $\mathbb{P}$ does *not* determine the line bundle $\mathscr{O}_{\mathbb{P}}(1)$: even if $C_Z(Y \times \mathbb{A}^1)$ is a vector bundle $\mathscr{E}$, twisting $\mathscr{E}$ by a line bundle $\mathscr{L}$ determines an isomorphic projective bundle: $\mathbb{P}(\mathscr{E} \otimes \mathscr{L}) \cong \mathbb{P}(\mathscr{E})$, but modifies the tautological line bundle by a corresponding twist. In this sense the notation $\mathscr{O}_{\mathbb{P}}(1)$ is ambiguous, as different realizations of the scheme $\mathbb{P}$ affect $\mathscr{O}_{\mathbb{P}}(1)$ by a twist by a line bundle. Implementing this additional degree of freedom leads to classes that share many of the standard properties of Segre classes, but are in certain situations better behaved and easier to use.

Thus, we consider the datum of a subscheme $Z \subseteq Y$ as above, together with a line bundle $\mathscr{L}$ over Z.

Definition 1.1. The $\mathscr{L}$-*tensored* Segre class of Z in Y is

$$s(Z,Y)^{\mathscr{L}} := s(Z,Y) \otimes_{Y\times\mathbb{A}^1} \mathscr{L} \quad .$$

This notion is essentially a particular case of the *twisted Segre operator* defined and studied by Steven Kleiman and Anders Thorup in their work on mixed Buchsbaum-Rim multiplicities, [KT96, §4]; see §2.1. The $\otimes$ operation used in Definition 1.1 was introduced in [Alu94, Definition 2]; 'tensoring' the class by $\mathscr{L}$ essentially reproduces the effect of tensoring the tautological line bundle $\mathscr{O}_{\mathbb{P}}(1)$ by $\mathscr{L}^\vee$. In particular, the ordinary Segre class $s(Z,Y)$ agrees with the class tensored by the trivial bundle: $s(Z,Y) = s(Z,Y)^{\mathscr{O}}$. Definition 1.1 has the advantage that if $s(Z,Y)$ is known, computing the tensored class does not require an explicit realization of the normal cone $\mathbb{P}$. Also, good properties of the $\otimes$ notation significantly help in manipulations of tensored classes. For example, [Alu94, Proposition 2] implies that

$$s(Z,Y)^{\mathscr{L}_1\otimes\mathscr{L}_2} = s(Z,Y)^{\mathscr{L}_1} \otimes_{Y\times\mathbb{A}^1} \mathscr{L}_2 \tag{1.2}$$

for all line bundles $\mathscr{L}_1$, $\mathscr{L}_2$ on Z.

We will prove several basic properties of tensored classes:

(i) If $Z \subseteq Y$ is a regular embedding, with normal bundle N_ZY, then $s(Z,Y)^{\mathscr{L}} = (c(\mathscr{L})c(N_ZY\otimes\mathscr{L}))^{-1} \cap [Z]$. In particular, if $Z = D \subseteq Y$ is a Cartier divisor, then $s(D,Y)^{\mathscr{O}(-D)} = (1+D+D^2+\cdots)\cap[D]$.

(ii) The tensored Segre class $s(Z,Y)^{\mathscr{L}}$ is preserved by birational morphisms and by flat morphisms.

(iii) If $Y = V$ is a nonsingular variety, the class $c(TV\otimes\mathscr{L})\cap s(Z,V)^{\mathscr{L}}$ is determined by Z and $\mathscr{L}$, and is independent of V.

(iv) Residual intersection: Suppose Z contains a Cartier divisor D in Y, and R is the residual scheme to D in Z (in the sense of [Ful84, §9.2]). Then (omitting evident push-forwards)

$$s(Z,Y)^{\mathscr{L}} = s(D,Y)^{\mathscr{L}} + s(R,Y)^{\mathscr{O}(D)\otimes\mathscr{L}} \quad .$$

(v) Suppose $Y \subseteq \mathbb{P}^n$, and let H be a general hyperplane. Then

$$s(H\cap Z, H\cap Y)^{\mathscr{L}} = H\cdot s(Z,Y)^{\mathscr{L}} \quad .$$

Several of these properties were stated without proof in [Alu15], and used therein to streamline intersection-theoretic computations. They are analogues (and formal consequences) of properties satisfied by ordinary Segre classes. Because of these properties, tools normally used to compute Segre classes can be applied to compute tensored classes directly. For example, one may blow-up Y along Z, use (i) to compute the tensored Segre class of the exceptional divisor, and (ii) (birational invariance) to obtain the tensored Segre class of Z in Y. Also note that, by (iv), the ordinary residual formula for Segre classes takes the simple form

$$s(Z,X) = s(D,X) + s(R,X)^{\mathscr{O}(D)} \tag{1.3}$$

and by (i) and (iv),

$$s(Z,X)^{\mathscr{O}(-D)} = (1+D+D^2+\cdots)\cap[D] + s(R,X) \quad .$$

These examples illustrate the notational advantage of using tensored classes: the reader is invited to compare (1.3) with the standard formulation of the residual formula in [Ful84, Proposition 9.2].

Tensored Segre classes arise naturally in enumerative geometry. A template situation in characteristic 0 may be as follows. Consider the linear system of hypersurfaces of $\mathbb{P}^n$ of degree d containing a given scheme Z. For $k = 0, \dots, n$ we may ask for the number N_k of points of intersection of k general such hypersurfaces and $n-k$ general hyperplanes, *in the complement of Z.* (By Bertini's theorem, these intersection points will count with multiplicity 1.) We will prove:

Theorem 1.2. *With notation as above,*

$$\iota_*\left(s(Z,\mathbb{P}^n)^{\mathscr{O}(-d)}\right) = \sum_{k=0}^{n}(d^k - N_k)[\mathbb{P}^{n-k}] \tag{1.4}$$

where $\iota : Z \hookrightarrow \mathbb{P}^n$ is the inclusion.

For example, the problem of computing characteristic numbers of degree-r plane curves fits this template: the projective space $\mathbb{P}^n$, with $n = r(r+3)/2$, parametrizes degree-r plane curves; the linear system is spanned by the hypersurfaces of degree $d = 2r-2$ parametrizing curves that are tangent to lines; and $Z \subseteq \mathbb{P}^n$ is a scheme supported on the set of non-reduced curves. In this case, the numbers N_k are the 'characteristic numbers' of the family of degree r plane curves. (They are known for $r \le 4$, [Vak99]; the problem of their computation is completely open for $r \ge 5$.) It is well known that the problem is equivalent to the problem of computing the Segre class $s(Z,\mathbb{P}^n)$; Theorem 1.2 makes this fact completely transparent.

By the same token, Theorem 1.2 can be used to compute Segre classes, in low dimension: if the hypersurfaces are general elements of the linear system of hypersurfaces of degree d containing a given scheme Z, then a computer algebra system can be used to evaluate the numbers N_k, giving $\iota_*\left(s(Z,\mathbb{P}^n)^{\mathscr{O}(-d)}\right)$ by (1.4), and $\iota_* s(Z,\mathbb{P}^n)$ may then be obtained by tensoring by $\mathscr{O}(d)$, making use of (1.2). This strategy is reminiscent of the algorithm (via residual schemes) introduced by Eklund, Jost, Peterson ([EJP13]). An example will be given in §3 (Example 3.6).

We also note the following consequence of Theorem 1.2.

