TRIM RESOLUTIONS, STRINGY AND MATHER CLASSES, AND IC CHARACTERISTIC CYCLES

PAOLO ALUFFI

ABSTRACT. We introduce *trim* resolutions of complex algebraic varieties, a strengthening of the notion of small resolution. We prove that the characteristic cycle of the intersection cohomology sheaf of a variety admitting a trim resolution is irreducible and that for such varieties the stringy and Chern-Mather classes coincide.

1. INTRODUCTION

Recall that a proper surjective stratified map $\pi : Y \to S$ of complex algebraic varieties of the same dimension is *small* if for all proper strata Z of S, $2d(Z) < \operatorname{codim}_Z S$, where $d(Z)$ denotes the (common) dimension of the fibers of π over points of Z. We say that π is a *small resolution* if Y is nonsingular and π is birational and small.

We introduce a strengthening of this condition, which we name 'trim' (Definition 2.2). Assume $\pi : Y \to S$ is a proper birational map, with Y nonsingular. For $d \geq 0$, denote by $Y_d \subseteq Y$ the locally closed subset along which the rank of the differential of π is d. We say that π is *trim* if $\dim Y_d < d$ for all $d < \dim Y$.

Trim maps are small (Proposition 3.1). If $\pi : Y \to S$ is small, and for all strata Z of S the restriction $\pi^{-1}(Z) \to Z$ is a smooth morphism, then π is trim (§4). For instance, the standard Abel-Jacobi resolution of the theta divisor of a non-hyperelliptic curve (which is small, cf. [BB98]) is trim.

Our main result is the following.

Theorem 1.1. *Let S be a complex algebraic variety admitting a trim resolution. Then*

- *For $s \in S$, the local Euler obstruction $\mathrm{Eu}_S(s)$ equals the Euler characteristic of the fiber over s in any trim resolution of S.*
- *The* stringy *Chern class of S equals its Chern-Mather class: $c_{\mathrm{str}}(S) = c_{\mathrm{Ma}}(S)$.*
- *The characteristic cycle of the intersection cohomology sheaf of S is irreducible.*

The definition of characteristic cycle will be recalled below (§2). 'Stringy' Chern classes were defined in [Alu05, dFLNU07] (see [Alu07] for a lean account). If $\pi : Y \to S$ is a crepant resolution, $c_{\mathrm{str}}(S)$ equals the push-forward of the total Chern class of the tangent bundle of Y; the equality stated in the theorem is

$$f_*(c(TY) \cap [Y]) = c_{\mathrm{Ma}}(S) \tag{1.1}$$

in the Chow group of S. In general, stringy Chern classes are defined for normal varieties with a $\mathbb{Q}$-Cartier canonical divisor and at worst log-terminal singularities, and take value in the Chow group with rational coefficients. These additional stipulations are not needed for the notion considered in this note; we can adopt the left-hand side of (1.1) as the definition of stringy Chern class in the (integral) Chow group of S, compatibly with the more general definition. The fact that the class is independent of the choice of trim resolution is also a consequence of Theorem 1.1.

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We note that Theorem 1.1 implies that Batyrev's *stringy Euler number* ([Bat99]) of a variety admitting a trim resolution equals its Euler characteristic weighted by the local Euler obstruction.

Theorem 1.1 is a consequence of considerations concerning Sabbah's formalism of *conical Lagrangian cycles,* to which local Euler obstructions and Chern-Mather classes relate directly (see §2). Concerning intersection cohomology, recall that if $\pi : Y \to S$ is a small resolution, then the intersection cohomology sheaf of S is the push-forward of a shift of the constant sheaf on Y ([GM83, §6.2]). We evaluate the corresponding push-forward at the level of characteristic cycles after embedding S in a nonsingular variety X. More precisely, we prove (Proposition 2.3) that, for trim resolutions, the Lagrangian push-forward of the zero-section $T^*_Y Y$ of the cotangent bundle T^*Y equals the conormal cycle $T^*_S X$. The theorem follows from this more basic observation, as we show in §3.

