The Structure of Properly Convex Manifolds

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(joint with D. Long)

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Some Questions

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• What sort of structure do convex projective manifolds have?
  *Deformations of finite volume strictly convex manifolds are structurally similar to complete finite volume hyperbolic manifolds*
Projective Space

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- A projective hyperplane is the projectivization of an $n$-plane in $\mathbb{R}^{n+1}$. 
A Decomposition of $\mathbb{R}P^n$

- Let $H$ be a hyperplane in $\mathbb{R}^{n+1}$.
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- $\mathbb{RP}^n \setminus P(H)$ is called an affine patch.
What is convex projective geometry?

Motivation from hyperbolic geometry

- Let $\langle x, y \rangle = x_1 y_1 + \ldots x_n y_n - x_{n+1} y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C = \{ x \in \mathbb{R}^{n+1} | \langle x, x \rangle < 0 \}$
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- Let \( C = \{ x \in \mathbb{R}^{n+1} | \langle x, x \rangle < 0 \} \)
- \( P(C) \) is the *Klein model* of hyperbolic space.
- \( P(C) \) has isometry group \( \text{PSO}(n, 1) \leq \text{PGL}_{n+1}(\mathbb{R}) \)
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Nice Properties of Hyperbolic Space

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Nice Properties of Hyperbolic Space

- **Convex**: Intersection with projective lines is connected.
- **Properly Convex**: Convex and closure is contained in an affine patch $\iff$ Disjoint from some projective hyperplane.
- **Strictly Convex**: Properly convex and boundary contains no non-trivial projective line segments.
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Motivation from hyperbolic geometry

Convex projective geometry focuses on the geometry of manifolds that are locally modeled on properly (strictly) convex domains.
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**Hyperbolic Geometry**

\[ \mathbb{H}^n / \Gamma \]

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**Convex Projective Geometry**
\[ \Omega / \Gamma \]
\[ \Omega \text{ properly (strictly) convex} \]
\[ \Gamma \leq \text{PGL}(\Omega) \]
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What is Convex Projective Geometry

Examples

1. Hyperbolic manifolds
What is Convex Projective Geometry

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2. Let $T$ be the interior of a triangle in $\mathbb{R}P^2$ and let $\Gamma \leq \text{Diag}^+$ be a suitable lattice inside the group of $3 \times 3$ diagonal matrices with determinant 1 and distinct positive eigenvalues. $T/\Gamma$ is a properly convex torus.
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These are extreme examples of properly convex manifolds. Generic examples interpolate between these extreme cases.
Hilbert Metric

Let $\Omega$ be a properly convex set and $\text{PGL}(\Omega)$ be the projective automorphisms preserving $\Omega$.

\[
d_{\Omega}(x, y) = \log \left[ \frac{|x - a|}{|x - b|} \right] = \log \left( \frac{|y - a|}{|y - b|} \right)
\]

• When $\Omega$ is an ellipsoid $d_{\Omega}$ is twice the hyperbolic metric.

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Discrete subgroups of $\text{PGL}(\Omega)$ act properly discontinuously on $\Omega$. 
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Classification of Isometries
a la Cooper, Long, Tillmann

If $\Omega$ is open and properly convex then $\text{PGL}(\Omega)$ embeds in $\text{SL}^{\pm}_{n+1}(\mathbb{R})$ which allows us to talk about eigenvalues.
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If $\gamma \in \text{PGL}(\Omega)$ then $\gamma$ is

1. *elliptic* if $\gamma$ fixes a point in $\Omega$ (zero translation length + realized),
2. *parabolic* if $\gamma$ acts freely on $\Omega$ and has all eigenvalues of modulus 1 (zero translation length + not realized), and
3. *hyperbolic* otherwise (positive translation length)
Similarities to Hyperbolic Isometries

Strictly Convex Case

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4. When $\Omega$ is strictly convex, parabolic and hyperbolic elements in a common discrete subgroup do not share fixed points.
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4. When $\Omega$ is strictly convex, parabolic and hyperbolic elements in a common discrete subgroup do not share fixed points.
5. When $\Omega$ is strictly convex, a discrete torsion-free subgroup of elements fixing a geodesic is infinite cyclic.
Similarities to Hyperbolic Isometries

The General Case

A properly convex domain is a compact convex subset of $\mathbb{R}^n$ and so if $\gamma \in \text{PGL}(\Omega)$ then Brouwer fixed point theorem applies
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- Elliptic elements are all conjugate into $\text{O}(n)$.
- Parabolic elements have a connected fixed set in $\partial \Omega$.
- Hyperbolic elements have an attracting and repelling subspaces $A_+$ and $A_-$ in $\partial \Omega$. The action on these sets is orthogonal and their dimension is determined by the number of “powerful” Jordan blocks of $\gamma$. 
Margulis Lemma

Let $\Omega \subset \mathbb{R}P^n$ be an open properly convex domain and let $\Gamma \leq \text{PGL}(\Omega)$ be a discrete group. Then there exists a number $\mu_n$ (depending only on $n$) such that if $x \in \Omega$ then the group

$$\Gamma_x = \langle \gamma \in \Gamma | d_\Omega(x, \gamma x) < \mu_n \rangle$$

is virtually nilpotent.
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Result due to Gromov-Margulis-Thurston for $\mathbb{H}^n$ and Cooper-Long-Tillmann in general.
Rigidity and Flexibility

When $n \geq 3$ Mostow-Prasad rigidity tells us that complete finite volume hyperbolic structures are very rigid

**Theorem 1 (Mostow ’70, Prasad ’73)**

Let $n \geq 3$ and suppose that $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ both have finite volume. If $\Gamma_1$ and $\Gamma_2$ are isomorphic then $\mathbb{H}^n/\Gamma_1$ and $\mathbb{H}^n/\Gamma_2$ are isometric.
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There is no Mostow-Prasad rigidity for properly (strictly) convex domains.