Corollary 1.3. *Assume $Z \subseteq \mathbb{P}^n$ may be defined by an ideal generated by polynomials of degree d. Then $s(Z,\mathbb{P}^n)^{\mathscr{O}(-d)}$ is effective.*

Example 1.4. Let Z be the Veronese surface in $\mathbb{P}^5$. Then $\iota_* s(Z,\mathbb{P}^5) = 4[\mathbb{P}^2] - 18[\mathbb{P}^1] + 51[\mathbb{P}^0]$ is not effective, but as the Veronese surface is cut out by quadrics,

$$\iota_* s(Z,\mathbb{P}^5)^{\mathscr{O}(-2)} = 4[\mathbb{P}^2] + 14[\mathbb{P}^1] + 31[\mathbb{P}^0]$$

is effective. ⌟

Ampleness considerations imply that the class $(1+dH)^{n+1}s(Z,\mathbb{P}^n)$ is effective. Corollary 1.3 also implies this fact, as we note in §3; hence it is a stronger constraint. Further constraints on the degrees of the components of $s(Z,\mathbb{P}^n)^{\mathscr{O}(-d)}$ may be derived from Theorem 1.2 by applying a theorem of June Huh; see Remark 3.7.

Theorem 1.2 will follow from a re-writing of the Fulton-MacPherson intersection product for the intersection of a collection of linearly equivalent effective Cartier divisors $X_1, \dots, X_m$ in a variety V. Let $\mathscr{O}(X)$ be the (common) line bundle of these divisors, and assume that Z is a union of connected components of the intersection $X_1 \cap \dots \cap X_m$.

Theorem 1.5. *The contribution of Z to the intersection product $X_1 \cdots X_m$ equals the component of dimension* $\dim V - m$ *in $s(Z,V)^{\mathscr{O}(-X)}$.*

This result is a formal consequence of known formulae (cf. [Ful84, Example 6.1]) and of properties of the $\otimes$ operation from [Alu94, §2]. The reason we find Theorem 1.5 remarkable is that the class $s(Z,V)^{\mathscr{O}(-X)}$ *does not depend* on the number m of hypersurfaces: if Z is a collection of components of the intersection of more hypersurfaces from the same linear system, then its contribution to the corresponding intersection product is simply evaluated by terms of higher codimension in *the same* class $s(Z,V)^{\mathscr{O}(-X)}$. In fact, we can show (Theorem 3.4) that the contribution supported on subvarieties of Z for the intersection product of *any* number of general elements of the linear system equals the term of appropriate dimension in $s(Z,V)^{\mathscr{O}(-X)}$. This fact is responsible for the particularly simple form taken by Theorem 1.2.

A similar notational advantage occurs in computing Segre classes of joins with linear subspaces in projective space. Let Z be a subscheme of $\mathbb{P}^n$; we may assume Z is defined by a (possibly non-saturated) ideal generated by homogeneous polynomials $F_1,\dots,F_m$ *of the same degree* d in $k[x_0,\dots,x_n]$. For $N\ge n$, we consider the subscheme $Z_N^{(d)}$ of $\mathbb{P}^N$ defined by the ideal generated by the F_i, viewed as polynomials in $k[x_0,\dots,x_N]$. Geometrically, $Z_N^{(d)}$ is the join of $Z\subseteq\mathbb{P}^n\subseteq\mathbb{P}^N$ and a subspace $\mathbb{P}^m$, $m=N-n-1$ complementary to $\mathbb{P}^n$; but the scheme structure of $Z_N^{(d)}$ along the 'vertex' $\mathbb{P}^m$ depends on the choice of the degree d.

Example 1.6. Let Z be a point in $\mathbb{P}^1$. Then $Z_2^{(1)}$ is a reduced line in $\mathbb{P}^2$, while $Z_2^{(2)}$ is a line with an embedded point. ⌟

An analogous linear join operation may also be defined at the level of Chow groups: as above, let $\mathbb{P}^m\subseteq\mathbb{P}^N$ be a complementary subspace to $\mathbb{P}^n$; and if $W\subseteq Z\subseteq\mathbb{P}^n$ is a subvariety, let $W\vee\mathbb{P}^m$ denote the cone over W with vertex along $\mathbb{P}^m$, a subvariety of $Z_N^{(d)}$. This correspondence passes through rational equivalence, hence it defines a map $\alpha\mapsto\alpha\vee\mathbb{P}^m$ from A_*Z to $A_*Z_N^{(d)}$.

Theorem 1.7. *In the situation detailed above (in particular, with $N=n+m+1$)*

$$s(Z_N^{(d)},\mathbb{P}^N)^{\mathscr{O}(-dH)}=\frac{d^{n+1}[\mathbb{P}^m]}{1-dH}+s(Z,\mathbb{P}^n)^{\mathscr{O}(-dH)}\vee\mathbb{P}^m\quad. \tag{1.5}$$

A 'relative' version of Theorem 1.7, proven here in Theorem 4.1, is used in computations carried out in [Alu15]. Another reason that motivates our interest in Theorem 1.7 is that this statement implies that the push-forward of the ordinary Segre class to $\mathbb{P}^N$ is of the form

$$\iota_{N*}s(Z_N^{(d)},\mathbb{P}^N)=\frac{A(H)}{(1+dH)^{n+1}}\cap[\mathbb{P}^N]$$

where $A(H)$ is a certain polynomial *independent of* N and with nonnegative coefficients (Theorem 4.3). This is a particular case of the rationality of a 'Segre zeta function' which may be associated with any homogeneous ideal I of a polynomial ring; the case proven here is the case in which all generators of I have the same degree (or, more generally, are elements of a linear system in a suitable relative setting, cf. Theorem 4.1). This zeta function appears to be quite interesting. For example, its poles record the degree of some, but in general not all, the elements of a minimal generating set of the ideal. The general case of this rationality statement will be discussed elsewhere.

The basic properties of tensored Segre classes are proven in §2. Theorem 1.5 is proven in §3, together with its enumeratively-inspired consequence, Theorem 1.2. The 'relative' generalization of Theorem 1.7, and the rationality of the Segre zeta function in the particular case considered here, are discussed in §4.

As mentioned above, the class considered here may be viewed as an application of the *twisted Segre operator* defined by S. Kleiman and A. Thorup in [KT96]; see §2.1. Properties (ii) and (iv) listed above are particular cases of more general statements for these operators ((a), (b) in section (4.4), and Theorem 4.6 in [KT96], respectively). While 'twisted Segre classes' may have been a natural choice for the name of the classes considered here, we opted for 'tensored' for consistency with the terminology used in [Alu15] and since the term 'twisted Segre class' is used in a different context in [Wal06].

Leendert van Gastel also considered classes defined similarly: one can interpret Corollary 3.6 in [vG91] as showing that the class of the *Vogel cycle* may be expressed as a tensored Segre class, up to multiplication by the Chern class of a line bundle. It would be interesting to compare this result with Theorem 1.2.

Finally, we note that the Chern-Schwartz-MacPherson class of a hypersurface X in a nonsingular variety M may be written as

$$c_{\mathrm{SM}}(X) = c(TM) \cap \Big(s(X,M) + (s(JX,M)^{\mathscr{O}(-X)})^\vee \Big) \tag{1.6}$$

where JX denotes the *singularity subscheme* of X, and $(\cdot)^\vee$ is the operation that changes the sign of the components of the class $(\cdot)$ which have odd codimension in M. This follows immediately from the definition and from [Alu99, Theorem I.4].

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2. Basic properties

As in the introduction, Y denotes a variety over an algebraically closed field k, embeddable in a nonsingular scheme. (By [Ful84, Lemma 4.2], the material extends without substantial changes to the case in which Y is a pure-dimensional k-scheme.) In this section we prove the basic properties of tensored Segre classes listed in the introduction.