There is considerable interest in conditions implying that the characteristic cycle of the intersection cohomology sheaf is irreducible. Lusztig ([Lus91, 13.7, p. 414]) expressed the 'hope' that this may be the case for Schubert varieties in flag manifolds of type A, D, E. A counterexample was constructed by Kashiwara and Saito for type A in $F\ell(8)$ ([KS97], [Bra02]), while it holds in $F\ell(n)$ for $n \leq 7$. Irreducibility is also known for all Schubert varieties in the standard Grassmannian ([BFL90]), and more generally for all Schubert varieties in cominuscule Grassmannians of types A, D, and E, see [BF97, MS25].

We hope that Theorem 1.1 may help in streamlining such verifications. For instance, since the Abel-Jacobi resolution of the theta divisor of a non-hyperelliptic curve is trim, the irreducibility of the IC characteristic cycle in this case (first established in [BB98]) follows directly from Theorem 1.1.

The connection between the irreducibility of the IC characteristic cycle and the equality of Chern-Mather and stringy Chern classes was pointed out by B. Jones in [Jon10, Remark 3.3.2], ultimately as an application of the microlocal index formula of Dubson and Kashiwara. We freely borrow ideas from [Jon10] in §3.

A condition on fibers of small resolutions is considered in [GJL], including a proof that the condition implies the irreducibility of the IC characteristic cycles.

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2. Lagrangian push-forward

Let X be a nonsingular variety and denote by T^*X the cotangent bundle of X. The *conormal variety* $T^*_W X$ of a closed subvariety $W \subseteq X$ is the closure in T^*X of the conormal variety to the nonsingular part W° of V: $T^*_W X = \overline{T^*_{W^\circ} X}$. The conormal variety of X itself is the zero-section $T^*_X X$ of the cotangent bundle. All conormal varieties have dimension $\dim X$; they determine conormal *cycles* in $Z_{\dim X} T^*X$. It will also be convenient to take the projective completion of these constructions: we will denote by $\mathbb{T}^*X$ the projective completion $\mathbb{P}(T^*X \oplus 1\!\!1)$ of T^*X (here, $\mathbb{P}$ denotes the projective bundle of lines) and by $\mathbb{T}^*_W X$ the closure of the conormal variety in $\mathbb{T}^*X$.

For a nonsingular variety X, we denote by $\mathcal{L}(X)$ the free abelian group of *conical Lagrangian cycles* in the cotangent bundle T^*X. Conormal cycles are conical Lagrangian, and in fact (cf. [Ken90, Lemma 3]) $\mathcal{L}(X)$ may be realized as the free abelian group on conormal cycles. For a closed (and possibly singular) subvariety $V \subseteq X$, we denote by $\mathcal{L}(V)$ the subgroup of $\mathcal{L}(X)$ generated by the conormal cycles $T^*_W X$ with $W \subseteq V$. Thus, elements of $\mathcal{L}(V)$ may be viewed as finite integer linear combinations $\sum_W m_W T^*_W X$ ranging over closed subvarieties W of V. Clearly $\mathcal{L}(V)$ is isomorphic to the group of algebraic cycles of V, and in particular it is independent of the ambient nonsingular variety X.

Important invariants of V may be expressed directly in terms of Lagrangian cycles by means of intersection-theoretic operations, after taking the projective completion. As above, realize V as a closed subvariety of a nonsingular variety X; the results will be independent of the choice of X. Let $\pi : \mathbb{T}^*_V X \to V$ be the natural projection; and let $\mathcal{O}(1)$ be the tautological line bundle on $\mathbb{T}^*X = \mathbb{P}(T^*X \oplus 1\!\!1)$.

— The *local Euler obstruction* $\mathrm{Eu}_V : V \to \mathbb{Z}$ is

$$\mathrm{Eu}_V(p) = (-1)^{\dim X - \dim V} \int c(\pi^* TX|_V) c(\mathcal{O}(1))^{-1} \cap s(\pi^{-1}(p), \mathbb{T}^*_V X)$$

where $s(\pi^{-1}(p), \mathbb{T}^*_V X)$ denotes the *Segre class* in the sense of [Ful84, Chapter 4];

— The *Chern-Mather class* $c_{\mathrm{Ma}}(V) \in A_*V$ is

$$c_{\mathrm{Ma}}(V) = (-1)^{\dim X - \dim V} c(TX|_V) \cap \pi_* \left(c(\mathcal{O}(1))^{-1} \cap [\mathbb{T}^*_V X] \right) .$$

Equivalent results were established by C. Sabbah ([Sab85, 1.2.1, 1.2.2]); we also address the reader to [Ken90] and [PP01] for clear treatments of these formulas.

