Stringy Chern classes

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1 Stringy Euler number

Work of Dixon, Harvey, Vafa and Witten in the 80's ([DHVW85]) introduced a notion of Euler characteristic (for quotients of a torus by a finite group) which became known as the *physicist's orbifold Euler number.* In the 90's V. Batyrev introduced a notion of *stringy Euler number* ([Bat99b]) for 'arbitrary Kawamata log-terminal pairs', proving that this number agrees with the physicist's orbifold Euler number for algebraic varieties with a regular action of a finite group, and thereby proving a strong form of the McKay correspondence as conjectured by Miles Reid.

The stringy Euler number is one of a series of stringy invariants, defined in different contexts and at different levels of generality. Among many others, the work of Lev Borisov and Anatoly Libgober ([BL03], [BL]) stands out as probably the most advanced.

A natural question is whether invariants such as the stringy Euler number are numerical manifestations of more refined invariants. For example, recall that the conventional Euler characteristic of a compact nonsingular complex variety is the degree of the total Chern class of its tangent bundle (Poincaré-Hopf). Is there a 'stringy' Poincaré-Hopf theorem? That is, is the stringy Euler number the degree of a *stringy Chern class* naturally defined for singular varieties?

2 Celestial integrals and stringy Chern classes

Simultaneous work of de Fernex–Lupercio–Nevins–Uribe ([dFLNU]), and of the author ([Alu]) answered this question affirmatively, by explicitly constructing a class in the Chow group of a variety X (with at worst log-terminal singularities) whose degree is Batyrev's stringy Euler number.

The two approaches have points of contact, and produce the same class. De Fernex et al. make use of *motivic integration,* a tool which has provided a natural framework for defining and studying stringy invariants (an excellent survey is Wim Veys' paper [Vey]); and they include an explicit formula for quotient varieties amounting to a 'McKay correspondence' for this invariant.

It should be noted that the stringy Chern classes may also be obtained by specializing formulas of Lev Borisov and Anatoly Libgober for their *orbifold elliptic class.* Also, the 'relative' motivic framework in [dFLNU] is close in spirit to the one developed by Brasselet, Schürmann, and Yokura in [BSY].

My approach in [Alu] is only tangentially motivated by stringy invariants. My main motivation is to build a natural formalism for studying intersection theoretic invariants of birationally equivalent varieties; stringy invariants, and a close connection with the so-called *Chern-Schwartz-MacPherson classes,* are byproducts of the main construction.

To a variety X I associate a large group $A_*\mathcal{C}_X$ (containing the Chow group A_*X of X), by taking an inverse limit of Chow groups through the *system of modifications* (that is, proper birational maps onto X), organized by proper push-forwards. For certain data $\mathcal{D}$, $\mathcal{S}$ (arising, for example, from a divisor D and a constructible subset S of X) I define a distinguished element

$$\int_{\mathcal{S}} 1\!\!1(\mathcal{D})\, d\mathfrak{c}_X \in A_*\mathcal{C}_X \quad .$$

The definition relies crucially on the factorization theorem for birational maps of Abramovich et al., [AKMW02]; in particular, it relies on resolution of singularities, limiting the theory to characteristic zero for the time being.

The main property of these 'celestial[1] integrals' is that they satisfy a change-of-variable formula with respect to proper birational morphisms. This fact alone makes this formalism similar to motivic integration; and indeed some of the formulas arising in the theory of celestial integrals are similar to formulas from motivic integration, and some of its applications are similar to applications of motivic integration.

For example, celestial integrals may be used to compare Chern classes of birational varieties. Using the change-of-variable formula it is straightforward to prove (for example) that birational varieties in the same K-equivalence class (birational Calabi-Yau varieties provide examples) 'have the same Chern classes' in the Chow group of their common modification system (a fact proved 'by hand' in [Alu04]); this result should be seen as a parallel of an observation originally made by Batyrev ([Bat99a]).

As another application, it is equally straightforward to prove that if $\pi : Y \to X$ is a *crepant* resolution, then the push-forward $\pi_* c(TY) \cap [Y]$ is in fact independent of the resolution. This is the type of theorems which, at the level of Euler numbers, allowed Batyrev to define his stringy invariants.