We recall the definition of the tensored classes: for a closed embedding $\iota : Z \hookrightarrow Y$, and for a line bundle $\mathscr{L}$ on Z, we let

$$s(Z,Y)^{\mathscr{L}} = s(Z,Y) \otimes_{Y\times\mathbb{A}^1} \mathscr{L} \quad .$$

Here we identify Z with $Z \times \{0\} \subseteq Y \times \mathbb{A}^1$. The $\otimes$ notation, borrowed from [Alu94, §2], acts on a class α_k in $A_k Z$ by

$$\alpha_k \otimes_{Y\times\mathbb{A}^1} \mathscr{L} := c(\mathscr{L})^{-(\dim Y + 1 - k)} \cap \alpha_k \quad , \tag{2.1}$$

and this definition is extended by linearity to the whole Chow group A_*Z. Properties of this operation are proven in [Alu94, §2]. Recall that $\alpha \mapsto \alpha \otimes_{Y\times\mathbb{A}^1} \mathscr{L}$ defines an action of Pic on A_*Z; in particular, $\alpha = (\alpha \otimes_{Y\times\mathbb{A}^1} \mathscr{L}) \otimes_{Y\times\mathbb{A}^1} \mathscr{L}^\vee$. Hence, the ordinary Segre class may be recovered from a tensored one by tensoring by the dual line bundle:

$$s(Z,Y) = s(Z,Y)^{\mathscr{L}} \otimes_{Y\times\mathbb{A}^1} \mathscr{L}^\vee \quad .$$

We note that $s(Z,Y)^{\mathscr{O}} = s(Z,Y)$. Also, $s(Z,Z)^{\mathscr{L}} = c(\mathscr{L})^{-1} \cap [Z]$ if Z is pure-dimensional.

2.1. Twisted Segre operators.

As mentioned in §1, $s(Z,Y)^{\mathscr{L}}$ may be expressed in terms of the *twisted Segre operator* of [KT96]: with notation as in [KT96, §4.4],

$$s(Z,Y)^{\mathscr{L}} = s(Z, \mathscr{L}^\vee)[Y] \quad .$$

Indeed, according to [KT96, (4.4.2)] and with notation as in §1,

$$s(Z,\mathscr{L}^\vee)[Y] = q_* \left(\sum_{i\ge 0} c_1(\mathscr{O}_{\mathbb{P}}(1)\otimes q^*\mathscr{L}^\vee)^i \cap [\mathbb{P}] \right) = q_* \left(c(\mathscr{O}_{\mathbb{P}}(-1)\otimes q^*\mathscr{L})^{-1} \cap [\mathbb{P}] \right)$$

(cf. (1.1)). By [Alu94, Proposition 2], and denoting by $\hat{Y}$ the blow-up of $Y \times \mathbb{A}^1$ along $Z \times \{0\}$ (so that $[\mathbb{P}]$ is a divisor in $\hat{Y}$),

$$\begin{aligned} q_*\left(c(\mathscr{O}_{\mathbb{P}}(-1)\otimes q^*\mathscr{L})^{-1}\cap[\mathbb{P}]\right) &= q_*\left([\mathbb{P}]\otimes_{\hat{Y}}(\mathscr{O}_{\mathbb{P}}(-1)\otimes q^*\mathscr{L})\right)\\ &= q_*\left(([\mathbb{P}]\otimes_{\hat{Y}}\mathscr{O}_{\mathbb{P}}(-1))\otimes_{\hat{Y}} q^*\mathscr{L}\right)\\ &= q_*\left(([\mathbb{P}]\otimes_{\hat{Y}}\mathscr{O}_{\mathbb{P}}(-1))\right)\otimes_{Y\times\mathbb{A}^1}\mathscr{L}\\ &= s(Z,Y)\otimes_{Y\times\mathbb{A}^1}\mathscr{L}\\ &= s(Z,Y)^{\mathscr{L}} \quad . \end{aligned}$$

2.2. **Regular embeddings.**

(i) If $Z \subseteq Y$ is a regular embedding, with normal bundle $N_Z Y$, then

$$s(Z,Y)^{\mathscr{L}} = (c(\mathscr{L})c(N_Z Y\otimes\mathscr{L}))^{-1}\cap[Z] \quad . \tag{2.2}$$

Proof. If $Z \subseteq Y$ is a regular embedding, then $s(Z,Y) = c(N_Z Y)^{-1}\cap[Z]$ by [Ful84, Proposition 4.1(a)]. Also note that in this case Z is pure-dimensional (as Y is pure-dimensional by assumption). Applying [Alu94, Proposition 1], we obtain

$$\begin{aligned} s(Z,Y)^{\mathscr{L}} &= s(Z,Y)\otimes_{Y\times\mathbb{A}^1}\mathscr{L} = (c(N_Z Y)^{-1}\cap[Z])\otimes_{Y\times\mathbb{A}^1}\mathscr{L}\\ &= c(\mathscr{L})^{\operatorname{codim}_Z Y} c(N_Z Y\otimes\mathscr{L})^{-1}\cap([Z]\otimes_{Y\times\mathbb{A}^1}\mathscr{L})\\ &= c(\mathscr{L})^{\dim Y-\dim Z} c(N_Z Y\otimes\mathscr{L})^{-1}\cap(c(\mathscr{L})^{-(\dim Y+1-\dim Z)}\cap[Z]) \end{aligned}$$

with the stated result. □

Example 2.1. Let Z be the complete intersection of r linearly equivalent effective Cartier divisors $X_1,\dots,X_r$; let D be the common divisor class of the hypersurfaces X_i. Then $N_Z Y = \mathscr{O}(D)^{\oplus r}$, hence $N_Z Y\otimes\mathscr{O}(-D)\cong\mathscr{O}^{\oplus r}$, and (2.2) gives

$$s(Z,Y)^{\mathscr{O}(-D)} = c(\mathscr{O}(-D))^{-1}\cap[Z] = (1+D+D^2+\cdots)\cap[Z] = s(Z,Z)^{\mathscr{O}(-D)} \quad ,$$

independently of r. In particular, if $Z = D$ is a Cartier divisor, then $s(D,Y)^{\mathscr{O}(-D)} = s(D,D)^{\mathscr{O}(-D)} = D + D^2 + D^3 + \cdots$ as stated in §1. ⌟

2.3. **Behavior under morphisms.**

(ii) Let $\pi : Y' \to Y$ be a morphism of varieties, let $\rho : Z' := \pi^{-1}(Z) \to Z$ be the induced morphism, and $\widetilde{\mathscr{L}} = \rho^*\mathscr{L}$. Then
- If π is proper and onto, then $\rho_*(s(Z',Y')^{\widetilde{\mathscr{L}}}) = \deg(Y'/Y)s(Z,Y)^{\mathscr{L}}$.
- If π is flat, then $\rho^*(s(Z,Y)^{\mathscr{L}}) = s(Z',Y')^{\widetilde{\mathscr{L}}}$.

Proof. Both statements follow immediately from the analogous properties of ordinary Segre classes, proven in[Ful84, Proposition 4.2(a)], and from the projection formula, which implies that

$$\rho_*(\alpha\otimes_{Y'\times\mathbb{A}^1}\rho^*\mathscr{L}) = \rho_*(\alpha)\otimes_{Y\times\mathbb{A}^1}\mathscr{L}$$

if $\dim Y' = \dim Y$, as is immediate from the definition of the $\otimes$ operation. □

For the corresponding (more general) facts for twisted Segre operators, see [KT96, (4.4)].

Note that as a consequence of the first formula, tensored Segre classes are invariant under birational maps, in the sense that $\rho_*\left(s(\pi^{-1}(Z),Y')^{\widetilde{\mathscr{L}}}\right) = s(Z,Y)^{\mathscr{L}}$ if π is a proper birational morphism.

2.4. **Independence of a nonsingular ambient variety.**

(iii) If $Y = V$ is a nonsingular variety, the class $c(TV|_Z \otimes \mathscr{L}) \cap s(Z,V)^{\mathscr{L}}$ is independent of V; it is determined by Z and $\mathscr{L}$.