Let $\mathcal{F}(V)$ denote the abelian group of constructible functions $V \to \mathbb{Z}$. Following [PP01, §1], the relation between conormal cycles and local Euler obstructions is recorded by the homomorphism

$$\mathrm{CC} : \mathcal{F}(V) \to \mathcal{L}(V)$$

defined by prescribing

$$\mathrm{Eu}_W \mapsto (-1)^{\dim W} T^*_W X$$

for all closed subvarieties W of V. In fact, CC is an *isomorphism:* it simply matches a basis of $\mathcal{F}(V)$ with a basis of $\mathcal{L}(V)$.

Definition 2.1. The *characteristic cycle* $\mathrm{CC}(\alpha)$ of a constructible function α on V is the image of α in $\mathcal{L}(V)$ under this isomorphism. ⌟

Likewise, the formula for Chern-Mather classes motivates the introduction of a homomorphism

$$c_* : \mathcal{L}(V) \to A_*(V)$$

defined on generators by

$$T^*_W X \mapsto (-1)^{\dim W} c_{\mathrm{Ma}}(W) .$$

Then the composition $c_* \circ \mathrm{CC} : \mathcal{F}(V) \to A_*(V)$ agrees with the value at V of MacPherson's natural transformation $\mathcal{F} \rightsquigarrow A_*$, where $\mathcal{F}$ is taken as a functor with push-forward defined by Euler characteristics of fibers; see [Mac74] and [Ful84, Example 19.1.7]. For instance, the *Chern-Schwartz-MacPherson class* of a (possibly singular) algebraic variety V is the image $c_*(\mathrm{CC}(1\!\!1_V)) \in A_*V$ of the characteristic cycle in $\mathcal{L}(V)$ of the constant function $1\!\!1_V$. Sabbah provides an alternative proof of the naturality of this assignment, which is the main result of [Mac74], by defining a covariant *push-forward*

$$\varphi_* : \mathcal{L}(V') \to \mathcal{L}(V'')$$

for every proper map $\varphi : V' \to V''$, making $\mathcal{L}$ into a functor, in such a way that the above homomorphisms define natural transformations

$$\mathcal{F} \rightsquigarrow \mathcal{L} \quad , \quad \mathcal{L} \rightsquigarrow A_*$$

whose composition agrees with MacPherson's natural transformation.

We are interested in explicit formulas for this *Lagrangian push-forward.* After embedding V'' in a nonsingular variety X and replacing V' by a resolution Y, we can reduce to the case of a proper morphism $f : Y \to X$ of nonsingular varieties. We are specifically interested in the image $f_*(T^*_Y Y)$ of the zero-section in this situation.

Theorem 1.1 will be a consequence of the following result. We recall the definition of 'trim' given in the introduction.

Definition 2.2. Let $\pi : Y \to S$ be a proper birational morphism of varieties, with Y nonsingular. For $0 \le d \le \dim Y$, let Y_d denote the locus where the rank of the differential $d\pi$ equals d. Then π is a *trim resolution* if $\dim Y_d < d$ for all $0 \le d < \dim Y$. ⌟

Proposition 2.3. *Let $f : Y \to X$ be a proper morphism of nonsingular varieties, such that $Y \to f(Y)$ is a trim resolution. Then $f_*(T^*_Y Y) = T^*_{f(Y)} X$.*

Remark 2.4. Let $\pi : Y \to S$ be a proper surjective morphism, with Y nonsingular, and $\iota : S \hookrightarrow X$ a closed embedding, with X nonsingular; and let $f = \iota \circ \pi : Y \to X$ be the composition. As explained above, $T^*_{f(Y)} X = T^*_S X$ may be viewed as an element of $\mathcal{L}(S)$; Proposition 2.3 states that if π is trim, then this is the image $\pi_*(T^*_Y Y)$ under $\pi_* : \mathcal{L}(Y) \to \mathcal{L}(S)$. ⌟

The rest of this section is devoted to the proof of Proposition 2.3.

