How many smooth plane cubics with given j-invariant are tangent to 8 lines in general position?

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Abstract. We employ a variety of 'complete cubics' to give formulas for the characteristic numbers of families parametrized by hypersurfaces F in the $\mathbb{P}^9$ of plane cubics, in terms of information easily accessible given the equation of F. As examples, we obtain explicit results for families of cubics with given j-invariant and for other families arising naturally from the geometry of plane cubics.

§0. Introduction.

The answer to the question posed in the title is 50,448 for $j \neq 0, 1728$; 16,816 for $j = 0$; and 25,224 for $j = 1728$ (Theorem III, §3). These are 'characteristic numbers' for the corresponding families of plane cubics: in general, if a family of cubics is parametrized by a subvariety F of the $\mathbb{P}^9$ of all plane cubics, then its 'k-th characteristic number' (denoted $F(k)$ in the following), is the number of elements in the family that are tangent *at smooth points* to k lines and contain $\dim F - k$ points in general position in the plane.

In [**A1**] we have studied a five blow-up construction over $\mathbb{P}^9$ (also considered by U. Sterz, [**St**]) yielding a smooth variety of 'complete cubics', and employed it to compute the characteristic numbers for the family of *all* smooth cubics (verifying classic results of Maillard [**M**] and Zeuthen [**Z**]). Such a variety should in fact contain *in nuce* the answer to all enumerative questions about contacts of (reduced) plane cubics. Unfortunately, our analysis in [**A1**] doesn't provide one with such a clear picture as say the famous variety of 'complete conics', and applying the construction to obtain the characteristic numbers for a given family of cubics requires in general a rather involved analysis of the behavior of the family through the blow-up stages. Examples of such computations are worked out in [**A1**], §5 for families of cubics tangent to given lines at given points, and in [**A2**] for various families of singular cubics.

In this note we will discuss one case in which the process is most transparent and the answer can be expressed most explicitly: the case of families parametrized by *hypersurfaces* of $\mathbb{P}^9$. We will see that for such families the characteristic numbers can be written explicitly in terms of just *three* pieces of information: the degree of the hypersurface parametrizing the family, and two numbers recording the local structure of the family along the set of 'triple lines' and along the set of cubics consisting of a line and a 'double line' (see Theorem I, §1). This is a modern version of a formula of Zeuthen's (in [**Z**]); Kleiman and Speiser [**KS**] also prove a similar statement, from a viewpoint closer to Zeuthen's (we see Theorem I in §1 as the meeting point of the two approaches of [**A1**] and [**KS**]). We want to stress the new element in our result: the analysis of [**A1**] makes the set of data straightforward to

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obtain if the hypersurface is given explicitly (Theorem II, §2). In fact, we will give in an appendix a Maple procedure that will compute all characteristic numbers of an 8-dimensional family of smooth plane cubics, given the equation of the hypersurface parametrizing it. We hope that this tool will be of some use in probing the field in search of general properties of these numbers, or as a cross-check for other approaches.

The numbers listed in the first paragraph can be obtained by applying the procedure to the equations of the degree-4 and degree-6 invariants of plane cubics (which we give explicitly in §3), giving the answer for $j = 0, 1728$ respectively, and extending the result to all other finite j's with a simple argument. The case $j = \infty$, i.e. the discriminant hypersurface in $\mathbb{P}^9$, parametrizing all *singular* cubics, is special: contributions to the characteristic numbers may come in this case from configurations in which the singular point lies on one of the lines. We have studied this case in detail in [**A2**], so here we will deal with it only in passing.

Other concrete examples we have chosen to illustrate the procedure are hypersurfaces expressing special positions of a flex or of a flex line of the cubic: the computation of the characteristic numbers for these families becomes an elementary exercise (see the appendix); one of the results also settles the computation of a pair of constants left unknown in [**Z**].

On a different track, Theorem I exposes general features of the characteristic numbers for 8-dimensional families of cubics; some of these could be expected from the general set-up of the problem, some others seem to us quite remarkable. For example, the 8-th characteristic number of a family of plane cubics parametrized by a hypersurface F of $\mathbb{P}^9$ depends only on the degree of F and on its local structure along the set of triple lines: that this number doesn't depend on the behavior along the larger set of non-reduced cubics reflects the fact that the limit of the dual of a curve as it approaches the union of a line and a (distinct) double line forms a set of dimension < 8 in the image of the dual map (this is also a result in [**KS**], or can be derived from [**K**], Examples 3.4 d,e). The fact that the characteristic numbers depend on just three numbers reflects the fact that the Picard group of the normalization of the graph of the dual map (dominated by the variety constructed in [**A1**]) has three basic generators, a result of [**KS**].

Another curious consequence of the formulas in Theorem I in §1 is that the characteristic numbers of *any* family parametrized by a hypersurface of a given degree d are congruent to d mod 3. This also explains why the 10 characteristic numbers for smooth cubics are all congruent to 1 mod 3: these can all be expressed as characteristic numbers of the degree-4 hypersurface of cubics tangent to a given line, and of the hyperplane of cubics containing a given point.

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§**1. The main formula.** As in [**A1**], we work over an algebraically closed field of characteristic $\neq 2, 3$.

Our main formula comes from specializing the results of [**A1**] (specifically Theorem IV) to the case in which the family considered is parametrized by a codimension-1 (maybe non-complete) subvariety F of $\mathbb{P}^9$. As observed in [**A1**], §1, the characteristic numbers don't change if F is replaced by its closure in $\mathbb{P}^9$; so we just refer to F as a hypersurface, unless this might create ambiguities; also, we assume that the closure of F does not contain the discriminant hypersurface. As shown in [**A1**], computing the characteristic numbers for F amounts then to computing five 'full intersection classes'

$$B_i \circ \ F_i = c(N_{B_i} V_i) s(B_i \cap F_i, F_i) \quad ,$$

where $B_0, \dots, B_4$ are the varieties described in Theorem III in [**A1**], $V_0 = \mathbb{P}^9$, V_i is the blow-up of V_{i-1} along B_{i-1}, and F_i denotes the proper transform in V_i of the closure F_0 of F in $\mathbb{P}^9$.

