NEWTON-OKOUNKOV BODIES AND SEGRE CLASSES

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Abstract. Given a homogeneous ideal in a polynomial ring over $\mathbb{C}$, we adapt the construction of Newton-Okounkov bodies to obtain a convex subset of Euclidean space such that a suitable integral over this set computes the *Segre zeta function* of the ideal. That is, we extract the numerical information of the Segre class of a subscheme of projective space from an associated (unbounded) Newton-Okounkov convex set. The result generalizes to arbitrary subschemes of projective space the numerical form of a previously known result for monomial schemes.

1. Introduction

1.1. Seminal work of R. Lazarsfeld, M. Mustaţă ([LM09]) and K. Kaveh, A. Khovanskii ([KK12]) began the systematic study of *Newton-Okounkov bodies* associated with (for instance) linear systems on a variety. One of the remarkable features of the theory is a very compelling expression for the *intersection index* of a linear system. Roughly speaking, the intersection index of a linear system L on an n-dimensional variety V is the number of points of intersection of n general elements of L, where one discards intersections occurring along the base locus of L. Kaveh and Khovanskii prove ([KK12, Theorem 4.9]) that, modulo important technicalities, this index equals the normalized volume of the corresponding Newton-Okounkov body. This theorem may be viewed as a vast generalization of the classical Kushnirenko theorem on the number of solutions in a torus of a system of general equations with given Newton polytope.

On the other hand, the intersection index of a linear system admits a transparent interpretation in terms of standard intersection theory (this is observed in [KK10, §7]). The Fulton-MacPherson approach to intersection theory may then be used to express this index in terms of the *Segre class* ([Ful84, Chapter 4]) of the base scheme of the system in any completion of V. In fact, Segre classes are a considerably more refined type of information: for example, arbitrary intersection products (not just intersection *numbers*) may be defined by means of Segre classes ([Ful84, Proposition 6.1(a)]). In view of the theorem of Kaveh and Khovanskii mentioned above, it is natural to ask whether these more refined objects can be computed from a suitably constructed Newton-Okounkov body.

In the case of the Kushnirenko theorem, this is indeed the case. The main result of [Alu16] expresses the Segre class of a *monomial* ideal in terms of a certain integral evaluated on the Newton polytope determined by the ideal. The result yields (yet) another proof of Kushnirenko's theorem, generalizing it in a different direction than the Kaveh-Khovanskii result. Thus, we have two generalizations of Kushnirenko's theorem:

$$\begin{tikzcd}
\text{Kaveh-Khovanskii} \arrow[r, Rightarrow] & \text{Kushnirenko} \\
\boxed{???} \arrow[u, Rightarrow, dashed] \arrow[r, Rightarrow, dashed] & \text{[Alu16]} \arrow[u, Rightarrow]
\end{tikzcd}$$

The goal of this note is to fill in this diagram for arbitrary subschemes of projective space. Given a homogeneous ideal I of $\mathbb{C}[x_0,\dots,x_n]$, defining a subscheme X of $\mathbb{P}^n=\mathbb{P}^n_{\mathbb{C}}$, we construct an associated Newton-Okounkov body $\Delta(I)\subseteq\mathbb{R}^{n+1}$ and prove that the push-forward of $s(X,\mathbb{P}^n)$ to $\mathbb{P}^n$ is evaluated by a suitable integral over this body. The result may be viewed as a common generalization of [KK12, Theorem 4.9] and of a 'numerical' form of [Alu16], for subschemes of projective spaces. (We note that [KK12, Theorem 4.9] is a key ingredient in the proof.) More precisely, the result is as follows.

Theorem 1.1. *Denote by ι the inclusion $X\hookrightarrow\mathbb{P}^n$, and let h be the hyperplane class in $\mathbb{P}^n$. Define a power series $\sum\rho_i t^i$ by the identity*

$$\int_{\Delta(I)}\frac{(n+1)!t^{n+1}da_0\cdots da_n}{(1+(a_0+\cdots+a_n)t)^{n+2}}=\sum_{i\ge 0}\rho_i t^i\quad.$$

Then

$$\iota_*s(X,\mathbb{P}^n)=\left(1-\sum_{i=0}^n\rho_i h^i\right)\cap[\mathbb{P}^n]\quad.$$

The integral in this statement is computed formally, treating t as a positive real parameter. According to Theorem 1.1, the first $n+1$ coefficients $\rho_0,\dots,\rho_n$ of its expansion in t determine and are determined by the push-forward of $s(X,\mathbb{P}^n)$.

In fact, *all* coefficients ρ_i admit a Segre class interpretation. We prove that the integral appearing in Theorem 1.1 computes the *Segre zeta function* $\zeta_I(t)$ of the ideal I. More precisely:

Theorem 1.2. *With notation as above,*

$$\int_{\Delta(I)}\frac{(n+1)!t^{n+1}da_0\cdots da_n}{(1+(a_0+\cdots+a_n)t)^{n+2}}=1-\zeta_I(t)\quad.$$

The Segre zeta function $\zeta_I(t)$ is a power series evaluating the push-fowards of the Segre classes $s(X^{(N)},\mathbb{P}^N)$ of the cones $X^{(N)}$ over X in $\mathbb{P}^N$ for $N\ge n$. In [Alu17] it is shown that $\zeta_I(t)$ is a *rational* function; so must be the integral appearing in Theorems 1.1 and 1.2. In fact,

$$\zeta_I(t)=\frac{A(t)+d_0\cdots d_r t^{r+1}}{(1+d_0t)\cdots(1+d_rt)}$$

where the integers d_i are a subset of the degree sequence for I and $A(t)$ is a polynomial of degree $\le r$ with nonnegative integer coefficients. It would be interesting to provide an interpretation or a different proof for these facts in terms of the Newton-Okounkov body $\Delta(I)$.

The paper is organized as follows. In §2 we discuss preliminaries, including short summaries of the definitions of Segre classes and of ordinary Newton-Okounkov bodies, and of the Kaveh-Khovanskii intersection index of a linear system. We explain how the intersection index may be computed from the Segre class of an associated scheme (Corollary 2.4). We expand on this relation in the case of subschemes of projective space, and prove (Proposition 2.9) that knowledge of the intersection indices determined by linear systems associated with the graded pieces I_s of a homogeneous ideal I, for $s\gg 0$, is in fact equivalent to knowledge of the push-forward $\iota_*s(X,\mathbb{P}^n)$ of the Segre class of the subscheme X of $\mathbb{P}^n$ defined by I. In §3 we construct the Newton-Okounkov body $\Delta(I)$. The construction is an adaptation of the 'global Okounkov body' of [LM09, §4], and depends on the choice of a valuation on $\mathbb{C}[x_1,\dots,x_n]$ (as well as on the choice of a dehomogenizing parameter). The body $\Delta(I)$

is a closed convex cone in $\mathbb{R}^{n+1}$; it maps to $\mathbb{R}^1$ by the function $(a_0,\dots,a_n)\mapsto\sum_i a_i$, and we prove (Proposition 3.4) that, for integer $s\gg 0$, the fiber over s of this map equals the (conventional) Newton-Okounkov body of the graded piece I_s. This fact allows us to relate the volume of the fiber over $r\in\mathbb{R}$, $r\gg 0$, to the degrees of the components of $s(X,\mathbb{P}^n)$. In §4 this relation is used to prove Theorem 1.1. The argument is refined in §5 to yield the proof of the more precise Theorem 1.2.