A priori, $f_*(T^*_Y Y)$ is an integer linear combination of conormal cycles $\sum_Z m_Z T^*_Z X$, with $Z \subseteq f(Y)$. One of the summands is $T^*_{f(Y)} X$. The task is to show that if $Y \to f(Y)$ is trim and $Z \subsetneq f(Y)$ is a *proper* subvariety of $f(Y)$, then the coefficient of $T^*_Z X$ in $f_*(T^*_Y Y)$ is 0. Following Sabbah, we write f as the composition

$$Y \xrightarrow{\gamma} Y \times X \xrightarrow{\rho} X$$

where γ is the graph of f and ρ is the (smooth) projection. Let $p : \rho^*(T^*X) \to T^*X$ be the natural mophism. Then (cf. [Sab85, §2]) the components $T^*_Z X$ appearing in the decomposition of $f_*(T^*_Y Y)$ are the irreducible components of the image

$$p\left(\rho^*(T^*X) \cap T^*_{\gamma(Y)}(Y \times X)\right) \tag{2.1}$$

where we view both $\rho^*(T^*X)$ and $T^*_{\gamma(Y)}(Y \times X)$ as subschemes of $T^*(Y \times X)$. Let $T^*_Z X$ be one such component, and let z be a general point of Z and ξ a general covector in the fiber $(T^*_Z X)_z$. For (z, ξ) to be in the image (2.1), there must be a point $(y, z) \in \gamma(Y)$ mapping to z, such that $(0, \xi) \in T^*_y Y \oplus T^*_z X \cong T^*_{(y,z)}(Y \times X)$ vanishes on $T_{(y,z)}\gamma(Y)$.

Now, $(y, z) \in \gamma(Y)$ if and only if $y \in f^{-1}(z)$, while $(0, \xi)$ vanishes on $T_{(y,z)}\gamma(Y)$ if and only if ξ vanishes on $df(T_y Y)$, if and only if $f^*(\xi)$ vanishes on $T_y Y$. Therefore, $T^*_Z X$ is a component of (2.1) if and only if for a general $z \in Z$ and general $\xi \in (T^*_Z X)_z$, the *microlocal fiber* (cf. [BFL90, Definition 1.2])

$$F_{Z,z,\xi} := (\text{zero-scheme of } f^*(\xi)) \subseteq f^{-1}(z)$$

is nonempty. In order to prove Proposition 2.3, it suffices to prove that for every proper subvariety Z of $f(Y)$, the microlocal fiber $F = F_{Z,z,\xi}$ is empty for general $z \in Z$ and $\xi \in (T^*_Z X)_z$.

Now let Z be a subvariety of X and let Z° be its nonsingular part. Let $W^\circ = f^{-1}(Z^\circ)$ and consider the fiber product

$$W^\circ \times_{Z^\circ} T^*_{Z^\circ} X\,,$$

that is, the pull-back of the conormal bundle $T^*_{Z^\circ}X$ to W°. We denote points of this pull-back by pairs (w,ξ), where $w\in W^\circ$ and $\xi\in(T^*_{Z^\circ}X)_{f(w)}$ is a conormal vector to Z° at $f(w)$. Note that we have morphisms

$$W^\circ\times_{Z^\circ}T^*_{Z^\circ}X\hookrightarrow f^*T^*X|_{W^\circ}\longrightarrow T^*Y|_{W^\circ}\,;$$

we let $\mathcal{F}$ be the zero-scheme of this composition. Set-theoretically,

$$\mathcal{F}=\{(w,\xi)\,\text{s.t.}\,\xi|_{df(T_wY)}\equiv 0\}\,.$$

By construction we have projections $\mathcal{F}\to W^\circ$, $\mathcal{F}\to T^*_{Z^\circ}X$. The microlocal fiber F is naturally identified with the general fiber of the projection $\mathcal{F}\to T^*_{Z^\circ}X$. Therefore, in order to prove Proposition 2.3, it suffices to show that if Z is a proper subvariety of $f(Y)$, then $\dim\mathcal{F}<\dim(T^*_{Z^\circ}X)=\dim X$.