In fact, we will show that

$$c(TV|_Z \otimes \mathscr{L}) \cap s(Z,V)^{\mathscr{L}} = c(\mathscr{L})^{\dim V} \cap (c_{\mathrm{F}}(Z) \otimes_{V\times\mathbb{A}^1} \mathscr{L}) \tag{2.3}$$

where $c_{\mathrm{F}}(Z)$ is the class defined in [Ful84, Example 4.2.6]. This class only depends on Z, and if α_k is a class of dimension k, then

$$c(\mathscr{L})^{\dim V} \cap (\alpha_k \otimes_{V\times\mathbb{A}^1} \mathscr{L}) = c(\mathscr{L})^{\dim V} \cap (c(\mathscr{L})^{-(\dim V+1-k)} \cap \alpha_k) = c(\mathscr{L})^{k-1} \cap \alpha_k$$

is independent of V, so indeed (2.3) verifies (iii).

Proof of (2.3). By [Alu94, Proposition 1],

$$\begin{aligned} c(\mathscr{L})^{\dim V} \cap (c_{\mathrm{F}}(Z) \otimes_{V\times\mathbb{A}^1} \mathscr{L}) &= c(\mathscr{L})^{\dim V} \cap ((c(TV) \cap s(Z,V)) \otimes_{V\times\mathbb{A}^1} \mathscr{L}) \\ &= c(\mathscr{L})^{\dim V} \cap (c(\mathscr{L})^{-\dim V} c(TV\otimes\mathscr{L}) \cap (s(Z,V) \otimes_{V\times\mathbb{A}^1} \mathscr{L})) \\ &= c(TV \otimes \mathscr{L}) \cap s(Z,V)^{\mathscr{L}} \end{aligned}$$

as needed. □

2.5. **Residual intersection.**

(iv) Suppose Z contains a Cartier divisor D in Y, and let R be the residual scheme to D in Z. Then

$$s(Z,Y)^{\mathscr{L}} = s(D,Y)^{\mathscr{L}} + s(R,Y)^{\mathscr{O}(D)\otimes\mathscr{L}} \quad . \tag{2.4}$$

Proof. This follows from the usual residual intersection formula, i.e., [Ful84, Proposition 9.2], in the formulation given in [Alu94, Proposition 3]:

$$s(Z,Y) = s(D,Y) + c(\mathscr{O}(D))^{-1} \cap (s(R,Y) \otimes_Y \mathscr{O}(D)) \quad .$$

This gives

$$\begin{aligned} s(Z,Y)^{\mathscr{L}} &= s(Z,Y) \otimes_{Y\times\mathbb{A}^1} \mathscr{L} \\ &= s(D,Y) \otimes_{Y\times\mathbb{A}^1} \mathscr{L} + c(\mathscr{L})^{-1} \frac{c(\mathscr{L})}{c(\mathscr{O}(D)\otimes\mathscr{L})} \cap ((s(R,Y) \otimes_Y \mathscr{O}(D)) \otimes_Y \mathscr{L}) \\ &\overset{*}{=} s(D,Y)^{\mathscr{L}} + c(\mathscr{O}(D)\otimes\mathscr{L})^{-1}(s(R,Y) \otimes_Y (\mathscr{O}(D)\otimes\mathscr{L})) \\ &= s(D,Y)^{\mathscr{L}} + s(R,Y)^{c(\mathscr{O}(D)\otimes\mathscr{L})} \end{aligned}$$

as stated. Equality $*$ follows from [Alu94, Proposition 2]. □

The 'additivity' formula (2.4) will be used in the proof of Theorem 4.1. It may also be obtained as a particular case of additivity for twisted Segre operators, [KT96, Theorem 4.6, (4.7.1)].

2.6. **General hyperplane sections.**

(v) Suppose $Y \subseteq \mathbb{P}^n$, and let H be a general hyperplane. Then

$$s(Z\cap H, Y\cap H)^{\mathscr{L}} = H \cdot s(Z,Y)^{\mathscr{L}} \quad .$$

Proof. More generally, we can prove that if D is a Cartier divisor of Y intersecting properly every component of the normal cone of Z in Y, then

$$s(D\cap Z, D)^{\mathscr{L}} = D \cdot s(Z,Y)^{\mathscr{L}} \quad .$$

Indeed, it is easy to see that this is the case for ordinary Segre classes ([AF15, Lemma 4.1]), so we only need to verify that if $\alpha \in A_*Z$ and D is a divisor of Y, then for all line bundles $\mathscr{L}$

$$(D\cdot\alpha) \otimes_{D\times\mathbb{A}^1} \mathscr{L} = D \cdot (\alpha \otimes_{Y\times\mathbb{A}^1} \mathscr{L}) \quad .$$

This is immediately checked for a pure-dimensional class by using (2.1), hence it holds for all classes by linearity. □

3. Intersection product

In this section we use tensored Segre classes to give a reformulation of the Fulton-MacPherson intersection product. This will again be a formal consequence of the usual formulation of the product, but in some situations the use of tensored classes yields particularly simple expressions, cf. Theorems 1.2 and 1.5.

The reformulation relies on the following observation concerning the $\otimes$ operation used to define tensored classes.

Lemma 3.1. *Let A be a Chow class in a subscheme Z of a pure-dimensional scheme X, and let $\mathscr{L}$ be a line bundle on Z. Then the term of dimension $\dim X - c$ in*

$$c(\mathscr{L})^{c-1} \cap (A \otimes_X \mathscr{L})$$

equals the term of dimension $\dim X - c$ in A (in particular, it is independent of $\mathscr{L}$).

Proof. Letting $A^{(i)}$ denote the part of A of dimension $\dim X - i$,

$$\begin{aligned} c(\mathscr{L})^{c-1} \cap (A \otimes_X \mathscr{L}) &= c(\mathscr{L})^{c-1} \left(\frac{A^{(0)}}{c(\mathscr{L})^0} + \frac{A^{(1)}}{c(\mathscr{L})^1} + \frac{A^{(2)}}{c(\mathscr{L})^2} + \cdots \right) \\ &= c(\mathscr{L})^{c-1} \cap A^{(0)} + c(\mathscr{L})^{c-2} \cap A^{(1)} + \cdots + A^{(c-1)} + c(\mathscr{L})^{-1} \cap A^{(c)} + \cdots \end{aligned}$$

It is clear that the term of dimension $\dim X - c$ in this expression is $A^{(c)}$, independently of $\mathscr{L}$. □

Remark 3.2. If $A = \frac{c(\mathscr{E})}{c(\mathscr{F})} \cap [X]$, with $\mathscr{E}$, $\mathscr{F}$ vector bundles of ranks e, f respectively, then Lemma 3.1 asserts that the term of codimension $c = e - f + 1$ in A does not change if we tensor both $\mathscr{E}$ and $\mathscr{F}$ by a line bundle $\mathscr{L}$. Indeed, using [Alu94, Proposition 1]

$$\begin{aligned} c(\mathscr{L})^{c-1} \cap (A \otimes_X \mathscr{L}) &= c(\mathscr{L})^{e-f} \cap \left(\left(\frac{c(\mathscr{E})}{c(\mathscr{F})} \cap [X] \right) \otimes \mathscr{L} \right) \\ &= c(\mathscr{L})^{e-f} \cap \left(\frac{c(\mathscr{E} \otimes \mathscr{L})}{c(\mathscr{L})^{e-f} c(\mathscr{F} \otimes \mathscr{L})} \cap [X] \right) = \frac{c(\mathscr{E} \otimes \mathscr{L})}{c(\mathscr{F} \otimes \mathscr{L})} \cap [X] \quad . \end{aligned}$$

This recovers the result of [AF95]. ⌟

Now we consider a standard intersection template. Let V be a variety, $B \subseteq V$ a closed subscheme, and assume that the inclusion $B \hookrightarrow Y$ is a regular embedding. Let $f : Y \to V$ be a morphism, and assume $Z \subseteq Y$ is a collection of connected components of $f^{-1}(B)$. Let $g : Z \to B$ be the induced morphism.