In general, the integral

$$\int_{\mathcal{X}} 1\!\!1(0)\, d\mathfrak{c}_X$$

obtained by specializing to the divisor $\mathcal{D} = 0$, and computed on the whole modification system $\mathcal{X}$ of X, satisfies the basic requirement mentioned above:

[1]Terminology suggested by Prof. Matilde Marcolli, in view of the fact that modification systems are reminiscent of Hironaka's *voûte étoilée.*

that is, its degree agrees with Batyrev's stringy Euler number (and computing it for suitable divisors recovers Batyrev's stringy Euler number for Kawamata pairs). Thus, this integral is an appropriate definition of a *stringy Chern class* of a singular variety. In fact, the stringy Chern class in the Chow group *of* X is only one manifestation of this integral: the information carried by a class in the Chow group *of the modification system* $\mathcal{X}$ *of* X amounts to a distinguished choice of a stringy Chern class for X in the Chow group of *every variety birational to* X*,* compatibly with proper push-forwards.

One subtlety in the construction is that the integral depends on a coherent choice of a 'relative canonical divisor' through the system. The most conventional choice leads to the class mentioned above, but limits its scope to varieties with log-terminal singularities. This 'defect' is shared by other approaches to stringy invariants. A different choice for the relative canonical divisor leads to a flavor of the stringy class that is defined for arbitrary singularities; it would be interesting to establish a 'McKay correspondence' for this other flavor.

3 Schwartz-MacPherson classes as celestial integrals

An alternative celestial approach to obtaining 'natural' Chern classes for a singular variety X consists of embedding X in a nonsingular ambient variety M, and then computing

$$\int_{\mathcal{X}} \mathbb{1}(0)\, d\mathfrak{c}_M \quad :$$

the integral of 0 over the constructible subset $\mathcal{X}$ determined by X *in the modification system of* M. I prove in [Alu] that this class agrees with a known invariant, the *Chern-Schwartz-MacPherson* class of X. These invariants were introduced independently > 30 years ago by Marie-Hélène Schwartz ([Sch65b], [Sch65a]) and Robert MacPherson ([Mac74]); MacPherson's work provided a proof of a conjecture of Grothendieck and Deligne on the existence of a functorial notion of Chern classes for singular varieties. Celestial integrals provide a new construction of these classes.

In fact, it can be shown that *every* celestial integral may be reconstructed using MacPherson's natural transformation, from a well-defined constructible function depending on the specific data $\mathcal{D}$, $\mathcal{S}$. In particular, the stringy Chern class must correspond via MacPherson's natural transformation to a 'stringy constructible function', which would be very interesting to study further. The approach of de Fernex, Lupercio, Nevins, Uribe to stringy Chern classes gives one alternative construction of the stringy constructible function.

It would be likewise interesting to use the same formalism and translate into the language of constructible functions other items appearing in the theory of integrals over modification systems (and/or in the theory of motivic integrals), such as various flavors of *zeta functions.*

4 Deligne-Grothendieck conjecture

MacPherson's natural transformation links a functor of constructible functions (with push-forward defined by Euler characteristic of the fibers) to a homology functor, in such a way that on nonsingular varieties the constant function $1\!\!1$ is mapped to the (Poincaré dual of the) total Chern class of the tangent bundle. The existence of this natural transformation had been conjectured by Deligne and Grothendieck. In the particular case of maps to a point it reduces to a Poincaré-Hopf-type theorem for singular varieties, with respect to the conventional Euler number; incidentally, this had essentially been Schwartz's motivation in her work (which predates MacPherson's by several years). MacPherson's classes were shown to agree with Schwartz's in [BS81].

The algebraic version of the theory lifts MacPherson's natural transformation to the Chow group, while paying the price of limiting the allowed maps to be *proper.* While celestial integrals provide a new construction of MacPherson's classes, the class of maps allowed in the theory is further restricted to being proper and birational. The change-of-variable formula should morally be a manifestation of MacPherson's natural transformation, but it seems hard to transform this heuristic consideration into solid mathematics.

In recent work I take a different, but in some ways similar, approach: I construct Chow groups again by taking inverse limits, but on the category of 'maps to complete varieties'. This yields a notion of Chow group which agrees with the conventional notion for complete varieties, but is functorial with respect to arbitrary map.

Using this definition it is possible to systematically glue *local data* (say, defined on elements of a stratification of a given variety X) into *global data* for X, yielding another tool to produce intersection-theoretic data for (possibly) singular varieties, in terms of data specified for nonsingular (but possibly noncomplete) ones.

This yields a new construction of MacPherson's transformation, incorporating an independent proof of its naturality as prescribed by the Deligne-Grothendieck conjecture, and extending its 'algebraic' version to arbitrary maps (in characteristic zero). Specifically, Chern-Schwartz-MacPherson classes arise as the global version of local data amounting to the Chern classes of a bundle of differential forms with logarithmic poles along a divisor at infinity.

A natural question at this point is: what kind of local data glues to *stringy* global data? I can only speculate that the local-to-global formalism may produce a simpler approach to Borisov-Libgober's theory of orbifold elliptic classes, in particular leading to a new path to stringy Chern classes.

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