Given F, denote by m_i, $i = 0, \dots, 4$ the multiplicity of F_i along the center B_i of the $(i+1)$-th blow-up; then let

$$M = 2m_0 + m_1 + m_2, \quad N = m_3 + m_4.$$

So with each hypersurface F of $\mathbb{P}^9$ there are associated three numbers: the degree d of F and the two numbers M, N.

THEOREM I. *Suppose (the closure of) F does not contain the discriminant hypersurface. Then, with d, M, N as above, the characteristic numbers for F are*

$$F(k) = \begin{cases} d & k = 0 \\ 4d & k = 1 \\ 16d & k = 2 \\ 64d & k = 3 \\ 256d - 24N & k = 4 \\ 976d - 240N & k = 5 \\ 3424d - 885N - 360M & k = 6 \\ 9766d - 1470N - 2520M & k = 7 \\ 21004d - 8400M & k = 8 \end{cases}$$

NOTE. These imply formula (1) in [**Z**], p. 727. Zeuthen proceeds then to find the values for M, N $(= B/40$, A in his notations[1]) for the hypersurface formed by the cubics tangent to a given line, by a very beautiful interplay of different relations with other enumerative results. The point of Theorem I here is not so much to give a modern version of Zeuthen's formulas (for which we could quote [**KS**], Corollary 3.2 and Propositions 5.5, 6.2), but the fact that the blow-ups of [**A1**] give the integers M, N *explicitly.* We'll exploit this in §3, and give a method to compute M, N directly for any given hypersurface of $\mathbb{P}^9$.

[1] This denominator '40' is nicely explained at the end of the introduction of [**KS**].

Proof: We can assume that F is irreducible, and that the general element of F is non-singular as a plane cubic: so (in the terminology of [**A1**]) the tangencies will be automatically *proper*, and by Theorem I in [**A1**] elements of F will contribute with multiplicity one. By Theorem IV in [**A1**], the k-th characteristic number for F is given by

$$F(k) = 4^k \cdot d - \sum_{i=0}^{4} \int_{B_i} \frac{(B_i \circ P_i)^{8-k}(B_i \circ L_i)^k (B_i \circ F_i)}{c(N_{B_i} V_i)}$$

where $B_i \circ P_i$, $B_i \circ L_i$, $c(N_{B_i} V_i)$ are given in Theorem III in [**A1**]. Also, $B_i \circ F_i = m_i[B_i] + B_i \cdot F_i$ (by [**A1**], §2): it's clear then that all the information is there. As an illustration, the computation for $k = 6$ runs:

$$\begin{aligned} F(6) &= 4^6 \cdot d - \int_{B_0} \frac{(3h)^2(2+12h)^6(1+h)^3(m_0+3dh)}{(1+3h)^{10}} - \int_{B_1} \text{etc.} \\ &= 4096d - (576m_0) - (81m_0 + 279m_1) - (639m_0 + 369m_1 + 648m_2) \\ &\quad - (390d + 1092m_3 - 360m_0 - 180m_1 - 180m_2) - (282d - 207m_3 \\ &\quad + 885m_4 - 216m_0 - 108m_1 - 108m_2) \\ &= 3424d - 885(m_3 + m_4) - 360(2m_0 + m_1 + m_2) \\ &= 3424d - 885N - 360M \quad . \blacksquare \end{aligned}$$

The fact that all contributions of the m_i's will group in each case to contribution of $M = 2m_0 + m_1 + m_2$ and $N = m_3 + m_4$ seems rather magic, but finds partly an explanation in the Picard group of the normalization of the graph of the dual map having three basic generators (see [**KS**], particularly section 2): indeed, the characteristic numbers compute the pull-back of nine intersection products from the graph (which is dominated by $\widetilde{V}$), so they all depend only on the three numbers specifying the class in the graph of the divisor determined by F.

We quote a couple of immediate consequences of Theorem I here, since they raise questions that seem rather interesting to us.

Corollary 1. *The maximum characteristic numbers for a hypersurface F of $\mathbb{P}^9$ of degree d are achieved by all and only the hypersurfaces not containing the set of triple lines, and they are in such case*

$$d, 4d, 16d, 64d, 256d, 976d, 3424d, 9766d, 21004d$$

Proof: $M, N \geq 0$ always; for hypersurfaces not containing the locus of triple lines, $M = N = 0$. $\blacksquare$

Can one give a lower bound? Is there a hypersurface F of some degree d for which $M = 5d/2$? Such a family would have the impressively low $F(8) = 4d$. Can this be achieved?

Corollary 2. *The characteristic numbers of a family of cubics parametrized by a hypersurface of degree d of $\mathbb{P}^9$ are congruent to d modulo 3.*

Proof: Just read Theorem I modulo 3. ∎

It is tempting to conjecture that such a pleasant symmetry must be an instance of a very general statement. The obvious guess is that the the statement of Corollary 2 holds for plane curves of any degree, modulo a suitable integer. Unfortunately this is in contrast with known results about quartic curves, so such general statement must be discarded. What is the right conjecture?

§2. The blow-ups in coordinates. As shown in §1, the characteristic numbers for a family parametrized by a hypersurface F of $\mathbb{P}^9$ are determined by the degree of F and by two numbers encoding the behavior of F through the five blow-ups constructing the variety of complete cubics. Computing these two numbers from the equation of F will be easy once the blow-ups are explicitly written out in coordinates, over suitable open sets of the V_i's (the only requirement on these open sets is not to be disjoint from the B_i's).