If $I\subseteq\mathbb{C}[x_0,\dots,x_n]$ is a *monomial* ideal in the indeterminates x_i, the construction presented in §3 simply reproduces the Newton polytope of I, and Theorem 1.1 reproduces the corresponding consequence of the result from [Alu16]. We note that the result from [Alu16] is stronger, in the sense that it is a statement about classes in the Chow group, rather than about their degrees; further, it holds for monomial schemes based on a set of divisors meeting with 'regular crossings' (a substantial weakening of the normal crossing condition) on any (not necessarily smooth) variety. We expect that a corresponding strengthening of Theorems 1.1 and 1.2 should hold, providing a computation of the Segre class $s(X,V)$ as an integral on a Newton-Okounkov body generalizing the construction given here, interpreted as a class in A_*X in the same way as is done in the monomial case in [Alu16]. A proper generalization should put little or no requirements on the ambient variety V. The proof of such a result would likely have to rely on different techniques—we do not expect the 'volume' considerations that lead to the proof of Theorems 1.1 and 1.2 in this paper to be adequate to deal with classes in the Chow group. A proof of the stronger result would likely ultimately rely on the birational invariance of Segre classes and on the study of the effect of blow-ups on a suitable generalization of the Newton-Okounkov body $\Delta(I)$ constructed here.

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2. Preliminaries

2.1. We work over $\mathbb{C}$. This is necessary as the results from [KK12] we will use in the proof are stated over $\mathbb{C}$; we expect that the results of this paper should hold over arbitrary algebraically closed fields. For example, considerations about *intersection indices* certainly hold in this generality (cf. [KK14]), as well as their relation to Segre classes established in §2.6.

2.2. Let $X\subsetneq V$ be a closed embedding of schemes; for convenience we assume V to be a variety. The *Segre class* $s(X,V)$ is an element of the Chow group A_*X of X, characterized by the following properties:

- *Birational invariance:* If $f:V'\to V$ is a proper birational map, then
$$s(X,V)=f_*s(f^{-1}(X),V')\quad .$$
- *Divisors:* If X is a Cartier divisor in V, then $s(X,V)=c(\mathscr{O}(X))^{-1}\cap[X]$.

It is clear that these properties determine $s(X,V)$ for any closed subscheme $X\subsetneq V$: by the first point we may blow-up V along X, reducing to the case in which $f^{-1}(X)$ is a divisor in V, and the second point determines the class in this case.

It can be shown that the second point generalizes to arbitrary regular embeddings: if $X\subsetneq V$ is a regular embedding, with normal bundle $\mathscr{N}$, then $s(X,V)=c(\mathscr{N})^{-1}\cap[X]$. For a thorough treatment of Segre classes, the reader is addressed to [Ful84, Chapter 4].

The Fulton-MacPherson intersection product may be defined in terms of Segre classes. If X and Y are subvarieties of a variety V, with X regularly embedded in V with normal bundle $\mathscr{N}$, consider the fiber diagram

$$\begin{tikzcd} X \cap Y \arrow[r] \arrow[d, "j"'] & Y \arrow[d] \\ X \arrow[r] & V \end{tikzcd}$$

Then we may set

$$X \cdot Y := \{c(j^*\mathscr{N}) \cap s(X \cap Y, Y)\}_{\dim X + \dim Y - \dim V}$$

where $\{\cdot\}_k$ stands for the component of dimension k in the class within braces; cf. [Ful84, Proposition 6.1(a)]. This definition satisfies all expected properties of an intersection product, and may be used to give A_*V the structure of a ring when V is nonsingular ([Ful84]).

Thus, Segre classes may be viewed as a key tool in intersection theory, and this is our motivation in seeking alternative ways to compute them. We also reproduce here the following result, which will be relevant to our discussion of the 'intersection index' in §2.4. Let L be a finite dimensional vector space of sections of a line bundle $\mathscr{L}$ on a compact n-dimensional variety $\overline{V}$. We have the associated 'Kodaira' rational map $\varphi : \overline{V} \dashrightarrow \mathbb{P}(L^\vee)$, and we let $\widetilde{V} \to \overline{V}$ be any resolution of the indeterminacies of φ, i.e., a proper birational morphism such that the composition with φ determines a *regular* map

$$\tilde{\varphi} : \widetilde{V} \longrightarrow \mathbb{P}(L^\vee) \quad .$$

For example, $\widetilde{V} \to \overline{V}$ could be the blow-up of $\overline{V}$ along the base scheme B of L (i.e., the intersection of all divisors of V corresponding to sections in L).

Proposition 2.1. *Let H be the hyperplane class in $\mathbb{P}(L^\vee)$. Then with notation as above*

$$\int (\tilde{\varphi}^* H)^n \cdot [\widetilde{V}] = \int c_1(\mathscr{L})^n \cap [\overline{V}] - \int c(\mathscr{L})^n \cap s(B, \overline{V}) \quad .$$

Proof. Every resolution $\widetilde{V}$ factors through the blow-up $B\ell_B\overline{V}$. Therefore we may in fact assume that $\widetilde{V}$ is the blow-up, and then the statement is [Ful84, Proposition 4.4]. □

2.3. Let V be an algebraic variety of dimension n (not necessarily nonsingular or complete). We can associate vector spaces of rational functions on V with linear systems on any completion $\overline{V}$ of V: if $\mathscr{L}$ is a line bundle and $L \subseteq H^0(\overline{V}, \mathscr{L})$ is a subspace, fix a nonzero section s_0 of $\mathscr{L}$ and associate $s \in L$ with the rational function $\frac{s}{s_0}$. The choice can be performed compatibly with products: for example, we may choose s_0^t to identify the space $L^t \subseteq H^0(\overline{V}, \mathscr{L}^{\otimes t})$ spanned by products of t-tuples of elements of L with a space of rational functions. This will be implicitly assumed in the following; we will abuse notation and use the same notation for a linear system L and for a corresponding space of rational functions.

We fix a $\mathbb{Z}^n$-valued valuation v on the field of rational functions on V. For example, v could be the valuation associated with an 'admissible flag' as in [LM09, §1]; for a general discussion of valuations in the context needed here, see [KK12, §2.2].

For $t \in \mathbb{Z}^{\geq 0}$, the spaces L^t determine subsets $v(L^t) := v(L^t \smallsetminus 0)$ of $\mathbb{Z}^n$. By construction, $v(L^t) \subseteq v(L^u)$ for $t \leq u$. The Newton-Okounkov body $\Delta(L)$ captures the asymptotic behavior of $v(L^t)$ as $t \to \infty$. To construct $\Delta(L)$, consider the set

$$U = \{(\underline{a}, u) \in \mathbb{R}^n \times \mathbb{R}^1 \mid u \in \mathbb{Z}^{\geq 0}, \underline{a} \in v(L^u)\} \quad ;$$

U is the *graded semigroup* in the terminology of [LM09]. Let $\Sigma(U)$ be the closed convex cone spanned by U. The Newton-Okounkov body of L is obtained by setting the last coordinate u to 1 in $\Sigma(U)$.

Definition 2.2 ([LM09], Definition 1.8; [KK12], §1)**.** The *Newton-Okounkov body* of L is the convex set

$$\Delta(L) := \Sigma(U) \cap (\mathbb{R}^n \times \{1\}) \quad ,$$

viewed as a subset of $\mathbb{R}^n$. ⌟

The Newton-Okounkov body of a space L is a closed, convex, and compact subset of $\mathbb{R}^n$. Its definition depends on the valuation v and, for linear systems, on the chosen identification with spaces of rational functions. These choices will be inessential in what follows, so they are omitted from the notation.