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ B \arrow[r] & V \end{tikzcd}$$

Proposition 3.3. *For all line bundles $\mathscr{L}$ on Z, the contribution $(B \cdot Y)_Z$ of Z to the Fulton-MacPherson intersection product $B \cdot Y$ is given by*

$$(B \cdot Y)_Z = \{c(g^* N_B V \otimes \mathscr{L}) \cap s(Z, Y)^{\mathscr{L}}\}_d \tag{3.1}$$

where $\{\cdot\}_d$ denotes the term of dimension d, and $d = \dim Y - \operatorname{codim}_B V$.

The point of this statement is that the contribution of Z to $B \cdot Y$, and hence the right-hand-side of (3.1), is *independent of $\mathscr{L}$;* thus, we may have the flexibility of choosing a specific line bundle to simplify this expression. Theorem 1.5 will precisely be obtained in this fashion.

Proof. By [Ful84, §6.1], $(B\cdot Y)_Z=\{c(g^*N_BV)\cap s(Z,Y)\}_d$. Applying Lemma 3.1 to $A=c(g^*N_BV)\cap s(Z,Y)$, $X=Y\times\mathbb{A}^1$, and $c=\dim Y+1-d$, we obtain

$$\begin{aligned}(B\cdot Y)_Z&=\{c(g^*N_BV)\cap s(Z,Y)\}_d\\&=\{c(\mathscr{L})^{\dim Y-d}\cap((c(g^*N_BV)\cap s(Z,Y))\otimes_{Y\times\mathbb{A}^1}\mathscr{L})\}_d\\&\overset{*}{=}\{c(\mathscr{L})^{\dim Y-d}c(\mathscr{L})^{-\operatorname{codim}_BV}c(g^*N_BV\otimes\mathscr{L})\cap(s(Z,Y)\otimes_{Y\times\mathbb{A}^1}\mathscr{L})\}_d\\&=\{c(g^*N_BV\otimes\mathscr{L})\cap s(Z,Y)^{\mathscr{L}}\}_d\end{aligned}$$

as stated. Equality $*$ holds by [Alu94, Proposition 1]. □

Next, we verify that Theorem 1.5 follows from Proposition 3.3. Let $X_1,\dots,X_m$ be effective Cartier divisors in a variety V; assume $\mathscr{O}(X_i)$ is independent of i, and let $\mathscr{O}(X)$ denote this line bundle. We define the intersection product $X_1\cdots X_m$ by applying the Fulton-MacPherson definition ([Ful84, §6.1]) to the following Cartesian diagram:

$$\begin{tikzcd} X_1\cap\cdots\cap X_m \arrow[r] \arrow[d, "i"'] & V \arrow[d, "\Delta"] \\ X_1\times\cdots\times X_m \arrow[r] & V\times\cdots\times V \end{tikzcd}$$

where the vertical map Δ is the diagonal embedding.

Let Z be a union of connected components of the intersection $X_1\cap\cdots\cap X_m$. Theorem 1.5 states that the contribution $(X_1\cdots X_m)_Z$ of Z to the intersection product $X_1\cdots X_m$ is given by

$$(X_1\cdots X_m)_Z=\{s(Z,V)^{\mathscr{O}(-X)}\}_{\dim V-m}\quad .$$

Proof of Theorem 1.5. We have $Z\subseteq X_1\cap\cdots\cap X_m\overset{i}{\hookrightarrow}X_1\times\cdots\times X_m$. Denote by g this inclusion. We have

$$g^*N_{X_1\times\cdots\times X_m}(V\times\cdots\times V)=\bigoplus_j N_{X_j}V|_Z\cong\mathscr{O}(X)^{\oplus m}|_Z\quad .$$

It follows that

$$(g^*N_{X_1\times\cdots\times X_m}(V\times\cdots\times V))\otimes\mathscr{O}(-X)|_Z\cong\mathscr{O}_Z^{\oplus m}\quad ,$$

and hence

$$c(g^*N_{X_1\times\cdots\times X_m}(V\times\cdots\times V)\otimes\mathscr{O}(-X))=1\quad .$$

The statement then follows immediately from Proposition 3.3. □

Next, we assume that Z is cut out by a linear system $L\subseteq H^0(Y,\mathscr{L})$ and that Y is projective; let H be the hyperplane class on Y. For a class $\alpha\in A_kY$, we will let $\deg\alpha$ denote $\int_Y H^k\cdot\alpha$, i.e., the degree of the push-forward of α to projective space. If $X_1,X_2,\dots$ are general elements of L, we are interested in the contribution to $\deg(X_1\cdots X_c)$ supported on (subvarieties of) Z, for all $c\geq 0$. As mentioned in §1, this situation is motivated by enumerative geometry: typically, the non-complete variety $Y\smallsetminus Z$ may parametrize some type of geometric object, and the degree of the part of $X_1\cdots X_c$ supported on $Y\smallsetminus Z$ will have enumerative significance. This degree can be obtained by taking the contribution supported on Z away from the total degree of $X_1\cdots X_c$; thus, this operation may be viewed as 'performing intersection theory in the non-complete variety $Y\smallsetminus Z$'.

Theorem 3.4. *For all $c\geq 0$, the contribution to* $\deg(X_1\cdots X_c)$ *supported on Z equals* $\deg\{s(Z,Y)^{\mathscr{L}^\vee}\}_{\dim Y-c}$.

Proof. Let $H_1,\dots,H_{\dim Y-c}$ be general hyperplanes, and let

$$Z^{(c)}=H_1\cap\cdots\cap H_{\dim Y-c}\cap Z\quad ,\quad Y^{(c)}=H_1\cap\cdots\cap H_{\dim Y-c}\cap Y\quad ;$$

note that $\dim Y^{(c)} = c$. The intersections $X_i^{(c)} := X_i \cap Y^{(c)}$ are general representatives of the restriction of the linear system L to $Y^{(c)}$; this system cuts out $Z^{(c)}$. We have

$$\deg(X_1 \cdots X_c) = \int_Y (X_1^{(c)} \cdots X_c^{(c)}) \quad .$$

By Theorem 1.5, the contribution of $Z^{(c)}$ to $X_1^{(c)} \cdots X_c^{(c)}$ is given by

$$\int_Y \{s(Z_-^{(c)}, Y^{(c)})^{\mathscr{L}^\vee}\}_0 \quad ,$$

where $Z_-^{(c)}$ is the part of $X_1^{(c)} \cap \cdots \cap X_c^{(c)}$ supported within $Z^{(c)}$. By [Alua, Theorem 1.1(b)], $s(Z_-^{(c)}, Y^{(c)}) = s(Z^{(c)}, Y^{(c)})$, and hence $s(Z_-^{(c)}, Y^{(c)})^{\mathscr{L}^\vee} = s(Z^{(c)}, Y^{(c)})^{\mathscr{L}^\vee}$. Further, by property (v) of tensored Segre classes (cf. §2.6),

$$s(Z^{(c)}, Y^{(c)})^{\mathscr{L}^\vee} = H^{\dim Y - c} \cdot s(Z, Y)^{\mathscr{L}^\vee} \quad .$$

It follows that the contribution to $X_1 \cdots X_c$ supported on Z has degree

$$\int_Y H^{\dim Y - c} \cdot s(Z, Y)^{\mathscr{L}^\vee} = \deg\{s(Z, Y)^{\mathscr{L}^\vee}\}_{\dim Y - c}$$

as stated. □

Theorem 1.2 is a special case of Theorem 3.4, where $Y = \mathbb{P}^n$ and $\mathscr{L} = \mathscr{O}(d)$.