A description of the first three blow-ups was already needed in [**A1**], and we simply reproduce it here. We give homogeneous coordinates $(x_0 : x_1 : x_2)$ to $\mathbb{P}^2$ and $(a_0 : a_1 : \cdots : a_9)$ to $\mathbb{P}^9$, so that the cubic of coordinates $(a_0 : \cdots : a_9)$ has equation

$$a_0x_0^3 + a_1x_0^2x_1 + a_2x_0^2x_2 + a_3x_0x_1^2 + a_4x_0x_1x_2 \\ + a_5x_0x_2^2 + a_6x_1^3 + a_7x_1^2x_2 + a_8x_1x_2^2 + a_9x_2^3 = 0 \quad .$$

Then we have coordinates $(a_1, \ldots, a_9)$ for the open set $\{a_0 \neq 0\}$ in $\mathbb{P}^9$, and one can give coordinates $(b_1, \ldots, b_9)$ in V_1, $(c_1, \ldots, c_9)$ in V_2, and $(d_1, \ldots, d_9)$ in V_3 such that ([**A1**], §§3.1,2,3)

$$\text{(1)} \qquad \begin{array}{ccc} b_1 = a_1 & b_2 = a_2 & b_3 = 3a_3 - a_1^2 \\ b_4b_3 = 3a_4 - 2a_1a_2 & b_5b_3 = 3a_5 - a_2^2 & b_6b_3 = 9a_6 - a_1a_3 \\ b_7b_3 = 3a_7 - a_2a_3 & b_8b_3 = 3a_8 - a_1a_5 & b_9b_3 = 9a_9 - a_2a_5 \end{array}$$

$$\text{(2)} \qquad \begin{array}{ccc} c_1 = b_1 & c_2 = b_2 & c_3c_6 = b_3 \\ c_4 = b_4 & c_5 = b_5 & c_6 = 3b_6 - 2b_1 \\ c_7c_6 = 3b_7 - b_1b_4 & c_8c_6 = 3b_8 - b_2b_4 & c_9c_6 = 3b_9 - 2b_2b_5 \end{array}$$

$$\text{(3)} \qquad \begin{array}{ccc} d_1 = c_1 & d_2 = c_2 & d_3 = c_3 \\ d_4 = c_4 & d_5 = c_5 & d_6d_3 = c_6 \\ d_7 = c_7 & d_8 = c_8 & d_9 = c_9 \end{array} \quad .$$

For the fourth and fifth blow-ups, recall that the centers B_3, B_4 are isomorphic to the blow-up of $\mathbb{P}^2 \times \mathbb{P}^2$ along its diagonal: we give coordinates $(\alpha_1, \alpha_2, u, t)$ in B_3, B_4 so that the blow-up map to $\mathbb{P}^2 \times \mathbb{P}^2$ is

$$(\alpha_1, \alpha_2, u, t) \mapsto ((\alpha_1 + u, \alpha_2 + ut), (\alpha_1, \alpha_2)) \quad .$$

With this description, the map $B_3 \hookrightarrow V_3$ can be written ([**A1**], §3.3)

$$(\alpha_1, \alpha_2, u, t) \mapsto (3\alpha_1 + u, 3\alpha_2 + ut, \frac{u}{2}, 2t, t^2, -4, t, t^2, t^3) \quad .$$

Equations for B_3 in (this open set of) V_3 are therefore

$$\begin{cases} 4d_5 - d_4^2 = 0 \\ d_6 + 4 = 0 \\ 2d_7 - d_4 = 0 \\ 4d_8 - d_4^2 = 0 \\ 8d_9 - d_4^3 = 0 \end{cases} ,$$

and we can choose coordinates $(e_1, \dots, e_9)$ for (an affine open set of) V_4 so that

$$\begin{array}{ccc} e_1 = d_1 & e_2 = d_2 & e_3 = d_3 \\ e_4 = d_4 & e_5 = 4d_5 - d_4^2 & e_6 e_5 = d_6 + 4 \\ e_7 e_5 = 2d_7 - d_4 & e_8 e_5 = 4d_8 - d_4^2 & e_9 e_5 = 8d_9 - d_4^3 \end{array} \tag{4}$$

To obtain equations for B_4 in V_4, recall its construction from [**A1**], §3.4. If a point $(\alpha_1, \alpha_2, u, t) \in B_3$, and $u \neq 0$, then a neighborhood of its image in V_3 is isomorphic to a neighborhood of the cubic

$$(x_0 + (\alpha_1 + u)x_1 + (\alpha_2 + ut)x_2)(x_0 + \alpha_1 x_1 + \alpha_2 x_2)^2$$

in $\mathbb{P}^9$, consisting of the line $x_0 + (\alpha_1 + u)x_1 + (\alpha_2 + ut)x_2 = 0$ and of the double line supported on $x_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$. The tangent space to B_3 at $(\alpha_1, \alpha_2, u, t)$ is then identified with the four-dimensional space of cubics consisting of the line $x_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$ and of a conic containing the point $(\alpha_1 t - \alpha_2 : -t : 1)$ where the two lines intersect. The five-dimensional space of cubics containing the line $x_0 + \alpha_1 x_1 + \alpha_2 x_2 = 0$ determines then a point in the exceptional divisor E_4 over $(\alpha_1, \alpha_2, u, t)$: and B_4 is the set of all such points obtained as $(\alpha_1, \alpha_2, u, t)$ moves in B_3. To get a parametrization of B_4, consider the direction (in $\mathbb{P}^9$)

$$s \mapsto (x_0 + \alpha_1 x_1 + \alpha_2 x_2)^2 (x_0 + (\alpha_1 + u)x_1 + (\alpha_2 + ut)x_2) + s(x_0 + \alpha_1 x_1 + \alpha_2 x_2)x_2^2.$$

This is normal to B_3 and lies in the five-dimensional space defined above, so it determines the point in B_4 above $(\alpha_1, \alpha_2, u, t)$. Tracing the coordinates, this gives the curve

$$s \mapsto (3\alpha_1 + u, 3\alpha_2 + ut, \frac{u}{2}, 2t, -12\frac{s}{u^2}, 0, 0, \frac{1}{2}, 3t)$$

in V_4, converging to

$$(3\alpha_1 + u, 3\alpha_2 + ut, \frac{u}{2}, 2t, 0, 0, 0, \frac{1}{2}, 3t)$$

as $s \to 0$. This parametrization globalizes over $\{u = 0\}$ as well, so equations for B_4 are

$$\begin{cases} e_5 = 0 \\ e_6 = 0 \\ e_7 = 0 \\ 2e_8 - 1 = 0 \\ 2e_9 - 3e_4 = 0 \end{cases},$$

and we can choose coordinates $(f_1, \ldots, f_9)$ in V_5 so that

$$\begin{array}{ccc} f_1 = e_1 & f_2 = e_2 & f_3 = e_3 \\ f_4 = e_4 & f_5 = e_5 & f_6 f_5 = e_6 \\ f_7 f_5 = e_7 & f_8 f_5 = 2e_8 - 1 & f_9 f_5 = 2e_9 - 3e_4 \end{array} . \tag{5}$$

The equation of the exceptional divisor in $V_5 = \widetilde{V}$ is then $f_5 = 0$.