2.4. Kaveh and Khovanskii associate an *intersection index,* denoted $[L, \dots, L]$ in [KK12] with every space L of rational functions as above, and more generally with any choice of n spaces $L_1, \dots, L_n$. The index $[L_1, \dots, L_n]$ equals the number of solutions in V of a system of equations $\ell_1 = \cdots = \ell_n = 0$, where each ℓ_i is a general element in L_i, and one neglects intersections at which for some i all functions in L_i vanish, and those occurring where some of the functions ℓ_i have poles. We will be primarily interested in the case in which $L_1 = \cdots = L_n = L$. By construction, the intersection index of L does not change if we replace V by a dense open subset, so we may in fact assume that all functions in L are regular. If the space arises from a linear system L as in §2.3, the index is clearly independent of the chosen identification; we will therefore use the notation $[L, \dots, L]$ in this case as well.

We will first recall an intersection-theoretic interpretation of the Kaveh-Khovanskii intersection index; this is a minimal adaptation of the treatment in [KK10, §7]. Any choice $f_0, \dots, f_r$ of generators of L determines a rational map

$$\varphi : V \dashrightarrow \mathbb{P}^r \quad ,$$

mapping $p \in V$ to $(f_0(p) : \ \dots : f_r(p))$ provided $f_i(p) \neq 0$ for some p. Note that there is a canonical injection of $\mathbb{P}(L^\vee)$ into $\mathbb{P}^r$, and φ factors

$$\varphi : \quad V \dashrightarrow \mathbb{P}(L^\vee) \hookrightarrow \mathbb{P}^r$$

where $V \dashrightarrow \mathbb{P}(L^\vee)$ is the standard Kodaira rational map. Extend φ to any completion $\overline{V}$ of V, and let $\pi : \widetilde{V} \to \overline{V}$ be any birational map resolving the indeterminacies of φ:

$$\begin{tikzcd} & \widetilde{V} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{\varphi}"] & \\ \overline{V} \arrow[rr, dashed, "\varphi"] & & \mathbb{P}^r \end{tikzcd}$$

For example, $\widetilde{V}$ could be the closure of the graph of φ, or equivalently the blow-up of $\overline{V}$ along the base scheme of L. Finally, let H be the hyperplane class in $\mathbb{P}^r$. The following is essentially a particular case of [KK10, Corollary 7.7].

Lemma 2.3. *With notation as above,*

$$[L, \dots, L] = \int (\tilde{\varphi}^* H)^n \cdot [\widetilde{V}] \quad .$$

Proof. Since V and $\widetilde{V}$ share a dense open subset, the index $[L,\dots,L]$ may be computed on $\widetilde{V}$. There L corresponds to a base point free linear system, whose elements are preimages of hyperplanes from $\mathbb{P}^r$. The index then agrees with the ordinary intersection product, and this is the statement. □

In fact, identify L with a linear system of sections of a line bundle $\mathscr{L}$ on $\overline{V}$, and let B be its base locus, i.e., the intersection of all divisors defined by nonzero sections in the system.

Corollary 2.4. *With notation as above,*

$$[L,\dots,L]=\int c_1(\mathscr{L})^n\cap[\overline{V}]-\int c(\mathscr{L})^n\cap s(B,\overline{V}) \tag{2.1}$$

Proof. This follows from Lemma 2.3 and Proposition 2.1. □

Remark 2.5. We could adopt (2.1) as the *definition* of the intersection index $[L,\dots,L]$, and use it to give a treatment of the index over more general fields. In the following, we will exploit a relation between the intersection index and Segre classes of which (2.1) is the most straightforward manifestation; see especially Proposition 2.9 and Corollary 2.10. ⌟

2.5. Composing the morphism $\varphi: V \dashrightarrow \mathbb{P}^r$ with an a-Veronese embedding, we see that

$$[L^a,\dots,L^a]=\int(\tilde{\varphi}^*(aH))^n\cdot[\widetilde{V}]=a^n\int(\tilde{\varphi}^*H)^n\cdot[\widetilde{V}]=a^n[L,\dots,L] \tag{2.2}$$

by Lemma 2.3. And indeed, the Kaveh-Khovanskii intersection index is 'multiadditive' ([KK12, Theorem 4.7(1)]).

If L, M are two nonzero finite dimensional subspaces of rational functions on a variety V, choices of generators $f_0,\dots,f_r$ for L and $g_0,\dots g_s$ for M determine a set of generators f_ig_j for LM. The corresponding rational map $\psi: V \dashrightarrow \mathbb{P}^{rs+r+s}$ factors through the Segre embedding:

$$\psi:\quad V \overset{\varphi}{\dashrightarrow} \mathbb{P}^r\times\mathbb{P}^s \hookrightarrow \mathbb{P}^{rs+r+s} \quad .$$

The hyperplane class in $\mathbb{P}^{rs+r+s}$ pulls back to the sum $h+k$ of the (pull-backs of the) hyperplane classes in the two factors. By Lemma 2.3,

$$[LM,\dots,LM]=\int(\tilde{\varphi}^*(h+k))^n\cdot[\widetilde{V}]=\sum_{i=0}^n\binom{n}{i}\int(\tilde{\varphi}^*h^ik^{n-i})\cdot[\widetilde{V}]\quad ,$$

where again $\tilde{\varphi}:\widetilde{V}\to V$ resolves the indeterminacies of φ. It follows easily that

$$[\underbrace{L,\dots,L}_{i},\underbrace{M,\dots,M}_{n-i}]=\int(\tilde{\varphi}^*h^ik^{n-i})\cdot[\widetilde{V}]\quad . \tag{2.3}$$

The same technique may be used to express $[L_1,\dots,L_n]$ as an ordinary intersection product for any choice of n finite dimensional vector spaces of rational functions, generalizing Lemma 2.3. (See e.g., [KK10, Corollary 7.7].) Here we note the following observation, for later use.

Lemma 2.6. *With notation as above, assume that the Kodaira rational map associated with M is generically injective. Then the Kodaira rational map associated with LM is generically injective.*

Proof. Still using notation as above, the morphism $V \dashrightarrow \mathbb{P}^s$ is generically injective by hypothesis, therefore so is $\varphi: V\dashrightarrow\mathbb{P}^r\times\mathbb{P}^s$. It follows that $\psi: V\dashrightarrow\mathbb{P}^{rs+r+s}$ is generically injective, and then so must be the corresponding Kodaira rational map, since ψ factors through it. □

2.6. Now let $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous ideal: $I = \oplus_{s\ge 0} I_s$, where I_s is the piece in degree s. The *degree sequence* of I is the list $(d_0, \dots, d_r)$ of degrees of a minimal set of homogeneous generators for I; this depends only on I. In particular, so does the largest element $d := d_r$; we will call this the *generating degree* for I. Note that $I_d \cdot \mathbb{C}[x_0, \dots, x_n] = \oplus_{s\ge d} I_s$. We record the following elementary fact.

Lemma 2.7. *Let $L = I_d$, where d is the generating degree of I, and let $M = \langle x_0, \dots, x_n\rangle$. Then for $a, b \in \mathbb{Z}$ such that $a \ge 1$ and $b \ge da$ we have*

$$(I^a)_b = L^a M^{b-da} \quad . \tag{2.4}$$

For example, it follows that for every $c \ge 0$,

$$((I^a)_b)^c = (L^a M^{b-da})^c = L^{ac} M^{(b-da)c} = (I^{ac})_{bc} \tag{2.5}$$

if $a \ge 1$ and $b \ge da$.