Example 3.5. For the problem of characteristic numbers of plane conics we have $Y = \mathbb{P}^5$, $d = 2$; and Z consists of the Veronese surface in $\mathbb{P}^5$ with its reduced structure. Denoting by h the hyperplane in $Z \cong \mathbb{P}^2$, the pull-back of H to Z equals $2h$, and we have

$$c(N_Z\mathbb{P}^5) = \frac{(1+2h)^6}{(1+h)^3} \quad .$$

By property (i) of tensored classes (cf. §2.2) we have that

$$\begin{aligned} s(Z, \mathbb{P}^5)^{\mathscr{O}(-2H)} &= (c(\mathscr{O}(-4h))c(N_Z\mathbb{P}^5 \otimes \mathscr{O}(-4h)))^{-1} \cap [Z] = \frac{(1+h-4h)^3}{(1-4h)(1+2h-4h)^6} \cap [Z] \\ &= \frac{(1-3h)^3}{(1-4h)(1-2h)^6} \cap [Z] = (1 + 7h + 31h^2) \cap [Z] \end{aligned}$$

and it follows that

$$\iota_* s(Z, \mathbb{P}^5)^{\mathscr{O}(-2H)} = 4[\mathbb{P}^2] + 14[\mathbb{P}^1] + 31[\mathbb{P}^0] \quad .$$

Using Theorem 1.2, this says that the characteristic numbers N_k for smooth plane conics, that is, the number of conics tangent to k general lines and containing $5-k$ general points, must be $1, 2, 2^2, 2^3 - 4, 2^4 - 14, 2^5 - 31 = 1, 2, 4, 4, 2, 1$ for $k = 0, \dots, 5$. ⌟

Example 3.6. As mentioned in §1, results such as Theorems 3.4 or 1.2 may be used to compute Segre classes. For example, consider the monomial scheme Z defined by the ideal $I = (x_1^2x_2^6, x_1^3x_2^4, x_1^4x_2^3, x_1^5x_2, x_1^7)$ in $\mathbb{P}^3$. Let f_1, f_2, f_3 be general degree-8 polynomials in I, and let

$$J_1 = (f_1) : I^\infty \quad , \quad J_2 = (f_1, f_2) : I^\infty \quad , \quad J_3 = (f_1, f_2, f_3) : I^\infty \quad .$$

Macaulay2 ([GS]) can compute these ideals for 'random' polynomials f_i, and the degrees of the residual schemes R_1, R_2, R_3 defined by the ideals J_1, J_2, J_3:

$$\deg R_1 = 6 \quad , \quad \deg R_2 = 14 \quad , \quad \deg R_3 = 30 \quad . \tag{3.2}$$

According to Theorem 1.2 (assuming that the f_i's are random enough),

$$\iota_* s(Z, \mathbb{P}^3)^{\mathscr{O}(-8)} = (8-6)[\mathbb{P}^2] + (8^2 - 14)[\mathbb{P}^1] + (8^3 - 30)[\mathbb{P}^0] = 2[\mathbb{P}^2] + 50[\mathbb{P}^1] + 482[\mathbb{P}^0] \quad .$$

Letting H denote the hyperplane class, it follows that

$$\iota_* s(Z,\mathbb{P}^3) = \frac{2[\mathbb{P}^2]}{(1+8H)^2} + \frac{50[\mathbb{P}^1]}{(1+8H)^3} + \frac{482[\mathbb{P}^0]}{(1+8H)^4} = 2[\mathbb{P}^2] + 18[\mathbb{P}^1] - 334[\mathbb{P}^0] \quad .$$

Cf. [Alub, Example 1.2] for a different computation of the same class. We note that by (1.2)

$$\begin{aligned} s(Z,\mathbb{P}^3)^{\mathscr{O}(-d)} &= s(Z,\mathbb{P}^3)^{\mathscr{O}(-8)} \otimes_{\mathbb{P}^3\times\mathbb{A}^1} \mathscr{O}(8-d) \\ &= \frac{2[\mathbb{P}^2]}{(1+(8-d)H)^2} + \frac{50[\mathbb{P}^1]}{(1+(8-d)H)^3} + \frac{482[\mathbb{P}^0]}{(1+(8-d)H)^4} \\ &= 2[\mathbb{P}^2] + (4d+18)[\mathbb{P}^1] + (6d^2+54d-334)[\mathbb{P}^0] \quad , \end{aligned}$$

and by Theorem 1.2 we can deduce that the degrees of the corresponding residual schemes for degree-d general polynomials in I must be

$$\deg R_1' = d-2 \quad , \quad \deg R_2' = d^2-4d-18 \quad , \quad \deg R_3' = d^3-6d^2-54d+334 \quad .$$

Therefore, the residual degrees for any one d ($d=8$ in this case) determine the residual degrees for every d. (Macaulay2 can confirm low degree specializations of this formula (e.g., $d=9$) in this example.) ⌟

Corollary 1.3 follows from Theorem 1.2 and [Ful84, Theorem 12.3], which ensures that all contributions to an intersection product in projective space are nonnegative: with notation as in §1, $d^i - N_i \ge 0$ for all i, therefore the class

$$\iota_* s(Z,\mathbb{P}^n)^{\mathscr{O}(-d)} = (d^0-N_0)[\mathbb{P}^n] + (d^1-N_1)[\mathbb{P}^{n-1}] + \cdots + (d^n-N_n)[\mathbb{P}^0]$$

is effective. We also note that it follows that

$$\begin{aligned} (1+dH)^{n+1} \cap \iota_* s(Z,\mathbb{P}^n) &= (1+dH)^{n+1}\left(\iota_* s(Z,\mathbb{P}^n)^{\mathscr{O}(-dH)} \otimes_{\mathbb{P}^n\times\mathbb{A}^1} \mathscr{O}(dH)\right) \\ &= (1+dH)^{n+1}\left(\frac{(d^0-N_0)[\mathbb{P}^n]}{1+dH} + \frac{(d^1-N_1)[\mathbb{P}^{n-1}]}{(1+dH)^2} + \cdots + \frac{(d^n-N_n)[\mathbb{P}^0]}{(1+dH)^{n+1}}\right) \\ &= (d^0-N_0)(1+dH)^n[\mathbb{P}^n] + (d^1-N_1)(1+dH)^{n-1}[\mathbb{P}^{n-1}] + \cdots + (d^n-N_n)[\mathbb{P}^0] \end{aligned}$$

is necessarily an effective class.

Remark 3.7. Further constraints on the degrees of the components of $s(Z,\mathbb{P}^n)^{\mathscr{O}(-d)}$ follow from Theorem 1.2 and a theorem of June Huh. Specifically, assume that Z may be cut out by hypersurfaces of degree d in $\mathbb{P}^n$, and let

$$s(Z,\mathbb{P}^n)^{\mathscr{O}(-d)} = a_0[\mathbb{P}^n] + a_1[\mathbb{P}^{n-1}] + \cdots + a_n[\mathbb{P}^0] \quad .$$

Then the numbers $1-a_0$, $d-a_1$, $\dots$, d^n-a_n form a *log-concave sequence of nonnegative integers with no internal zeros.* Indeed, if Z may be cut out by hypersurfaces of degree d, then the blow-up of $\mathbb{P}^n$ along Z may be realized as a subvariety of $\mathbb{P}^n\times\mathbb{P}(\mathscr{O}(d)^{\oplus r}) \cong \mathbb{P}^n\times\mathbb{P}^{r-1}$ for some r, and the numbers N_i in Theorem 1.2 may be interpreted as the multidegrees of the class of this blow-up in $\mathbb{P}^n\times\mathbb{P}^{r-1}$. These numbers form a log-concave sequence with no internal zeros by [Huh12, Theorem 21], and the statement follows. ⌟

4. Segre classes of linear joins

Next we consider Segre classes of linear joins. The following situation generalizes slightly the one presented in §1; this generalization has been useful in applications. Let V be a variety, and $Z \subseteq Y = V\times\mathbb{P}^n$ a closed subscheme defined by a section s of $\mathscr{E}\otimes\mathscr{O}(d)$, where $\mathscr{E}$ is (the pull-back of) a vector bundle defined on V. The situation described in §1 corresponds to taking V to be a point. For any $N\ge n$, we embed $\mathbb{P}^n$ in $\mathbb{P}^N$, for example by $(x_0:\cdots:x_n)\mapsto(x_0:\cdots:x_n:0:\cdots:0)$; we may then define a subscheme $Z_N^{(d)}$ of $V\times\mathbb{P}^N$ by using 'the same' section s, interpreting the $\mathscr{O}(d)$ components of s as expressed in the first $(n+1)$ homogeneous coordinates of $\mathbb{P}^N$. Geometrically, the scheme $Z_N^{(d)}$ is supported on the join of $Z\subseteq V\times\mathbb{P}^n\subseteq V\times\mathbb{P}^N$ and $V\times\mathbb{P}^{N-n-1}$, where $\mathbb{P}^{N-n-1}$ is spanned by the last $N-n$ homogeneous coordinates. Thus $Z^{(d)}_n = Z$; but note that the scheme structure on $Z^{(d)}_N$ along the 'vertex' $V \times \mathbb{P}^{N-n-1}$ is not determined by Z alone—it also depends on the choice of d (cf. Example 1.6).