We will evaluate $\dim\mathcal{F}$ by considering the projection $\mathcal{F}\to W^\circ$.

For any $y\in Y$, let d_y be the rank of df at y, that is, the dimension $\dim(df(T_yY))$ of the image of T_yY in $T_{f(y)}X$. For $w\in W^\circ$, $df(T_wY)$ determines the subspace

$$T_w:=(df(T_wY)+T_{f(w)}Z^\circ)/T_{f(w)}Z^\circ$$

of the normal space $(TX/TZ^\circ)_{f(w)}$, with dimension $\geq d_w-\dim Z$. The condition that $\xi\in(T^*_{Z^\circ}X)_{f(w)}=(TX/TZ^\circ)^*_{f(w)}$ vanishes along $df(T_wY)$ is equivalent to the condition that it vanishes along T_w; hence

$$\dim\{\xi\in(T^*_{Z^\circ}X)_{f(w)}\,\text{s.t.}\,\xi|_{df(T_wY)}\equiv 0\}=(\dim X-\dim Z)-\dim T_w\leq\dim X-d_w\,. \tag{2.2}$$

This is the dimension of the fiber over w of the projection $\mathcal{F}\to W^\circ$.

Next, for $d\geq 0$ we let

$$W_d:=Y_d\cap W^\circ=\{w\in W^\circ\,\text{s.t.}\,d_w=d\}$$

and observe that $W^\circ=\cup_{0\leq d\leq\dim Y}W_d$. We have the bound $\dim W_d\leq\min(\dim Y_d,\dim W^\circ)$; in particular

- $\dim W_{\dim Y}\leq\dim W^\circ<\dim Y$, since Z is assumed to be a proper subvariety of $f(Y)$; and
- for $0\leq d<\dim Y$, we have $\dim W_d\leq\dim Y_d<d$ as $Y\to f(Y)$ is assumed to be trim.

Therefore, $\dim W_d<d$ for all d. It follows that $\dim\mathcal{F}<(\dim X-d)+d=\dim X$, and this concludes the proof of Proposition 2.3. □

3. Proof of Theorem 1.1

Let S be a variety admitting a trim resolution $\pi:Y\to S$; let $\iota:S\hookrightarrow X$ be an embedding in a nonsingular variety, and let $f=\iota\circ\pi:Y\to X$ be the composition. By the covariance of the Lagrangian push-forward and Proposition 2.3,

$$CC(f_*(1\!\!1_Y))=f_*(CC(1\!\!1_Y))=f_*((-1)^{\dim Y}T^*_YY)=(-1)^{\dim S}T^*_SX=CC(\mathrm{Eu}_S)$$

where we used the fact that Y is nonsingular (so that $1\!\!1_Y=\mathrm{Eu}_Y$) and that $\dim S=\dim Y$. Since CC is an isomorphism, this implies

$$f_*(1\!\!1_Y)=\mathrm{Eu}_S\,,$$

and by definition of push-forward of constructible functions, this means that

$$\mathrm{Eu}_S(s)=\chi(f^{-1}(s))\,,$$

where χ denotes the Euler characteristic. Since $f^{-1}(s)=\pi^{-1}(s)$ for $s\in S$, this proves the first assertion of Theorem 1.1.

The second assertion follows from the first by applying MacPherson's natural transformation. Alternatively, we can use the covariance of $\mathcal{L} \rightsquigarrow A_*$ (cf. Remark 2.4):

$$\begin{aligned} c_{\mathrm{Ma}}(S) &= c_*((-1)^{\dim S} T^*_S X) = c_*(\pi_*((-1)^{\dim S} T^*_Y Y)) = \pi_*(c_*((-1)^{\dim Y} T^*_Y Y)) \\ &= \pi_*(c_{\mathrm{Ma}}(Y))\,; \end{aligned}$$

since Y is nonsingular, $c_{\mathrm{Ma}}(Y) = c(TY) \cap [Y]$, so that $c_{\mathrm{str}}(S) := \pi_*(c(TY) \cap [Y]) = c_{\mathrm{Ma}}(S)$, concluding the proof.