Next, let $\iota : X \hookrightarrow \mathbb{P}^n$ be the closed subscheme defined by I in $\mathbb{P}^n$; we assume $X \subsetneq \mathbb{P}^n$. Our task here is to relate the push-forward of the Segre class of X to $\mathbb{P}^n$ to the asymptotic behavior of the graded pieces I_s of I, in terms of associated Kaveh-Khovanskii intersection indices. We will view each I_s as a linear system, determining a rational map

$$\varphi_s : \mathbb{P}^n \dashrightarrow \mathbb{P}^{N_s} := \mathbb{P}({I_s}^\vee) \quad . \tag{2.6}$$

The indeterminacies of φ_s are resolved by the closure Γ_s of the graph of φ_s, a subvariety of dimension n of $\mathbb{P}^n \times \mathbb{P}^{N_s}$. The class of Γ_s in the Chow group $A_n(\mathbb{P}^n \times \mathbb{P}^{N_s})$ may be written as

$$[\Gamma_s] = g_0^{(s)} H^{N_s} + \dots + g_n^{(s)} h^n H^{N_s - n} \quad ,$$

for integers $g_0^{(s)}, \dots, g_n^{(s)}$. Here, h and H denote the (pull-backs of the) hyperplane classes from the first and second factor. We let

$$G_s := g_0^{(s)} + g_1^{(s)} h \cdots + g_n^{(s)} h^n$$

be the 'shadow' of $[\Gamma_s]$ in $\mathbb{P}^n$. The integers $g_i^{(s)}$ are the *multidegrees* of the rational map φ_s; the class G_s packages the information of the multidegrees into a single class in $A_*\mathbb{P}^n$.

It follows easily from the definition that $g_0^{(s)} = 1$. By definition, the 'top' multidegree $g_n^{(s)}$ equals

$$g_n^{(s)} = H^n \cdot [\Gamma_s] \quad ; \tag{2.7}$$

thus, $g_n^{(s)}$ equals the degree of the closure of the image of φ_s if φ_s is generically injective, hence birational onto its image. By (2.7) and Lemma 2.3 we have

$$g_n^{(s)} = [I_s, \dots, I_s] \quad , \tag{2.8}$$

the Kaveh-Khovanskii intersection index of I_s. (More generally, $g_i^{(s)}$ equals the index $[\underbrace{I_s, \dots, I_s}_{i}, \underbrace{M, \dots, M}_{n-i}]$ for $i = 0, \dots, n$, cf. (2.3).)

2.7. We will use notation as in [Alu94, §2]: for $G = \sum g_i h^i \in A_*\mathbb{P}^n$ and $a \in \mathbb{Z}$, we let

$$G \otimes \mathscr{O}(ah) := \sum_{i=0}^{n} \frac{g_i h^i}{(1+ah)^i} \quad . \tag{2.9}$$

In fact, we consider the class

$$T_a(G) := \frac{1}{1-ah} \left(G \otimes \mathscr{O}(-ah)\right) \quad ;$$

explicitly,

$$T_a(G) = \sum_{i\ge 0}\left(\sum_{j=0}^{i}\binom{i}{j} a^{i-j} g_j\right) h^i \quad . \tag{2.10}$$

Lemma 2.8. *With notation as above:*

(i) *For $a, b \in \mathbb{Z}$, $T_a(T_b(G)) = T_{a+b}(G)$.*
(ii) *For $s \gg 0$, $T_{-s}(G_s)$ is independent of s.*
(iii) *In fact, $T_{-s}(G_s) = [\mathbb{P}^n] - \iota_* s(X, \mathbb{P}^n)$ for $s \ge d$, the generating degree of I.*

Proof. (i): This follows from the fact that the notation (2.9) defines an action of Pic on Chow ([Alu94, Proposition 2]). Alternately, the reader may verify that, associating with $G = \sum_i g_i h^i$ the column vector $(g_0, \dots, g_n)^t$, $T_a(G) = T^a \cdot (g_0, \dots, g_n)^t$, where T is the square matrix with (i,j)-entry $\binom{i}{j}$, $i, j = 0, 1, \dots$.

(ii) follows from (iii). To prove (iii), let $s \ge d$. Then X is defined scheme-theoretically by I_s, and (iii) follows from [Alu03, Proposition 3.1]. $\square$

2.8. For notational convenience, define integers σ_j, $j = 0, \dots, n$ for $X \subsetneq \mathbb{P}^n$ as above, so that

$$[\mathbb{P}^n] - \iota_* s(X, \mathbb{P}^n) = \sum_{j=0}^{n} \sigma_j [\mathbb{P}^{n-j}] \quad : \tag{2.11}$$

that is, $\sigma_0 = 1$ while $-\sigma_j$ is the degree of the component of $s(X, \mathbb{P}^n)$ of codimension j in $\mathbb{P}^n$, for $j > 0$.

Proposition 2.9. *Let d be the generating degree for I, and let $s \ge d$. Then*

$$[I_s, \dots, I_s] = \sum_{j=0}^{n} \binom{n}{j} \sigma_j s^{n-j} \tag{2.12}$$

Proof. By Lemma 2.8(i), $T_s \circ T_{-s} = \mathrm{id}$. Therefore, Lemma 2.8(iii) gives

$$G_s = T_s \circ T_{-s}(G_s) = T_s\big([\mathbb{P}^n] - \iota_* s(X, \mathbb{P}^n)\big)$$

for $s \ge d$. By (2.8), $[I_s, \dots, I_s]$ equals the coefficient of h^n in G_s, and this gives the statement by (2.10). $\square$

As a consequence of Proposition 2.9, the information carried by $\iota_* s(X, \mathbb{P}^n)$ is equivalent to the information carried by the intersection indices $[I_s, \dots, I_s]$ for $s \gg 0$. In one form or another, this observation is at the root of most methods used for the algorithmic computation of Segre classes, starting with [Alu03].

Our task is to extract $\iota_* s(X, \mathbb{P}^n)$ from a suitably constructed Newton-Okounkov body. For this purpose, we will need to generalize (2.12) to rational s. Let $q = \frac{b}{a} \in \mathbb{Q}$, with $a, b \in \mathbb{Z}^{>0}$, and assume $q \ge d$. It will be convenient to adopt the following notation:

$$[I_q, \dots, I_q] := \frac{1}{a^n}[(I^a)_b, \dots, (I^a)_b] \quad . \tag{2.13}$$

Note that $[I_q, \dots, I_q]$ is well-defined for rational $q \ge d$. Indeed, for $a, b \in \mathbb{Z}$ such that $a \ge 1$ and $b \ge da$ we have $(I^{ac})_{bc} = ((I^a)_b)^c$ for every $c \ge 0$, by (2.5); by (2.2), i.e., the multiadditivity of the intersection index,

$$[(I^{ac})_{bc}, \dots, (I^{ac})_{bc}] = [((I^a)_b)^c, \dots, ((I^a)_b)^c] = c^n [(I^a)_b, \dots, (I^a)_b] \quad ,$$

and therefore

$$\frac{1}{(ac)^n}[(I^{ac})_{bc},\dots,(I^{ac})_{bc}]=\frac{1}{a^n}[(I^a)_b,\dots,(I^a)_b]$$

as needed.

The formula obtained in Proposition 2.9 for the integral intersection index remains true for the fractional version.