These linear joins also define a map $\alpha \mapsto \alpha \vee (V \times \mathbb{P}^m)$, $(m = N-n-1)$ from A_*Z to $A_*Z^{(d)}_N$: if W is a subvariety of Z, the join $W \vee (V \times \mathbb{P}^m)$ is a subvariety of $Z^{(d)}_N$. Theorem 1.7 is the particular case corresponding to $V = \{pt\}$ of the following statement.

Theorem 4.1. *With notation as above, and letting H denote the hyperplane class:*

$$s(Z^{(d)}_N, V \times \mathbb{P}^N)^{\mathscr{O}(-dH)} = \frac{d^{n+1}[V \times \mathbb{P}^m]}{1-dH} + s(Z, \mathbb{P}^n)^{\mathscr{O}(-dH)} \vee (V \times \mathbb{P}^m) \quad . \tag{4.1}$$

Proof. We consider the projection $p : V \times \mathbb{P}^N \dashrightarrow V \times \mathbb{P}^n$ with center at $V \times \mathbb{P}^m$; the indeterminacy of p is resolved by blowing up $V \times \mathbb{P}^N$ along $V \times \mathbb{P}^m$. Let $\pi : \widetilde{Y} \to V \times \mathbb{P}^N$ be this blow-up, $\tilde{p}$ the lift of p to $\widetilde{V}$, and let E be the exceptional divisor.

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d, "\rho"'] & \widetilde{V} \arrow[d, "\pi"'] \arrow[dr, "\tilde{p}"] & \\
V \times \mathbb{P}^m \arrow[r, hook] & V \times \mathbb{P}^N \arrow[r, dashed, "p"] & V \times \mathbb{P}^n
\end{tikzcd}$$

By hypothesis, $Z^{(d)}_N$ is defined by the section s of $\mathscr{E} \otimes \mathscr{O}(d)$ whose zero-scheme is Z in $V \times \mathbb{P}^n$. It follows that $\pi^{-1}(Z^{(d)}_N) = E \cup \tilde{p}^{-1}(Z)$ set-theoretically; we will refine this statement to a scheme-theoretic one in a moment. First, let π', resp., $\tilde{p}'$ be the restrictions of π, resp., $\tilde{p}$ to $\pi^{-1}(Z^{(d)}_N)$, and note that

$$\pi'_* \tilde{p}'^*([W]) = [W] \vee \mathbb{P}^m$$

realizes the join operation $A_*Z \to A_*Z^{(d)}_N$.

Let $(x_0 : \cdots : x_N)$ be homogeneous coordinates in $\mathbb{P}^N$. On open sets U of a cover of V we may write $s = (F_1, \dots, F_m)$ $(m = \operatorname{rk} \mathscr{E})$ with

$$F_i = F_i(x_0, \dots, x_n) \in \mathscr{O}_V(U)[x_0, \dots, x_n]$$

homogeneous polynomials. The center $V \times \mathbb{P}^m$ of the blow-up is cut out by $x_0, \dots, x_n$, so the ideal of $\pi^{-1}(Z^{(d)}_N)$ in $\widetilde{V}$ is generated over U by

$$F_i(\eta x_j) = \eta^d F_i(x_j)$$

in the patch for $\widetilde{V}$ obtained by setting one of the x_j's to 1 and letting η be the coordinate corresponding to the exceptional divisor. It follows that

$$\pi^{-1}(Z^{(d)}_N) = dE \cup \tilde{p}^{-1}(Z)$$

(scheme theoretically). We apply the residual intersection formula ((iv), cf. §2.5) with $D = dE$, $R = \tilde{p}^{-1}(Z)$, and $\mathscr{L} = \mathscr{O}(-dH)$ to obtain

$$s(\pi^{-1}(Z^{(d)}_N), \widetilde{V})^{\mathscr{O}(-dH)} = s(dE, \widetilde{V})^{\mathscr{O}(-dH)} + s(\tilde{p}^{-1}(Z), \widetilde{V})^{\mathscr{O}(d(E-H))} \quad .$$

By birational invariance ((ii), §2.3) and the projection formula, this yields

$$s(Z^{(d)}_N, \mathbb{P}^{n+m+1})^{\mathscr{O}(-dH)} = \pi'_* \left(s(dE, \widetilde{V})^{\mathscr{O}(-dH)} + s(\tilde{p}^{-1}(Z), \widetilde{V})^{\mathscr{O}(d(E-H))} \right) \quad .$$

We have to compute the push-forward by π'_* of the two terms on the right-hand side.

Concerning the first term, we have $s(dE, \widetilde{V}) = \frac{d[E]}{1+dE}$, and π' restricts to ρ on E. We have

$$\rho_* \frac{[E]}{1+E} = s(V \times \mathbb{P}^m, V \times \mathbb{P}^{n+m+1}) = \frac{[V \times \mathbb{P}^m]}{(1+H)^{n+1}}$$

by the birational invariance of Segre classes and the fact that $V\times\mathbb{P}^m$ is regularly embedded in $V\times\mathbb{P}^N$, with normal bundle $\mathscr{O}(H)^{\oplus(n+1)}$. Hence

$$\rho_*\frac{d[E]}{1+dE}=\frac{d^{n+1}[V\times\mathbb{P}^m]}{(1+dH)^{n+1}}\quad .$$

It follows that

$$\pi'_*s(dE,\widetilde{V})^{\mathscr{O}(-dH)}=\left(\frac{d^{n+1}[V\times\mathbb{P}^m]}{(1+dH)^{n+1}}\right)^{\mathscr{O}(-dH)}=\frac{d^{n+1}[V\times\mathbb{P}^m]}{1-dH}$$

as a class in $A_*Z_N^{(d)}$. (Here we have again used [Alu94, Proposition 1].) As for the other term, since $\tilde{p}$ is flat, then (§2.3)

$$s(\tilde{p}^{-1}(Z),\widetilde{V})^{\mathscr{O}(d(H-E))}=\left(\tilde{p}'^*s(Z,V\times\mathbb{P}^n)\right)^{\mathscr{O}(d(H-E))}\quad ;$$

and since $H-E$ is the pull-back of the hyperplane class from $\mathbb{P}^n$ (which we also denote by H), we get

$$s(\tilde{p}^{-1}(Z),\widetilde{V})^{\mathscr{O}(d(E-H))}=\tilde{p}'^*(s(Z,V\times\mathbb{P}^n)^{\mathscr{O}(-dH)})\quad .$$

It follows that

$$\pi'_*s(\tilde{p}^{-1}(Z),\widetilde{V})^{\mathscr{O}(d(E-H))}=\pi'_*\tilde{p}'^*(s(Z,V\times\mathbb{P}^n)^{\mathscr{O}(-dH)})=s(Z,V\times\mathbb{P}^n)^{\mathscr{O}(-dH)}\vee(V\times\mathbb{P}^m)$$

and this concludes the proof. □

For $d=1$, (4.1) reproduces Lemma 4.2 in [Alu15], which was stated and used without proof in that reference. Theorem 1.7 follows from Theorem 4.1, by letting $V=$ a point. In this case, the information of s consists of $m=\operatorname{rk}E$ homogeneous polynomials $F_1,\dots,F_m\in k[x_0,\dots,x_n]$ of the same degree d.