Composing the maps described by the set of equations (1)–(5), we get a coordinate description of the map $\widetilde{V} \to \mathbb{P}^9$ from the variety of complete cubics to the $\mathbb{P}^9$ of ordinary cubics. Explicitly, one finds the rather unpleasant-looking list of equations:

$$\begin{aligned}
a_1 &= f_1, \quad a_2 = f_2, \quad a_3 = -\frac{4}{3} f_3^2 + \frac{1}{3} f_3^2 f_5^2 f_6 + \frac{1}{3} f_1^2, \\
a_4 &= -\frac{4}{3} f_4 f_3^2 + \frac{1}{3} f_4 f_3^2 f_5^2 f_6 + \frac{2}{3} f_1 f_2, \\
a_5 &= -\frac{1}{3} f_4^2 f_3^2 + \frac{1}{12} f_4^2 f_3^2 f_5^2 f_6 - \frac{1}{3} f_5 f_3^2 + \frac{1}{12} f_5^3 f_3^2 f_6 + \frac{1}{3} f_2^2 \\
a_6 &= \frac{16}{27} f_3^3 - \frac{8}{27} f_3^3 f_5^2 f_6 + \frac{1}{27} f_3^3 f_5^4 f_6^2 - \frac{4}{9} f_1 f_3^2 + \frac{1}{9} f_1 f_3^2 f_5^2 f_6 + \frac{1}{27} f_1^3, \\
a_7 &= \frac{8}{9} f_4 f_3^3 - \frac{4}{9} f_4 f_3^3 f_5^2 f_6 + \frac{1}{18} f_4 f_3^3 f_5^4 f_6^2 + \frac{8}{9} f_7 f_5^2 f_3^3 - \frac{4}{9} f_7 f_5^4 f_3^3 f_6 \\
&\quad + \frac{1}{18} f_7 f_5^6 f_3^3 f_6^2 - \frac{4}{9} f_1 f_4 f_3^2 + \frac{1}{9} f_1 f_4 f_3^2 f_5^2 f_6 - \frac{4}{9} f_2 f_3^2 + \frac{1}{9} f_2 f_3^2 f_5^2 f_6 + \frac{1}{9} f_2 f_1^2, \\
a_8 &= \frac{4}{9} f_4^2 f_3^3 - \frac{2}{9} f_4^2 f_3^3 f_5^2 f_6 + \frac{1}{36} f_4^2 f_3^3 f_5^4 f_6^2 + \frac{2}{9} f_8 f_5^2 f_3^3 - \frac{1}{9} f_8 f_5^4 f_3^3 f_6 \\
&\quad + \frac{1}{72} f_8 f_5^6 f_3^3 f_6^2 + \frac{2}{9} f_5 f_3^3 - \frac{1}{9} f_5^3 f_3^3 f_6 + \frac{1}{72} f_5^5 f_3^3 f_6^2 - \frac{4}{9} f_2 f_4 f_3^2 + \frac{1}{9} f_2 f_4 f_3^2 f_5^2 f_6 \\
&\quad - \frac{1}{9} f_1 f_4^2 f_3^2 + \frac{1}{36} f_1 f_4^2 f_3^2 f_5^2 f_6 - \frac{1}{9} f_1 f_5 f_3^2 + \frac{1}{36} f_1 f_5^3 f_3^2 f_6 + \frac{1}{9} f_1 f_2^2, \\
a_9 &= \frac{2}{27} f_4^3 f_3^3 - \frac{1}{27} f_4^3 f_3^3 f_5^2 f_6 + \frac{1}{216} f_4^3 f_3^3 f_5^4 f_6^2 + \frac{1}{27} f_9 f_5^2 f_3^3 - \frac{1}{54} f_9 f_5^4 f_3^3 f_6 \\
&\quad + \frac{1}{432} f_9 f_5^6 f_3^3 f_6^2 + \frac{1}{9} f_4 f_5 f_3^3 - \frac{1}{18} f_4 f_5^3 f_3^3 f_6 + \frac{1}{144} f_4 f_5^5 f_3^3 f_6^2 - \frac{1}{9} f_2 f_4^2 f_3^2 \\
&\quad + \frac{1}{36} f_2 f_4^2 f_3^2 f_5^2 f_6 - \frac{1}{9} f_2 f_5 f_3^2 + \frac{1}{36} f_2 f_5^3 f_3^2 f_6 + \frac{1}{27} f_2^3 \quad ;
\end{aligned}$$

these give the other main tool in the computation:

Theorem II. *(Notations of Theorem I) If $F(a_0 : \cdots : a_9) = 0$ is the equation of the hypersurface parametrizing the family, then the numbers M, N are resp. the highest power of f_3, f_5 dividing*

$$F(1 : f_1 : f_2 : -\frac{4}{3}f_3^2 + \frac{1}{3}f_3^2 f_5^2 f_6 + \frac{1}{3}f_1^2 : \dots) \quad .$$

Proof: The highest powers of f_3, f_5 dividing $F(1 : f_1 : f_2 : \dots)$ are resp. the coefficients of the third and fifth exceptional divisors in the inverse image of the hypersurface, and these are easily seen to be M, N. Or, simply trace (1)–(5) and the definition of the multiplicities $m_0, \dots, m_5$: for example, m_0 is the highest power of b_3 dividing $F(1 : b_1 : b_2 : \frac{1}{3}b_3 + \frac{1}{3}b_1^2 : \dots)$, therefore the highest power of c_3 dividing $F(1 : c_1 : c_2 : \dots)$; and $m_0 + m_1$ is the highest power of c_6 dividing $F(1 : c_1 : c_2 : \dots)$, so $M = 2m_0 + m_1 + m_2$ is the highest power of d_3 dividing $F(1 : d_1 : d_2 : \dots)$. The statement for M follows easily.