Corollary 2.10. *Let d be the generating degree for I, and let $q\in\mathbb{Q}$, $q\geq d$. Then with notation as above*

$$[I_q,\dots,I_q]=\sum_{j=0}^n\binom{n}{j}\sigma_j q^{n-j}\quad.$$

Proof. Let $\iota^{(a)}:X^{(a)}\hookrightarrow\mathbb{P}^n$ be the subscheme defined by the ideal I^a. Then $s(X^{(a)},\mathbb{P}^n)$ is obtained from $s(X,\mathbb{P}^n)$ by multiplying by a^i the component of codimension i in $\mathbb{P}^n$ ([Alu94]). Therefore

$$[\mathbb{P}^n]-\iota^{(a)}_*s(X^{(a)},\mathbb{P}^n)=\sum_{j=0}^n\sigma_j a^j[\mathbb{P}^{n-j}]\quad.$$

Further, note that the generating degree of I^a is ad. By Proposition 2.9, for $b\in\mathbb{Z}$, $b\geq ad$,

$$[(I^a)_b,\dots,(I^a)_b]=\sum_{j=0}^n\binom{n}{j}\sigma_j a^j b^{n-j}\quad.$$

With $q=\frac{b}{a}\geq d$, it follows that

$$[I_q,\dots,I_q]=\frac{1}{a^n}[(I^a)_b,\dots,(I^a)_b]=\frac{1}{a^n}\sum_{j=0}^n\binom{n}{j}\sigma_j a^j b^{n-j}=\sum_{j=0}^n\binom{n}{j}\sigma_j\frac{b^{n-j}}{a^{n-j}}\quad,$$

and this is the statement. □

3. Construction

3.1. In this section we provide our construction of the Newton-Okounkov body associated with a homogeneous ideal $I\subseteq\mathbb{C}[x_0,\dots,x_n]$. The construction is an adaptation of the construction of the 'global Okounkov body' given in [LM09, §4]; as in *loc. cit.* we abuse language and refer to the result of the construction as a *body,* although the object is not compact. It will be a closed convex subset of $\mathbb{R}^{n+1}$ with nonempty interior.

As in §2, we consider the homogeneous pieces of I and of its powers. The body will map to $\mathbb{R}^1$ in such a way that the fiber over $s\in\mathbb{Z}^{\gg 0}$ will be the (ordinary) Newton-Okounkov body associated with I_s. As in §2.3, we use a section of the line bundle $\mathscr{O}(s)$, chosen compatibly with products, to identify the linear systems ${I^t}_s\subseteq H^0(\mathbb{P}^n,\mathscr{O}(s))$ with spaces of rational functions on $\mathbb{P}^n$. For example, we can choose x_0^s; equivalently, we can de-homogenize by setting $x_0=1$. This will be done implicitly in what follows.

Fix a $\mathbb{Z}^n$-valued valuation v on $\mathbb{C}[x_1,\dots,x_n]$. Consider the subset of $\mathbb{R}^n\times\mathbb{R}^2$ defined by

$$U_I:=\{(\underline{a},s,t)\in\mathbb{R}^n\times\mathbb{R}^2\,|\,s\in\mathbb{Z}^{\geq 0},t\in\mathbb{Z}^{\geq 0},\underline{a}\in v((I^t)_s)\}\quad. \tag{3.1}$$

Let $\Sigma(U_I)$ be the closed convex cone generated by U_I. Setting the last coordinate to 1 defines a hyperplane $\{t=1\}\cong\mathbb{R}^n\times\mathbb{R}^1$. The intersection

$$\underline{\Delta}(I):=\Sigma(U_I)\cap\{t=1\}\subseteq\mathbb{R}^n\times\mathbb{R}^1 \tag{3.2}$$

is a closed convex set. The projection $\mathbb{R}^n\times\mathbb{R}^1\to\mathbb{R}^1$, $(\underline{a},s)\mapsto s$, defines a projection $\pi:\underline{\Delta}(I)\to\mathbb{R}^1$. We denote by $\underline{\Delta}_s$ the fiber $\pi^{-1}(s)$, viewed as a subset of $\mathbb{R}^n$.

Definition 3.1. The *Newton-Okounkov body of the ideal* I is the image $\Delta(I)$ of $\underline{\Delta}(I)$ via the isomorphism $\tau : \mathbb{R}^n \times \mathbb{R}^1 \to \mathbb{R}^{n+1}$, $((a_1, \dots, a_n), s) \mapsto (s - (a_1 + \dots + a_n), a_1, \dots, a_n)$. ⌟

Remark 3.2. In the proof of Proposition 3.4 it will be shown that $\Delta(I)$ has non-empty interior; in this sense it is a 'body'. ⌟

Remark 3.3. An alternative description of the body may be given in the style of [LM09, (1.5)]:

$$\Delta(I) = \text{closed convex hull } \left(\bigcup_{t' \geq 1} \frac{1}{t'} \cdot \tau(U_{I,t'}) \right) \quad ,$$

where $U_{I,t'} = U_I \cap \{t = t'\}$. ⌟

3.2. As a motivation for the construction presented above, and in particular for the use of the isomorphism τ in Definition 3.1, we note that if I is generated by *monomials* in the variables $x_0, \dots, x_n$, then the construction reduces to the usual Newton polytope if applied to the monomial valuation determined by the choice of n variables.

Indeed, for a polynomial $P \in \mathbb{C}[x_0, \dots, x_n]$, we may let $v(P) = (a_1, \dots, a_n)$ be the smallest exponent list of a monomial in $P|_{x_0=1}$, with respect to the lexicographic order; for example, $v(x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}) = (a_1, \dots, a_n)$. This leads easily to an effective description of the sets $v((I^t)_s)$:

$$(a_1, \dots, a_n) \in v((I^t)_s) \iff x_0^{s-a_1-\dots-a_n} x_1^{a_1} \cdots x_n^{a_n} \in (I^t)_s \quad .$$

It follows that

$$U_I := \{(\underline{a}, s, t) \,|\, s \in \mathbb{Z}^{\geq 0}, t \in \mathbb{Z}^{\geq 0}, \tau(\underline{a}, s) \in m(I^t)\}$$

where $m(I^t) \subseteq \mathbb{Z}^{n+1}$ is the set of exponents of monomials in I^t, and τ is the isomorphism defined in Definition 3.1. For a fixed $t = t'$, and with $U_{I,t'} = U_I \cap \{t = t'\}$, this shows that

$$\tau(U_{I,t'}) = m(I^{t'}) \quad .$$

The alternative description of $\Delta(I)$ given in Remark 3.3 then gives

$$\Delta(I) = \text{convex hull } \big(m(I)\big) \subseteq \mathbb{R}^{n+1} \quad ,$$

and this is the (unbounded) Newton polytope associated with I.

3.3. Recall that d denotes the generating degree for I.

Proposition 3.4. *Let $q \in \mathbb{Q}$, $q > d$, and write $q = \frac{b}{a}$ with $a, b \in \mathbb{Z}^{>0}$. Then $\underline{\Delta}_q = \frac{1}{a}\Delta((I^a)_b)$.*

(The right-hand side of this equality uses the notation introduced in Definition 2.2.)

Proof. First, assume that a and b are relatively prime. Let

$$U_{a,b} = \{(\underline{a}, u) \in \mathbb{R}^n \times \mathbb{R}^1 \,|\, u \in \mathbb{Z}^{\geq 0}\,,\, \underline{a} \in v(((I^a)_b)^u)\} \quad ;$$

so $\Delta((I^a)_b) = \Sigma(U_{a,b}) \cap \{u = 1\}$ by definition. By (2.5) we have $((I^a)_b)^u = (I^{au})_{bu}$ for $u \geq 0$ (since $b \geq da$), hence

$$U_{a,b} = \{(\underline{a}, u) \in \mathbb{R}^n \times \mathbb{R}^1 \,|\, u \in \mathbb{Z}^{\geq 0}\,,\, \underline{a} \in v((I^{au})_{bu})\} \quad .$$

On the other hand, consider the hyperplane $H_q \subseteq \mathbb{R}^n \times \mathbb{R}^2$ consisting of $(\underline{a}, s, t)$ with $s = qt$. Since $(s, t) \in \mathbb{Z}^2$ for $(\underline{a}, s, t)$ in U_I and a, b are relatively prime, along U_I the condition

$s = qt$ is equivalent to $(s,t) = (bu, au)$ for some $u \in \mathbb{Z}$. It follows that the linear map $(\underline{a}, u) \mapsto (\underline{a}, bu, au)$ induces a bijection

$$U_{a,b} \xrightarrow{\cong} U_I \cap H_q \quad , \tag{3.3}$$

which we will use to identify these two sets. (And note $t = au$ under this identification.)