Example 4.2. Let Z be a nonsingular conic in $\mathbb{P}^2$; then $Z_3^{(d)}$ is supported on a quadric cone in $\mathbb{P}^3$. By Theorem 1.7,

$$s(Z_3^{(d)},\mathbb{P}^3)^{\mathscr{O}(-dH)}=d^3[\mathbb{P}^0]+s(Z,\mathbb{P}^2)^{\mathscr{O}(-dH)}\wedge\mathbb{P}^0\quad .$$

We have

$$s(Z,\mathbb{P}^2)^{\mathscr{O}(-dH)}=(1-dH)^{-1}(1+(2-d)H)^{-1}\cap[Z]=[Z]+(2d-2)H\cdot[Z]=[Z]+(4d-4)[\mathbb{P}^0]$$

and therefore

$$s(Z_3^{(d)},\mathbb{P}^3)^{\mathscr{O}(-dH)}=2[\mathbb{P}^2]+(4d-4)[\mathbb{P}^1]+d^3[\mathbb{P}^0]$$

after push-forward to $\mathbb{P}^3$. The ordinary Segre class is immediately obtained from this:

$$s(Z_3^{(d)},\mathbb{P}^3)=\frac{2[\mathbb{P}^2]}{(1+dH)^2}+\frac{(4d-4)[\mathbb{P}^1]}{(1+dH)^3}+\frac{d^3[\mathbb{P}^0]}{(1+dH)^4}=2[\mathbb{P}^2]-4[\mathbb{P}^1]+d(d^2-6d+12)[\mathbb{P}^0]$$

after push-forward to $\mathbb{P}^3$. The case $d=2$ corresponds to the *reduced* quadric cone in $\mathbb{P}^3$. ⌟

In the rest of the section we will focus on the simpler Theorem 1.7. In our view, the most interesting feature of this statement is that the shape of the expression (1.5) is *independent* of $N\geq n$; thus it can be taken as an invariant of the ideal $I=(F_1,\dots,F_m)$. In terms of ordinary Segre classes, this observation takes the following form.

Theorem 4.3. *With notation as above, let $\iota_N:Z_N^{(d)}\hookrightarrow\mathbb{P}^N$ be the inclusion. Then*

$$\iota_{N*}s(Z_N^{(d)},\mathbb{P}^N)=\frac{A(H)}{(1+dH)^{n+1}}\cap[\mathbb{P}^N]\tag{4.2}$$

where $A(H)$ is a polynomial of degree $n+1$ with nonnegative coefficients, independent of N.

Proof. We push forward the class to $\mathbb{P}^N$, and write it 'cohomologically'. So

$$\iota_{N*}s(Z,\mathbb{P}^n)^{\mathscr{O}(-dH)} = a_0 + a_1 H + \cdots + a_n H^n$$

is a shorthand for the class $(a_0 + a_1 H + \cdots + a_n H^n) \cap [\mathbb{P}^n] = a_0[\mathbb{P}^n] + a_1[\mathbb{P}^{n-1}] + \cdots + a_n[\mathbb{P}^0]$. This class is effective (by Corollary 1.3), so the coefficients a_i are all nonnegative. By Theorem 1.7,

$$\iota_{N*}s(Z_N^{(d)},\mathbb{P}^N)^{\mathscr{O}(-dH)} = a_0 + a_1 H + \cdots + a_n H^n + \frac{d^{n+1}H^{n+1}}{1-dH} \quad .$$

The ordinary Segre class is obtained by tensoring by $\mathscr{O}(dH)$:

$$\iota_{N*}s(Z_N^{(d)},\mathbb{P}^N) = \frac{a_0}{1+dH} + \cdots + \frac{a_n H^n}{(1+dH)^{n+1}} + \frac{d^{n+1}H^{n+1}}{(1+dH)^{n+1}}$$

that is,

$$\iota_{N*}s(Z_N^{(d)},\mathbb{P}^N) = \frac{a_0(1+dH)^n + a_1 H(1+dH)^{n-1} + \cdots + a_n H^n + d^{n+1}H^{n+1}}{(1+dH)^{n+1}} \tag{4.3}$$

and this verifies the statement. □

Remark 4.4. The polynomial $A(H)$ has the following interpretation. Let $S_Z(H) \in \mathbb{Z}[H]$ be the polynomial of degree $\le n$ such that

$$\iota_* s(Z,\mathbb{P}^n) = S_Z(H) \cap [\mathbb{P}^n] \quad .$$

Then

$$A(H) = \left[(1+dH)^{n+1} S_Z(H)\right]_n + d^{n+1}H^{n+1} \quad , \tag{4.4}$$

where $[\cdot]_n$ denotes truncation to H^n of the polynomial within $[\cdot]$. (This is obtained from the numerator of (4.3) by a computation analogous to the one presented at the end of §3.) Thus, $A(H) - d^{n+1}H^{n+1}$ is the unique polynomial of degree $\le n$ such that

$$(A(H) - d^{n+1}H^{n+1}) \cap [\mathbb{P}^n] = (1+dH)^{n+1} \cap \iota_* s(Z,\mathbb{P}^n) \quad .$$

As observed at the end of §3, this is an effective class; and indeed $A(H)$ has nonnegative coefficients as proven in Theorem 4.3. Also note that $S_Z(H)$ is determined by Z as a subscheme of $\mathbb{P}^n$, while $Z_N^{(d)}$ depends on the choice of degree d for generators of an ideal defining Z. Expressions (4.2) and (4.4) clarify the dependence of the Segre class $s(Z_N^{(d)},\mathbb{P}^N)$ on the scheme Z and the choice of d.

Example 4.5. For the conic in Example 4.2, $A(H) = 2H + (6d-4)H^2 + d^3H^3$. ⌟

Remark 4.6. The class $\iota_{N*}s(Z^{(d)},\mathbb{P}^N)$ (for $N \gg 0$) is an invariant determined by the homogeneous ideal $I = (F_1,\dots,F_m)$ chosen to define Z scheme-theoretically in $\mathbb{P}^n$, subject to the condition that $\deg F_i = d$ for all i. By Theorem 4.3, this invariant of I may be interpreted as the result of setting $t =$ hyperplane class H in a well-defined rational function $\zeta_I(t)$ with a single pole at $-1/d$ of order $\le (n+1)$, and numerator of degree $\le (n+1)$ and with nonnegative coefficients.

Such a 'zeta function' $\zeta_I(t)$ can be defined for any homogeneous ideal $I \subseteq k[x_0,\dots,x_n]$, and we will prove elsewhere that the essential features verified here for ideals generated by polynomials of a fixed degree hold in general:

- $\zeta_I(t)$ is rational;
- The poles of $\zeta_I(t)$ are at $-1/d_i$, where d_i are degrees of polynomials in any generating set for I.
- The numerator of $\zeta_I(t)$ is a polynomial of degree equal to the degree of the denominator, with nonnegative coefficients, and with known leading term.

In general, not all generators of I will contribute poles to $\zeta_I(t)$. It would be very worthwhile providing a complete description of the poles of $\zeta_I(t)$ and an effective interpretation of the numerator of this function. At present, such descriptions are available for the case studied in this note (as discussed in Remark 4.4) and for ideals generated by monomials, where the information can be obtained from an associated Newton polytope. ⌟

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