Notice that the coordinate description does not cover the case $F = a_0$; but in this case $M = N = 0$, and the statements hold trivially. ∎

As an illustration, consider the family parametrized by

$$F(a_0 : \cdots : a_9) = a_3^2 - 3a_1 a_6 \quad ;$$

pulling-back to $\widetilde{V}$:

$$\begin{aligned} F(1 : f_1 : f_2 : \dots) &= (-\frac{4}{3}\,f_3^2 + \frac{1}{3}\,f_3^2 f_5^2 f_6 + \frac{1}{3}\,f_1^2)^2 - 3f_1(\frac{16}{27}\,f_3^3 - \frac{8}{27}\,f_3^3 f_5^2 f_6 \\ &\quad + \frac{1}{27}\,f_3^3 f_5^4 f_6^2 - \frac{4}{9}\,f_1 f_3^2 + \frac{1}{9}\,f_1 f_3^2 f_5^2 f_6 + \frac{1}{27}\,f_1^3) \\ &= -\frac{1}{9}f_3^2(-4 + f_5^2 f_6)(-f_3^2 f_5^2 f_6 + f_1 f_3 f_5^2 f_6 + f_1^2 + 4f_3^2 - 4f_1 f_3) \end{aligned}$$

Therefore $M = 2, N = 0$ by Theorem II, and the characteristic numbers for this family are

2, 8, 32, 128, 512, 1952, 6128, 14492, 25208

as $k = 0, \dots, 8$, by Theorem I.

As an other example,

$$F(a_0 : \cdots : a_9) = 4a_5^3 a_0 - 18a_9 a_2 a_5 a_0 - a_2^2 a_5^2 + 4a_2^3 a_9 + 27a_9^2 a_0^2$$

is the equation of the set of all cubics tangent to the line $x_1 = 0$; therefore its characteristic numbers will be the last 9 of the characteristic numbers for the family of all smooth cubics. For this equation

$$\begin{aligned} F(1 : f_1 : f_2 : \dots) = \frac{1}{6912} f_3^6 f_5^2 (f_5^2 f_6 - 4)^3 (f_9^2 f_5^4 f_6 + 6f_9 f_5^3 f_4 f_6 + 4f_4^3 f_9 f_5^2 f_6 \\ + 9f_4^2 f_6 f_5^2 - 4f_9^2 f_5^2 + 12f_4^4 f_5 f_6 - 24f_9 f_5 f_4 + 16f_5 + 4f_4^6 f_6 - 16f_4^3 f_9 + 12f_4^2) \end{aligned}$$

so $M=6, N=2$, and the characteristic numbers are indeed

$$4,\ 16,\ 64,\ 256,\ 976,\ 3424,\ 9766,\ 21004,\ 33616 \quad ,$$

as listed in [**Z**], [**KS**], or [**A1**].

§**3. Cubics with given j-invariant.** We want to illustrate Theorems I and II by applying them to families of smooth cubic curves with a given j-invariant. Recall then that the equation of such a family is

$$j = \frac{1728 C_4^3}{C_4^3 - C_6^2} \qquad\qquad (j \neq 0, 1728)$$

where C_4, C_6 are the classic degree-4 and degree-6 invariants of plane cubics, suitably normalized (see e.g. [**Si**], III, §1). For $j=0$ or 1728, the above equation becomes resp. $C_4^3=0$, $C_6^2=0$ (as the extra automorphisms of the corresponding curves cause these hypersurfaces to wrap on themselves); reduced equations are then $C_4=0$, $C_6=0$.

What are C_4, C_6 *explicitly* in the coordinates $a_0 : \cdots : a_9$ of §2? At a loss with a reference, we have to list them here! We will actually list $16C_4$ and $64C_6$, to avoid denominators:

- $16C_4$:

$$\begin{aligned}
&a_4^4 + 16a_2^2a_7^2 + 16a_1^2a_8^2 + 16a_3^2a_5^2 - 48a_2a_3^2a_9 - 48a_0a_3a_8^2 - 48a_1^2a_7a_9 - 48a_0a_5a_7^2\\
&- 16a_2a_3a_5a_7 + 144a_1a_2a_9a_6 + 24a_1a_3a_4a_9 - 216a_0a_4a_9a_6 + 24a_0a_4a_7a_8\\
&- 16a_1a_3a_5a_8 - 8a_2a_4^2a_7 + 24a_2a_4a_5a_6 - 8a_1a_4^2a_8 - 16a_1a_2a_7a_8 + 144a_0a_5a_8a_6\\
&+ 24a_2a_3a_4a_8 - 48a_1a_5^2a_6 + 24a_1a_4a_5a_7 - 8a_3a_4^2a_5 + 144a_0a_3a_7a_9 - 48a_2^2a_8a_6
\end{aligned}$$

- $64C_6$:

$$\begin{aligned}
&a_4^6 - 64a_5^3a_3^3 - 64a_2^3a_7^3 - 64a_8^3a_1^3 - 864a_5^3a_0a_6^2 - 576a_7^2a_0a_3a_5^2 + 36a_8a_0a_4^3a_7\\
&+ 864a_8^2a_0a_2a_4a_6 + 216a_9^2a_3^2a_1^2 - 1296a_8a_0a_6a_2a_5a_7 - 144a_5a_0a_8a_7^2a_1\\
&+ 720a_5a_0a_8a_3a_7a_4 - 72a_5a_0a_4^2a_7^2 + 288a_2a_0a_5a_7^3 - 144a_8a_0a_2a_4a_7^2i\\
&- 144a_8^2a_0a_7a_4a_1 - 144a_5a_8a_7a_4a_1^2 - 864a_8^3a_0^2a_6 - 144a_8^2a_0a_3a_7a_2 + 216a_8^2a_0^2a_7^2\\
&+ 48a_5a_8a_3a_7a_2a_1 + 864a_7a_0a_4a_6a_5^2 - 5832a_9^2a_0^2a_6^2 - 864a_9^2a_0a_3^3 - 12a_8a_4^4a_1\\
&+ 288a_9a_2a_5a_3^3 - 864a_9a_0^2a_7^3 - 12a_4^4a_3a_5 + 48a_4^2a_3^2a_5^2 + 288a_5^3a_3a_6a_1\\
&+ 96a_8a_2^2a_7^2a_1 - 864a_9^2a_6a_1^3 - 72a_5^2a_6a_4^2a_1 - 576a_8^2a_2^2a_6a_1 - 144a_8a_3^2a_2a_4a_5\\
&+ 36a_9a_3a_4^3a_1 + 216a_5^2a_2^2a_6^2 - 12a_2a_4^4a_7 + 864a_5a_0a_8^2a_6a_1 + 3888a_2a_0a_5a_9a_6^2\\
&- 864a_2^3a_9a_6^2 + 216a_8^2a_3^2a_2^2 - 576a_2^2a_3^2a_9a_7 - 72a_9a_3^2a_4^2a_2 + 864a_2^2a_4a_6a_9a_3\\
&+ 96a_5a_3a_2^2a_7^2 + 36a_2a_5a_4^3a_6 - 72a_8a_6a_4^2a_2^2 - 144a_8a_6a_2^2a_5a_3 + 96a_5^2a_3^2a_7a_2\\
&- 144a_3a_5^2a_6a_4a_2 + 288a_8a_9a_7a_1^3 + 48a_2^2a_4^2a_7^2 + 36a_8a_4^3a_2a_3 + 24a_3a_5a_4^2a_2a_7\\
&+ 48a_8^2a_4^2a_1^2 + 864a_8a_0a_3^2a_9a_4 + 216a_7^2a_5^2a_1^2 - 576a_5a_0a_8^2a_3^2 + 288a_8a_6a_2^3a_7
\end{aligned}$$