Since $\frac{b}{a} > d$, $(I^a)_b = L^a M^{b-da}$ with $L = I_d$, $M = \langle x_0, \dots, x_n \rangle$ (Lemma 2.7). Therefore the Kodaira map associated with $(I^a)_b$ is generically injective by Lemma 2.6. It follows that $\Delta((I^a)_b)$ and hence $\Sigma(U_{a,b})$ are full-dimensional (since $\Delta((I^a)_b)$ has nonzero volume, cf. [KK12, Proposition 4.8 and Theorem 4.9(a)]): i.e., $\Sigma(U_{a,b})$ has dimension $n+1$. By (3.3), $\Sigma(U_I) \cap H_q$ *contains* (a copy of) $\Sigma(U_{a,b})$. Therefore, hyperplane sections of $\Sigma(U_I)$ have dimension $n+1$, and it follows that $\Sigma(U_I)$ has dimension $n+2$, that is, it has nonempty interior. This also proves that $\underline{\Delta}(I)$ and hence $\Delta(I)$ have nonempty interior, as promised in Remark 3.2.

Further, the argument shows that the hyperplane H_q meets the interior of $\Sigma(U_I)$. The hypotheses of [LM09, Proposition A.1] are satisfied, therefore

$$\Sigma(U_I) \cap H_q = \Sigma(U_I \cap H_q) = \Sigma(U_{a,b}) \quad ,$$

and hence

$$\underline{\Delta}_q = \big(\Sigma(U_I) \cap \{t=1\}\big) \cap \{s=q\} = \Sigma(U_I) \cap H_q \cap \{t=1\} = \Sigma(U_{a,b}) \cap \{au = 1\} \quad .$$

Since $\Sigma(U_{a,b})$ is a cone,

$$\Sigma(U_{a,b}) \cap \{au = 1\} = \frac{1}{a} \left(\Sigma(U_{a,b}) \cap \{u=1\}\right) = \frac{1}{a} \Delta((I^a)_b) \quad ,$$

concluding the verification if a and b are relatively prime. The general case follows, since $\frac{1}{ac}\Delta((I^{ac})_{bc}) = \frac{1}{a}\Delta((I^a)_b)$ for all $c > 0$, again since $\Sigma(U_{a,b})$ is a cone. □

3.4. Recall the definition of the integers σ_j, from (2.11). In the following statement, Vol stands for the 'normalized' n-dimensional volume, that is, $n!$ times the ordinary Euclidean volume in dimension n (denoted Vol_n in [KK12]).

Corollary 3.5. *Let $r \in \mathbb{R}$, $r > d$. Then* $\mathrm{Vol}(\underline{\Delta}_r) = \sum_{i=0}^n \binom{n}{i} \sigma_{n-i} r^i$.

Proof. By continuity, it suffices to verify the given formula for $r = q \in \mathbb{Q}$, $q > d$. Let then $q \in \mathbb{Q}$, $q = \frac{b}{a} > d$, with a, b positive integers.

Let $L = I_d$, $M = \langle x_0, \dots, x_n \rangle$. By Lemma 2.7 we have $(I^a)_b = L^a M^{b-da}$; note that $b - da > 0$ since $q > d$. By Lemma 2.6, the Kodaira map associated with $(I^a)_b$ is birational onto its image. By [KK12, Theorem 4.9(2)],

$$[(I^a)_b, \dots, (I^a)_b] = \mathrm{Vol}(\Delta((I^a)_b))$$

(we are normalizing the volume by the factorial of the dimension), and therefore

$$\mathrm{Vol}\left(\frac{1}{a}\Delta((I^a)_b)\right) = \frac{1}{a^n}\mathrm{Vol}(\Delta((I^a)_b)) = \frac{1}{a^n}[(I^a)_b, \dots, (I^a)_b] = [I_q, \dots, I_q]$$

adopting (2.13). The stated formula follows by Proposition 3.4 and Corollary 2.10. □

Remark 3.6. As the referee suggests, it is conceivable that the coefficients of $\mathrm{Vol}(\underline{\Delta}_r)$ may limit to Segre-type information if $\underline{\Delta}$ is constructed using more general bi-graded/filtered linear series (not necessarily finitely generated). ⌟

4. From the Newton-Okounkov body to the Segre class

Let $X \subseteq \mathbb{P}^n$ be a closed subscheme, and let $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ be any homogeneous ideal defining X scheme-theoretically.

We have associated with I a 'Newton-Okounkov body' (Definition 3.1) $\Delta(I)$. This body depends on the defining ideal I, on the dehomogenizing factor, and the choice of a valuation. The following result expresses the degrees of the components of the Segre class of X in $\mathbb{P}^n$ in terms of the truncation of a series computed as an integral over $\Delta(I)$.

By definition, the body $\Delta(I)$ is a subset of $\mathbb{R}^{n+1}$. We let $a_0, \dots, a_n$ denote coordinates in this space. The integral appearing in the result is the ordinary 'calculus' integral, depending on a parameter t (which we may take to range in $\mathbb{R}^{>0}$).

Theorem 4.1. *Denote by ι the inclusion $X \hookrightarrow \mathbb{P}^n$, and let h be the hyperplane class in $\mathbb{P}^n$. Define a power series $\sum \rho_i t^i$ by the identity*

$$\int_{\Delta(I)} \frac{(n+1)! t^{n+1} da_0 \cdots da_n}{(1+(a_0+\cdots+a_n)t)^{n+2}} = \sum_{i\geq 0} \rho_i t^i \quad . \tag{4.1}$$

Then

$$\iota_* s(X, \mathbb{P}^n) = \left(1 - \sum_{i=0}^n \rho_i h^i\right) \cap [\mathbb{P}^n] \quad .$$

In other words, the coefficients $\rho_0, \dots, \rho_n$ of the series defined in (4.1) agree with the numbers $\sigma_0, \dots, \sigma_n$ defined in (2.11).

Proof. We perform a change of variables, using the isomorphism τ from Definition 3.1:

$$\tau((a_1, \dots, a_n), s) = (s - (a_1 + \cdots + a_n), a_1, \dots, a_n) \quad .$$

The absolute value of the jacobian is 1, therefore

$$\begin{aligned}\int_{\Delta(I)} \frac{(n+1)! t^{n+1} da_0 \cdots da_n}{(1+(a_0+\cdots+a_n)t)^{n+2}} &= \int_{\underline{\Delta}(I)} \frac{(n+1)! t^{n+1} da_1 \cdots da_{n-1} ds}{(1+(s-(a_1+\cdots+a_n)+a_1+\cdots+a_n)t)^{n+2}} \\ &= \int_{\underline{\Delta}(I)} \frac{(n+1)! t^{n+1} da_1 \cdots da_{n-1} ds}{(1+st)^{n+2}} \quad .\end{aligned}$$

We have $s \geq 0$ on $\underline{\Delta}(I)$, so this integral equals

$$\int_0^\infty \left(\int_{\underline{\Delta}_s} da_1 \cdots da_n\right) \frac{(n+1)!\, t^{n+1}}{(1+st)^{n+2}}\, ds = (n+1) \int_0^\infty \mathrm{Vol}(\underline{\Delta}_s) \frac{t^{n+1}}{(1+st)^{n+2}}\, ds \quad .$$