Concerning the third point in Theorem 1.1, recall that the characteristic cycle of a complex of sheaves on a variety S embedded in a nonsingular variety X is the characteristic cycle (in $\mathcal{L}(X)$) of its stalk Euler characteristic. (This may be adopted as the definition of the characteristic cycle, or as an application of the local index formula, cf. [Dim04, Theorem 4.3.25(i)].) As explained in §2, the characteristic cycle may be viewed as an element of $\mathcal{L}(S)$. The third point of theorem 1.1 is a statement about the characteristic cycle of the intersection cohomology sheaf $\mathrm{IC}^\bullet_S$ of S.

Proposition 3.1. *Let $\pi : Y \to S$ be a trim proper birational map. Then π is small.*

Proof. Let $\pi : Y \to S$ be a trim proper birational map, let Z be a proper stratum of an adapted stratification of S, and let $W = \pi^{-1}(Z)$. Also, let $d(Z)$ be the common dimension of $\pi^{-1}(z)$ for $z \in Z$; thus, $\dim W = \dim Z + d(Z)$. The differential vanishes along directions tangent to the fibers, therefore $\dim \ker d_w\pi \geq d(Z)$ for a general w in a component of maximal dimension in W. Equivalently, $\mathrm{rk}\, d_w\pi \leq \dim Y - d(Z)$. Since π is trim and Z is a proper stratum, $\dim W < \mathrm{rk}\, d_w f$; hence $\dim W < \dim Y - d(Z)$. Therefore,

$$\dim Z + d(Z) < \dim Y - d(Z)$$

i.e.,

$$2d(Z) < \dim Y - \dim Z = \dim S - \dim Z = \mathrm{codim}_S Z\,.$$

Since this inequality holds for all proper strata Z of S, π is small. □

Now let S be a variety admitting a trim resolution $\pi : Y \to S$.

By Proposition 3.1, π is small. It follows (cf. [GM83, §6.2]) that the intersection cohomology sheaf $\mathrm{IC}^\bullet_S$ is the direct image of the intersection cohomology sheaf of Y. Since Y is nonsingular, the latter is a shift of the constant sheaf. Therefore,

$$\mathrm{IC}^\bullet_S = R\pi_*\mathbb{Q}_Y[\dim Y]\,.$$

(Concerning the shift, we follow the modern convention as in e.g., [dCM09, Remark 4.2.4].) The following formula for the stalk Euler characteristic of $\mathrm{IC}^\bullet_S$ at $z \in S$ is a consequence of standard properties of (derived) direct images:

$$\chi_z(\mathrm{IC}^\bullet_S) = \chi_z(R\pi_*\mathbb{Q}_Y[\dim Y]) = (-1)^{\dim S} \sum_i (-1)^i \dim H^i(\pi^{-1}(z); \mathbb{Q}) = (-1)^{\dim S} \chi(\pi^{-1}(z))\,.$$

By the first assertion in Theorem 1.1, this implies

$$\chi_z(\mathrm{IC}^\bullet_S) = (-1)^{\dim S}\,\mathrm{Eu}_S(z)\,.$$

Now embed S in a nonsingular variety X. The characteristic cycle of $\mathrm{IC}^\bullet_S$ is

$$CC(\chi_z(\mathrm{IC}^\bullet_S)) = CC((-1)^{\dim S}\,\mathrm{Eu}_S(z)) = T^*_S X\,,$$

and this verifies it is irreducible, completing the proof of Theorem 1.1.

4. Further remarks

1. In this note we have considered the notion of 'trim' only for proper birational morphisms because that is the case relevant to our application in Theorem 1.1. It could be taken as a template for more general proper maps, and it would be interesting to investigate corresponding generalizations of the main result. For instance, if $f : Y \to X$ is generically finite of degree m onto its image and satisfies the same dimensional constraints $\dim Y_d < d$ for $d < \dim Y$, then $f_*(T^*_Y Y) = mT^*_{f(Y)}X$, with the same argument given for Proposition 2.3.