$$
\begin{aligned}
&+ 864a_8a_0a_6a_5^2a_3 - 144a_2^2a_4a_6a_5a_7 + 864a_5a_6a_9a_4a_1^2 - 576a_9a_7^2a_2a_1^2 \\
&+ 864a_8a_6a_9a_2a_1^2 + 96a_8^2a_7a_2a_1^2 + 96a_5a_8^2a_3a_1^2 + 96a_8a_3^2a_5^2a_1 - 144a_8a_9a_4a_3a_1^2 \\
&+ 540a_9a_0a_6a_4^3 - 144a_5a_3^2a_9a_4a_1 - 144a_8a_3^2a_9a_2a_1 + 36a_5a_4^3a_7a_1 \\
&- 144a_8a_3a_4a_2^2a_7 - 576a_8a_6a_5^2a_1^2 - 648a_9a_6a_4^2a_2a_1 + 864a_2^2a_6a_9a_7a_1 \\
&+ 720a_2a_4a_3a_9a_7a_1 - 1296a_5a_6a_9a_2a_3a_1 - 144a_5^2a_6a_2a_7a_1 - 144a_5a_2a_4a_7^2a_1 \\
&+ 720a_8a_6a_4a_2a_5a_1 - 72a_8^2a_0a_3a_4^2 + 288a_8^3a_0a_3a_1 - 72a_9a_4^2a_7a_1^2 \\
&- 1296a_8a_0a_6a_9a_4a_1 - 144a_5a_3a_9a_7a_1^2 - 144a_5^2a_3a_7a_4a_1 - 144a_8^2a_3a_4a_2a_1 \\
&+ 24a_8a_4^2a_2a_7a_1 + 24a_8a_4^2a_5a_3a_1 - 1296a_8a_0a_6a_9a_2a_3 + 3888a_8a_0^2a_6a_9a_7 \\
&- 1296a_8a_0a_3a_9a_7a_1 + 864a_5a_0a_3^2a_9a_7 + 864a_9a_0a_3a_7^2a_2 - 648a_9a_0a_4^2a_7a_3 \\
&+ 864a_9a_0a_7^2a_4a_1 - 648a_8a_0a_6a_4^2a_5 + 3888a_9^2a_0a_3a_6a_1 - 1296a_5a_0a_3a_6a_9a_4 \\
&- 1296a_5a_0a_6a_9a_7a_1 - 1296a_2a_0a_4a_6a_9a_7 \quad .
\end{aligned}
$$

Manipulating such (seemingly huge) polynomials is well within reach of today's personal computers. We used the Maple implementation on a Cadmus computer to apply Theorem II and get

$$
\begin{aligned}
&\text{for } C_4\text{:} \qquad M = 8,\ N = 4 \quad ; \\
&\text{for } C_6\text{:} \qquad M = 12,\ N = 6 \quad .
\end{aligned}
$$

Thus Theorem I gives immediately

THEOREM III(1). *The characteristic numbers for the families $F_{(0)}$, $F_{(1728)}$ of cubic curves with j-invariant $= 0, 1728$ are*

$$
F_{(0)}(k) = \begin{cases} 4 \\ 16 \\ 64 \\ 256 \\ 928 \\ 2944 \\ 7276 \\ 13024 \\ 16816 \end{cases} \qquad F_{(1728)}(k) = \begin{cases} 6 & k=0 \\ 24 & k=1 \\ 96 & k=2 \\ 384 & k=3 \\ 1392 & k=4 \\ 4416 & k=5 \\ 10914 & k=6 \\ 19536 & k=7 \\ 25224 & k=8 \end{cases}
$$

For all other j, the equation is

$$
(*) \qquad (j - 1728)C_4^3 - jC_6^2 = 0
$$

Now, the initial form with respect to f_3, f_5 of the pull-backs of C_4, C_6 to $\widetilde{V}$, in the coordinates $(f_1, \ldots, f_9)$ are

–for C_4:

$$
\frac{64}{81} f_3^8 f_5^4 (16 f_4^2 f_7^2 + f_6 - 8 f_7 f_9 + 4 f_8^2 - 8 f_4 f_7 f_8)
$$

–for C_6:

$$
\begin{aligned}
&-\frac{512}{729}f_3^{12}f_5^6(-108f_7^2-6f_8f_6+64f_4^3f_7^3-f_9^2f_6+8f_8^3+24f_4f_7f_6\\
&-24f_8^2f_4f_7-9f_8^2f_6f_4^2-48f_4^2f_8f_7^2-36f_4^4f_6f_7^2+36f_4^3f_8f_6f_7+96f_9f_4f_7^2\\
&-12f_9f_6f_4^2f_7-24f_8f_9f_7+6f_8f_6f_9f_4)
\end{aligned}
$$

One can then write the initial form for (*) in $(f_1, \dots, f_9)$, and check that it doesn't vanish for any j. By Theorem II, we can conclude that for all $j \neq 0, 1728$

$$M = 24,\ N = 12 \quad .$$

Theorem I yields then

THEOREM III(2). *The characteristic numbers for the family $F_{(j)}$ of plane cubic curves with given j-invariant $\neq 0, 1728$ are*

$$
F_{(j)}(k) = \begin{cases}
12 & k = 0\\
48 & k = 1\\
192 & k = 2\\
768 & k = 3\\
2784 & k = 4\\
8832 & k = 5\\
21828 & k = 6\\
39072 & k = 7\\
50448 & k = 8
\end{cases}
$$