Now recall that d denotes the generating degree for I. The function $\mathrm{Vol}(\underline{\Delta}_s)$ is bounded on $s \in [0, d]$ (by Corollary 3.5) and is independent of t, therefore

$$\int_0^d \mathrm{Vol}(\underline{\Delta}_s) \frac{t^{n+1}}{(1+st)^{n+2}}\, ds \equiv 0 \mod (t^{n+1})$$

as a series in t. It follows that

$$(n+1) \int_d^\infty \mathrm{Vol}(\underline{\Delta}_s) \frac{t^{n+1}}{(1+st)^{n+2}}\, ds = \rho_0 + \rho_1 t + \cdots + \rho_n t^n + \text{h.o.t.} \quad :$$

that is, since we are only interested in the coefficients ρ_i with $i \leq n$, we can perform the integration over $[d, \infty)$. By Corollary 3.5,

$$\mathrm{Vol}(\underline{\Delta}_s) = \sum_{i=0}^n \binom{n}{i} \sigma_{n-i} s^i$$

for $s > d$. Therefore,

$$\rho_0 + \rho_1 t + \cdots + \rho_n t^n + \text{h.o.t.} = (n+1)\int_d^\infty \sum_{i=0}^n \binom{n}{i}\sigma_{n-i}s^i \frac{t^{n+1}}{(1+st)^{n+2}}\,ds$$

and by the same token used above, we may change the integration range back to $[0, \infty)$:

$$\rho_0 + \rho_1 t + \cdots + \rho_n t^n + \text{h.o.t.} = (n+1)\int_0^\infty \sum_{i=0}^n \binom{n}{i}\sigma_{n-i}s^i \frac{t^{n+1}}{(1+st)^{n+2}}\,ds \tag{4.2}$$

The right-hand side is evaluated by using the following lemma.

Lemma 4.2. *For $0 \le i \le n$,*

$$(n+1)\int_0^\infty \frac{s^i}{(1+st)^{n+2}}ds = \frac{1}{\binom{n}{i}t^{i+1}} \quad . \tag{4.3}$$

Proof. Apply the change of variable $z = 1/(1+st)$, i.e., $s = \frac{1}{t}(\frac{1}{z}-1)$ (treating t as a positive parameter):

$$\int_0^\infty \frac{s^i}{(1+st)^{n+2}}ds = -\int_0^1 \frac{\frac{1}{t^i}\left(\frac{1}{z}-1\right)^i}{\frac{1}{z^{n+2}}}\left(-\frac{1}{tz^2}\right)dz = \frac{1}{t^{i+1}}\int_0^1 z^{n-i}(1-z)^i dz \quad .$$

The last integral is an instance of the *Beta function,* so we obtain

$$\begin{aligned}\int_0^\infty \frac{s^i}{(1+st)^{n+2}}ds &= \frac{1}{t^{i+1}}B(n-i+1, i+1) = \frac{1}{t^{i+1}}\frac{\Gamma(n-i+1)\Gamma(i+1)}{\Gamma(n+2)} \\ &= \frac{1}{t^{i+1}}\frac{(n-i)!i!}{(n+1)!}\end{aligned}$$

with the stated consequence. □

With this understood, (4.2) gives

$$\rho_0 + \rho_1 t + \cdots + \rho_n t^n \equiv \sum_{i=0}^n \sigma_{n-i}t^{n-i} \mod (t^{n+1}) \quad ,$$

and this proves the needed equality $\rho_i = \sigma_i$ for $i = 0, \dots, n$. The stated relation with the Segre class follows from (2.11). □

5. From the Newton-Okounkov body to Segre zeta functions

5.1. Theorem 4.1 only gives information on the first $n+1$ coefficients of the expansion

$$\int_{\Delta(I)} \frac{(n+1)!t^{n+1}da_0\cdots da_n}{(1+(a_0+\cdots+a_n)t)^{n+2}} = \sum_{i\ge 0}\rho_i t^i \quad .$$

This begs the question of what the other coefficients may mean. In this section we will prove that the integral computes the 'Segre zeta function' of the given ideal.

Let $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous ideal. For $N \ge n$, let $X^{(N)}$ denote the subscheme of $\mathbb{P}^N$ determined by the extension $I^{(N)}$ of I to $\mathbb{C}[x_0, \dots, x_N]$, and denote by $\iota^{(N)}$ the inclusion $X^{(N)} \hookrightarrow \mathbb{P}^N$. (Thus $I = I^{(n)}$, $X = X^{(n)}$.) In [Alu17] it is shown that there exists a power series $\zeta_I(t) = \sum_{i\ge 0} s_i t^i$ such that *for all $N \ge n$*

$$\zeta_I(h) \cap [\mathbb{P}^N] = \sum_{i=0}^\infty s_i h^i \cap [\mathbb{P}^N] = \iota_*^{(N)} s(X^{(N)}, \mathbb{P}^N) \quad , \tag{5.1}$$

where we denote by h the hyperplane class in $\mathbb{P}^N$. Of course $h^i \cap [\mathbb{P}^N] = 0$ for $i > N$; thus, for any given N only $s_0, \dots, s_N$ contribute nonzero components in (5.1). It is also shown in [Alu17] that $\zeta_I(t)$ is rational, with poles constrained by the degree sequence of I.

We can let

$$1 - \zeta_I(t) = \sum_{j=0}^{\infty} \sigma_j t^j \quad ,$$

where for any $N \geq n$ the coefficients σ_j are defined as in (2.11): that is, the definition of σ_j is independent of n for $n \geq j$ (this follows from [Alu17, Lemma 5.2]) and we can assemble the information simultaneously for all j into a single power series. Our last goal is to prove that the integral appearing in Theorem 4.1 agrees with this function.

Theorem 5.1. *Let $I \subseteq \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous ideal, and let $\Delta(I) \subseteq \mathbb{R}^{n+1}$ be the corresponding Newton-Okounkov body. Then*

$$\int_{\Delta(I)} \frac{(n+1)!\, t^{n+1} da_0 \cdots da_n}{(1 + (a_0 + \cdots + a_n)t)^{n+2}} = 1 - \zeta_I(t) \quad . \tag{5.2}$$

In other words, with notation as above and as in (4.1), $\rho_i = \sigma_i$ for *all* $i \geq 0$. The parameter t is implicitly assumed to be a positive real in computing the integral. The content of Theorem 5.1 is that the integral then equals a well-defined rational function of t, and this function equals $1 - \zeta_I(t)$. In particular, the integral is independent of the choices (of a valuation and of a dehomogenizing term) used to define $\Delta(I)$.

5.2. Theorem 4.1 shows that (5.2) is true *modulo* t^{n+1}. In particular, we have

$$\int_{\Delta(I^{(N)})} \frac{(N+1)! t^{N+1} da_0 \cdots da_N}{(1 + (a_0 + \cdots + a_N)t)^{N+2}} \equiv \int_{\Delta(I)} \frac{(n+1)! t^{n+1} da_0 \cdots da_n}{(1 + (a_0 + \cdots + a_n)t)^{n+2}} \mod t^{n+1}$$

for all $N \geq n$. In order to prove Theorem 5.1, it suffices to prove that the two integrals are equal *modulo* t^{N+1}. Inductively, it suffices to show that

$$\int_{\Delta(I^{(n+1)})} \frac{(n+2)! t^{n+2} da_0 \cdots da_{n+1}}{(1 + (a_0 + \cdots + a_{n+1})t)^{n+3}} \equiv \int_{\Delta(I)} \frac{(n+1)! t^{n+1} da_0 \cdots da_n}{(1 + (a_0 + \cdots + a_n)t)^{n+2}} \mod t^{N+1} \tag{5.3}$$

for all $N \geq n$. Recall that the definition of the Newton-Okounkov body of an ideal depends on the choice of a valuation. We will assume that a valuation v has been chosen for $\mathbb{A}^n$, and we will show that there exists a corresponding valuation v' for $\mathbb{A}^{n+1}$ with respect to which (5.3) holds. Theorem 5.1 will follow from (5.3), and in particular it will follow that the integral is independent of the choice of valuation, at all orders.