Similarly, we have restricted attention to *complex* algebraic varieties to align with some standard literature, but the results should extend without change to algebraically closed fields of characteristic 0; see [Ken90] for a treatment of the Lagrangian functor in that generality. Also, the results should hold equivariantly; see [AMSS23, §3.2] for the relevant equivariant formalism of characteristic cycles and characteristic classes.

2. The relation between the notion of 'trim' and 'small' may be clarified by the following observation.

—A proper birational morphism $\pi : Y \to S$ is small if and only if the fiber product $Y \times_S Y$ has a unique component of dimension $\dim Y$ (cf. e.g., [dCM04, Remark 2.1.2]).

—A proper birational morphism $\pi : Y \to S$ is trim if and only if the linear fiber space associated with the sheaf of differentials $\Omega_{Y|S}$ has a unique component of dimension $\dim Y$.

Indeed, for all $y \in Y$ we have the exact sequence (tensor [Har77, II.8.11] by the residue field $\mathbb{C}(y)$)

$$T^*_{\pi(y)}S \longrightarrow T^*_y Y \longrightarrow \Omega_{Y|S} \otimes \mathbb{C}(y) \longrightarrow 0 \ ,$$

so the 'trim' condition is equivalent to the requirement that the *co*dimension of the locus where $\dim \Omega_{Y|S} \otimes \mathbb{C}(y)$ equals d is larger than d for all $d > 0$.

3. Let $\pi : Y \to S$ be a small stratified map such that the restriction $\pi^{-1}(Z) \to Z$ is a smooth morphism for all strata Z of S. (Here, $\pi^{-1}(Z)$ is the scheme-theoretic inverse image.) Then π is trim.

Indeed, we claim that under this hypothesis, the rank of the differential along $W = \pi^{-1}(Z)$ equals $\dim Y - d(Z)$. To verify this, consider the fiber square

$$\begin{tikzcd} W \arrow[r, hook, "j"] \arrow[d, "\rho"'] & Y \arrow[d, "\pi"] \\ Z \arrow[r, hook, "i"] & S \end{tikzcd}$$

and let $w \in W$, $z := \pi(w) \in Z$. Since the ideal sheaf $\mathcal{I}_Z$ of Z in X generates the ideal sheaf $\mathcal{I}_W$ of W in Y, there is a surjection $\rho^*(\mathcal{I}_Z/\mathcal{I}_Z^2) \twoheadrightarrow \mathcal{I}_W/\mathcal{I}_W^2$. Chasing the diagram

$$\begin{tikzcd} \rho^*(\mathcal{I}_Z/\mathcal{I}_Z^2) \arrow[r] \arrow[d] & j^*\pi^*\Omega_X \arrow[r] \arrow[d] & \rho^*\Omega_Z \arrow[r] \arrow[d] & 0 \\ \mathcal{I}_W/\mathcal{I}_W^2 \arrow[r] & j^*\Omega_Y \arrow[r] & \Omega_W \arrow[r] & 0 \end{tikzcd}$$

shows that the cokernels of the two right-most vertical maps are isomorphic. Tensoring by the residue field of $w \in W$ preserves cokernels, and dualizing shows that the kernels of the vertical maps in

$$\begin{tikzcd} T_wW \arrow[r, hook] \arrow[d, two heads, "\rho_*"'] & T_wY \arrow[d, "\pi_*"] \\ T_zZ \arrow[r, hook] & T_zX \end{tikzcd}$$

are equal. The induced map $\rho_* : T_wW \to T_zZ$ is surjective, since $\rho : W \to Z$ is smooth by assumption. It follows that $\ker \pi_* = \ker \rho_*$ has dimension $d(Z)$, and this proves our claim.

The smallness condition for a proper stratum Z is $2d(Z) < \dim Y - \dim Z$, which is then equivalent to

$$\dim W = \dim Z + d(z) < \dim Y - d(Z) = \operatorname{rk} d_w\pi$$

for all $w \in W$. For all $d < \dim Y$, the locus Y_d in Definition 2.2 is a (finite) union of inverse images of proper strata, so this shows that $\dim Y_d < d$ for $d < \dim Y$ and proves that π is trim.

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Mathematics Department, Florida State University, Tallahassee FL 32306, U.S.A.
E-mail address: `paluffi@fsu.edu`