It seems to us that the geometry behind these numbers should be as follows. Fix a general collection of 8 points and lines, and consider the smooth cubics with given j-invariant that contain the points and are tangent to the lines. As $j \to 0$ (or 1728), all these curves will move toward each other 3 by 3 (or 2 by 2), and as j hits 0 (or 1728), when the curves acquire an extra order-3 (or order-2) automorphism, they collide in groups of 3 (or 2). So

$$F_{(0)}(k) = \frac{1}{3}F_{(j)}(k), \quad F_{(1728)}(k) = \frac{1}{2}F_{(j)}(k)$$

for $j \neq 0, 1728$. What Theorem III indicates is that for no j do these curves fly off and converge to non-reduced cubics (is there an *a priori* reason why this should be the case?).

A word about the case $j = \infty$, i.e. the discriminant hypersurface. Similar computations as above reveal $M = 24$, $N = 12$ in this case as well (these are $G/40$, F in Zeuthen's notation for formula (4) in [**Z**], p. 727, derived on p. 728), so the list of Theorem III(2) holds for the discriminant (see also [**KS**], Proposition 7.4); but it loses enumerative significance, since curves that are not 'properly' tangent to the lines will contribute to these numbers. We discuss the situation in [**A2**], together with another (more 'geometric') derivation of the same list (Proposition 3.1, first column). It is interesting to observe that the intermediate multiplicities $m_0, \dots, m_4$ (see §1) are for all finite $j \neq 0, 1728$

$$m_0 = 8,\ m_1 = \mathbf{4},\ m_2 = \mathbf{4},\ m_3 = 6,\ m_4 = 6 \quad ;$$

while for $j = \infty$ they are

$$m_0 = 8,\ m_1 = \mathbf{5},\ m_2 = \mathbf{3},\ m_3 = 6,\ m_4 = 6$$

(see [**A2**]). As it happens, this difference does not influence M, N. Is this an accident, or is it the manifestation of a general principle?

Appendix: A Maple procedure. Here we work over the complex numbers.

The Maple[2] procedures that follow will compute the characteristic numbers of a family of plane cubics parametrized by a hypersurface F, given its equation. This simply implements Theorem I and II from §§1,2.

NOTE. The procedures as listed below are not 'exact': they employ Maple's random number generator to speed the computation of the highest power of f_3, f_5 dividing $F(1 : f_1 : f_2 \dots)$ (as requested by Theorem II). Of course it is possible that the 'random' choices produce a zero of the initial form of $F(1 : f_1 : \dots)$, and therefore a miscalculation of M, N. To reassure the reader of the statistical reliability of our shortcut, we should point out that the procedures below have *never* been caught wrong (of course all results listed in this paper have been checked with an exact—but slower—procedure): for example, in a test we have run them 5,000 times on the degree-4 invariant C_4 of §3, without observing a single mistake. However, to obtain exact procedures just replace the lines from `die := rand(1..500);` to the next `end;` with

```
multi := proc(exp)
expand(subs(blowup,exp));
[ldegree(",f3),ldegree(",f5)];
end;
```

In the version below, the procedures are quite fast: for example, the implementation of Maple on the Cadmus at the Math. Inst. of Erlangen processes C_4 in less than 5 seconds, and C_6 in less than 40.

```
blowup := {a0 = 1,a1 = f1,a2 = f2,
a3 = -4/3*f3**2+1/3*f3**2*f5**2*f6+1/3*f1**2,
a4 = -4/3*f4*f3**2+1/3*f4*f3**2*f5**2*f6+2/3*f1*f2,
a5 = -1/3*f4**2*f3**2+1/12*f4**2*f3**2*f5**2*f6-1/3*f5*f3**2+
1/12*f5**3*f3**2*f6+1/3*f2**2,
a6 = 16/27*f3**3-8/27*f3**3*f5**2*f6+1/27*f3**3*f5**4*f6**2
-4/9*f1*f3**2 +1/9*f1*f3**2*f5**2*f6+1/27*f1**3,
a7 = 8/9*f4*f3**3-4/9*f4*f3**3*f5**2*f6+8/9*f7*f5**2*f3**3
```

[2]Maple is a trademark of the University of Waterloo

```
+1/18*f4*f3**3*f5**4*f6**2-4/9*f7*f5**4*f3**3*f6
+1/18*f7*f5**6*f3**3*f6**2-4/9*f1*f4*f3**2
+1/9*f1*f4*f3**2*f5**2*f6-4/9*f2*f3**2+1/9*f2*f3**2*f5**2*f6
+1/9*f2*f1**2,
a8 = 4/9*f4**2*f3**3-2/9*f4**2*f3**3*f5**2*f6
+1/36*f4**2*f3**3*f5**4*f6**2+2/9*f8*f5**2*f3**3
-1/9*f8*f5**4*f3**3*f6+1/72*f8*f5**6*f3**3*f6**2+2/9*f5*f3**3
-1/9*f5**3*f3**3*f6+1/72*f5**5*f3**3*f6**2-4/9*f2*f4*f3**2
+1/9*f2*f4*f3**2*f5**2*f6-1/9*f1*f4**2*f3**2
+1/36*f1*f4**2*f3**2*f5**2*f6-1/9*f1*f5*f3**2
+1/36*f1*f5**3*f3**2*f6+1/9*f1*f2**2,
a9 = 2/27*f4**3*f3**3-1/27*f4**3*f3**3*f5**2*f6
+1/216*f4**3*f3**3*f5**4*f6**2+1/27*f9*f5**2*f3**3
-1/54*f9*f5**4*f3**3*f6+1/432*f9*f5**6*f3**3*f6**2
+1/9*f4*f5*f3**3-1/18*f4*f5**3*f3**3*f6
+1/144*f4*f5**5*f3**3*f6**2-1/9*f2*f4**2*f3**2
+1/36*f2*f4**2*f3**2*f5**2*f6-1/9*f2*f5*f3**2
+1/36*f2*f5**3*f3**2*f6+1/27*f2**3};

die := rand(1..500);
multi := proc (exp)
subs(f1 = die(),f2 = die(),f4 = die(),f6 = die(),f7 = die(),
f8 = die(),f9 = die(),f5 = die(),blowup);
subs(f1 = die(),f2 = die(),f4 = die(),f6 = die(),f7 = die(),
f8 = die(),f9 = die(),f3 = die(),blowup);
expand(subs("",exp)); expand(subs("",exp));
[ldegree("",f3),ldegree(",f5)];
end;

process:=proc (M, N, g)
[g,4*g,16*g, 64*g, 256*g-24*N, 976*g-240*N, 3424*g-885*N-360*M,
9766*g-1470*N-2520*M, 21004*g-8400*M];
end;

numbers:=proc (exp)
mult:=multi(exp);
answer:=process(op("),degree(exp));
end;
```