5.3. Let v be a $\mathbb{Z}^n$-valued valuation on $\mathbb{C}[x_1, \dots, x_n]$. We extend v to a valuation v' on $\mathbb{C}[x_1, \dots, x_{n+1}]$ as follows, cf. [KK12, Definition 2.26]. Given $f \in \mathbb{C}[x_1, \dots, x_{n+1}]$, write

$$f = x_{n+1}^e g$$

with $x_{n+1} \not| g$. Define

$$v'(f) = (v(g|_{x_{n+1}=0}), e) \quad .$$

(This is also the inductive step building valuations associated with flags in [LM09, §1.1].) Then v' is a $\mathbb{Z}^{n+1}$-valued valuation. Given a homogeneous ideal I in $\mathbb{C}[x_0, \dots, x_n]$, denote by I' the extension of I to $\mathbb{C}[x_0, \dots, x_{n+1}]$. We want to compare the bodies $\Delta(I) \subseteq \mathbb{R}^{n+1}$, $\Delta(I') \subseteq \mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R}^1$ defined by means of v, v' respectively. (Also recall that we are silently dehomogenizing by setting $x_0 = 1$; cf. §3.1.)

Lemma 5.2. *With notation as above, $\Delta(I') = \Delta(I) \times \mathbb{R}^{\geq 0}$.*

Proof. The key ingredients in the construction of $\Delta(I)$, $\Delta(I')$ are the sets

$$U_I := \{(\underline{a}, s, t) \in \mathbb{R}^n \times \mathbb{R}^2 \,|\, s \in \mathbb{Z}^{\geq 0}, t \in \mathbb{Z}^{\geq 0}, \underline{a} \in v({I^t}_s)\}$$
$$U_{I'} := \{(\underline{a}', s, t) \in \mathbb{R}^{n+1} \times \mathbb{R}^2 \,|\, s \in \mathbb{Z}^{\geq 0}, t \in \mathbb{Z}^{\geq 0}, \underline{a}' \in v'({I'^t}_s)\}$$

(cf. (3.1)). Here $\underline{a} = (a_1, \dots, a_n)$, $\underline{a}' = (a_1, \dots, a_{n+1}) = (\underline{a}, a_{n+1})$. Consider the map

$$\lambda : (\mathbb{R}^n \times \mathbb{R}^2) \times \mathbb{R} \to (\mathbb{R}^{n+1} \times \mathbb{R}^2)$$

given by

$$\lambda((\underline{a}, j, t), a_{n+1}) = ((\underline{a}, a_{n+1}), a_{n+1} + j, t) \quad .$$

We claim that

$$U_{I'} = \lambda(U_I \times \mathbb{Z}^{\geq 0}) \quad . \tag{5.4}$$

Indeed, the equality $I'_j = \bigoplus_{i=0}^j x_{n+1}^{j-i} I_i$ and the definition of v' imply that

$$v'(I'_j) = (v(I_j), 0) \cup (v(I_{j-1}), 1) \cup \cdots \cup (v(I_0), j) \quad .$$

Applying this to powers of I and of its extension I', we get

$$U_{I'} = \{(\underline{a}', s, t) = ((\underline{a}, a_{n+1}), s, t) \in \mathbb{Z}^{n+1} \times \mathbb{Z}^2 \,|\, s \geq 0, t \geq 0, a_{n+1} \geq 0, \underline{a} \in v({I^t}_{s-a_{n+1}})\}$$

and (5.4) is simply a restatement of this identity.

Since taking closed convex cones is preserved by linear maps, and λ maps $t = 1$ to $t = 1$, (5.4) implies that

$$\underline{\Delta}(I') = \lambda|_{t=1}(\underline{\Delta}(I) \times \mathbb{R}^{\geq 0}) \quad .$$

Lemma 5.2 follows by applying the isomorphisms τ used in Definition 3.1: we have the commutative diagram

$$\begin{tikzcd}
(\mathbb{R}^n \times \mathbb{R}^1) \times \mathbb{R}^1 \arrow[r, "\lambda|_{t=1}"] \arrow[d, "\tau \times \mathrm{id}"'] & \mathbb{R}^{n+1} \times \mathbb{R}^1 \arrow[d, "\tau"] \\
\mathbb{R}^{n+1} \times \mathbb{R}^1 \arrow[r, "\cong"] & \mathbb{R}^{n+2}
\end{tikzcd}$$

where the bottom map is simply $((a_0, \dots, a_n), a_{n+1}) \mapsto (a_0, \dots, a_{n+1})$, and this gives the stated identification $\Delta(I) \times \mathbb{R}^{\geq 0} = \Delta(I')$. $\square$

5.4. We can now complete the proof of Theorem 5.1. As we argued in §5.2, it suffices to prove (5.3). We will show that the two integrals appering in (5.3) agree modulo any power of t, if the valuation v' is used for the construction of $\Delta(I')$ (and it will then follow that they must be equal for any choice of valuation).

This is a straightforward consequence of Lemma 5.2. Since $\Delta(I') = \Delta(I) \times \mathbb{R}^{\geq 0}$,

$$\begin{aligned}
&\int_{\Delta(I')} \frac{(n+2)!\, t^{n+2}}{(1 + (a_0 + \cdots + a_{n+1})t)^{n+3}} \, da_0 \cdots da_{n+1} \\
&\quad = \int_{\Delta(I)} \left(\int_0^\infty \frac{(n+2)!\, t^{n+2}}{(1 + (a_0 + \cdots + a_{n+1})t)^{n+3}} da_{n+1} \right) da_0 \cdots da_n \\
&\quad = \int_{\Delta(I)} \frac{(n+1)!\, t^{n+1}}{(1 + (a_0 + \cdots + a_n)t)^{n+2}} \, da_0 \cdots da_n
\end{aligned}$$

as needed. $\square$

Remark 5.3. In the case of ideals of $\mathbb{C}[x_0, \dots, x_n]$ generated by monomials in the variables x_i, Theorem 5.1 recovers a 'numerical' form of the main result of [Alu16]. Indeed, as seen in §3.2, in this case $\Delta(I)$ is the ordinary Newton polytope of I, and the statement of Theorem 4.1 follows by setting $X_1 = \cdots = X_n = t$ (which amounts to taking degrees) in the integral appearing in [Alu16, Theorem 1.1]. As this result is independent of the number of variables, the integral in fact computes the Segre zeta function, yielding the monomial case of Theorem 5.1. ⌟

5.5. To summarize, let $f_0, \dots, f_r$ be homogeneous polynomials in $\le n$ variables. These polynomials generate an ideal $I \subseteq \mathbb{C}[x_0, \dots, x_n]$, and we have proved that the integral

$$\int_{\Delta(I)} \frac{(n+1)!\, t^{n+1}}{(1 + (a_0 + \cdots + a_n)t)^{n+2}}\, da_0 \cdots da_n$$

is *independent of* n, and in fact

$$1 - \int_{\Delta(I)} \frac{(n+1)!\, t^{n+1}}{(1 + (a_0 + \cdots + a_n)t)^{n+2}}\, da_0 \cdots da_n = \zeta_I(t)$$

is a rational function, with various properties studied in [Alu17]—for example, this function can only have poles at $-1/d_i$, where the integers d_i belong to the degree sequence for I, and its numerator is a polynomial with nonnegative coefficients. Some of the known properties of $\zeta_I(t)$ can probably be ascribed to properties of the Newton-Okounkov body $\Delta(I)$. It would be interesting to explore this connection.

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