The procedure `multi` computes M, N by applying Theorem II; the procedure `process` computes the characteristic numbers from M, N and the degree of F, by use of Theorem I; and `numbers` executes both procedures. At the end of the computation, the variable `answer` contains the list of characteristic numbers; the variable `mult` contains M, N.

Examples

```
> a9;
                                   a9
> numbers(");
          [1, 4, 16, 64, 256, 976, 3424, 9766, 21004]
```

These are the characteristic numbers for the family of plane cubics containing the point $(0 : 0 : 1)$; of course they give the first 9 characteristic numbers for the family of all smooth cubics.

```
          3                           2    2         3            2    2
     4 a5   a0 - 18 a9 a2 a5 a0 - a2   a5   + 4 a2   a9 + 27 a9   a0
> numbers(");
        [4, 16, 64, 256, 976, 3424, 9766, 21004, 33616]
```

This is the family of cubics tangent to the line $x_1 = 0$, cf. §2.

In case the equation is given by a determinant, the following modifications (replace the highlighted lines) will accelerate the computation considerably, as Maple won't have to compute the determinant until the last moment:

...

`die := rand(1..500); with(linalg,det);`

multi := proc (exp)

...

f8 = die(),f9 = die(),f3 = die(),blowup);

`det(subs("",op(exp))); det(subs("",op(exp)));`

[ldegree(" ",f3),ldegree(",f5)];

...

end;

`numbers:=proc (exp,g)`

mult:=multi(exp);

`answer:=process(op("),g);`

end;

In this case, provide the degree of the expression together with a matrix whose determinant gives the polynomial.

Examples

—*Characteristic numbers for the family of cubics with flex on a given line.*

We can choose the line. We require then the cubic $(a_0 : \cdots : a_9)$ and its hessian to vanish simultaneously somewhere on the line $x_0 = 0$, which amounts to the simultaneous vanishing of

$$C = a_6x^3 + a_7x^2 + a_8x + a_9$$

and

$$
\begin{aligned}
H = & -6a_4^2x^3a_6 - 8a_3^2x^3a_8 - 6a_4^2a_9 - 8a_1x^3a_7^2 - 8a_2a_8^2 - 8a_5^2a_7 + 24a_1x^3a_8a_6 \\
& + 8a_4x^3a_3a_7 + 24a_2a_9a_7 + 8a_5a_4a_8 - 8a_1x^2a_8a_7 + 72a_1x^2a_9a_6 + 24a_1xa_9a_7 \\
& - 8a_1xa_8^2 + 24a_2a_8x^2a_6 - 8a_2a_8xa_7 + 72a_2a_9a_6x - 8a_2a_7^2x^2 - 24a_4x^2a_5a_6 \\
& + 2a_4^2xa_8 - 24a_5^2a_6x + 16a_5a_3x^2a_7 + 16a_5a_3xa_8 - 24a_3^2x^2a_9 - 24a_3xa_4a_9 \\
& + 2a_4^2x^2a_7
\end{aligned}
$$

The equation is the resultant of these two polynomials with respect to x, a degree-12 polynomial. Its characteristic numbers are then:

```
> with(linalg,bezout):
> matr:=bezout(C,H,x):
> numbers(matr,12);
        [12, 48, 192, 768, 2856, 9552, 25563, 51042, 75648]
```

The combined multiplicities are in this case $M = 21, N = 9$.

—*Characteristic numbers for the family of cubics with flex line containing a given point.*

We can choose the point. We have to impose that the cubic with coordinates $(a_0 : \cdots : a_9)$ restricts to a triple point on some line between say $(1 : 0 : 0)$ and $(0 : 1 : s)$. The cubic restricts to the polynomial (in t)

$$
a_0t^3 + a_1t^2 + a_2t^2s + a_3t + a_4ts + a_5ts^2 + a_6 + a_7s + a_8s^2 + a_9s^3
$$

on such a line; requiring that its second derivative vanishes where the polynomial and its first derivative do amounts to the simultaneous vanishing of

$$
\begin{aligned}
Q = & 2a_1^3 + 6a_1^2a_2s + 6a_1a_2^2s^2 + 2a_2^3s^3 - 9a_3a_1a_0 - 9a_3a_2sa_0 - 9a_4a_1a_0s \\
& - 9a_4a_2s^2a_0 - 9a_5a_1a_0s^2 - 9a_5a_2s^3a_0 + 27a_6a_0^2 + 27a_7sa_0^2 + 27a_8s^2a_0^2 \\
& + 27a_9s^3a_0^2
\end{aligned}
$$

and

$$
R = -a_1^2 - 2a_1a_2s - a_2^2s^2 + 3a_3a_0 + 3a_4sa_0 + 3a_5s^2a_0 \quad .
$$

So the degree-9 equation for this hypersurface is the resultant of Q, R with respect to s, divided by its factor a_0^3. The characteristic numbers:

```
> with(linalg,bezout):
> matr:=bezout(Q,R,s):
> numbers(matr,9);
        [9, 36, 144, 576, 2232, 8064, 23841, 53244, 88236]
```

(By specifying that the degree is 9, the contribution of a_0^3 to the resultant is discarded, as it doesn't affect $M = 12, N = 3$.)

Did Zeuthen know these numbers? He considers this last family ('c'' in formulas (2) and (3) in [**Z**], p. 727) in deriving his relations, but he stops short of determining the key coefficients giving the characteristic numbers (C, D in his notations), maybe because he didn't need them for his immediate purposes. The result listed above implies $C = D = 1$.

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