There are examples of finite volume hyperbolic manifolds whose complete hyperbolic structure can be “deformed” to a non-hyperbolic convex projective structure.
Deformations

• Start with $M_0 = \Omega_0/\Gamma_0$ which is properly convex.

Ex: Let $\Omega_0 \sim \Xi_0 \leq \text{PSO}(n, 1)$, such that $\Omega_0/\Gamma_0$ is finite volume and contains an embedded totally geodesic hypersurface $\Sigma$. Let $\Gamma_1$ be obtained by ‘bending’ along $\Sigma$. 
Deformations

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- “Perturb” $\Gamma_0$ to $\Gamma_1 \leq \text{PGL}(\Omega_1) \leq \text{PGL}_{n+1}(\mathbb{R})$, where $\Gamma_0 \cong \Gamma_1$ and $\Omega_1$ is properly convex.
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Ex: Let $\Omega_0 \cong \mathbb{H}^n$, $\Gamma_0 \leq \text{PSO}(n, 1)$, such that $\Omega_0 / \Gamma_0$ is finite volume and contains an embedded totally geodesic hypersurface $\Sigma$. Let $\Gamma_1$ be obtained by “bending” along $\Sigma$. 
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Let $\mathbb{H}^n/\Gamma$ be a closed hyperbolic manifold.

- Since $\Gamma$ acts cocompactly by isometries on $\mathbb{H}^n$ we see that $\Gamma$ is $\delta$-hyperbolic group (Švarc-Milnor)
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- By compactness, we see that if $1 \neq \gamma \in \Gamma$ then $\gamma$ is hyperbolic.
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- By compactness, we see that if $1 \neq \gamma \in \Gamma$ then $\gamma$ is hyperbolic.

- In particular, if $1 \neq \gamma \in \Gamma$ then $\gamma$ is *positive proximal* (eigenvalues of minimum and maximum modulus are unique, real, and positive).
Structure of Convex Projective Manifolds
The Closed Case

Let \( M = \Omega / \Gamma \) be a closed properly convex manifold that is a deformation of a closed strictly convex manifold \( M_0 = \Omega_0 / \Gamma_0 \).
Structure of Convex Projective Manifolds

The Closed Case

Let $M = \Omega/\Gamma$ be a closed properly convex manifold that is a deformation of a closed strictly convex manifold $M_0 = \Omega_0/\Gamma_0$.

Theorem 2 (Benoist)

Suppose $\Omega/\Gamma$ is closed. $\Omega/\Gamma$ is strictly convex if and only if $\Gamma$ is $\delta$-hyperbolic.
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Proof sketch.

If $\Omega$ is not strictly convex then it will contain arbitrarily fat triangles and is thus not $\delta$-hyperbolic.
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Suppose $\Omega / \Gamma$ is closed. $\Omega / \Gamma$ is strictly convex if and only if $\Gamma$ is $\delta$-hyperbolic.

Proof sketch.

If $\Omega$ is not strictly convex then it will contain arbitrarily fat triangles and is thus not $\delta$-hyperbolic. Since $\Gamma$ acts cocompactly by isometries on $\Omega$, Švarc-Milnor tells us that $\Omega$ is q.i. to $\Gamma$ and is thus $\delta$-hyperbolic.
Structure of Convex Projective Manifolds
The Closed Case

Theorem 3 (Benoist)

Let \( 1 \neq \gamma \in \Gamma \) then \( \gamma \) is positive proximal.

Proof.

- Again by compactness we have that if \( 1 \neq \gamma \in \Gamma \) then \( \gamma \) is hyperbolic.
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Proof.

- Again by compactness we have that if $1 \neq \gamma \in \Gamma$ then $\gamma$ is hyperbolic.
- Since $\Omega$ is strictly convex and $\gamma$ is hyperbolic we see that $\gamma$ has exactly 2 fixed points in $\partial \Omega$ and acts as translation along the geodesic connecting them. $\gamma$ is thus positive proximal.
Let $M = \mathbb{H}^n / \Gamma$ be a finite volume hyperbolic manifold. We can decompose $M$ as

$$M = M_K \bigcup_{i} C_i,$$

where $M_K$ is a compact and $\pi_1(M_K) = \Gamma$ and $C_i$ are components of the thin part called *cusps*.
Let $M = \mathbb{H}^n / \Gamma$ be a finite volume hyperbolic manifold. We can decompose $M$ as

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where $M_K$ is a compact and $\pi_1(M_K) = \Gamma$ and $C_i$ are components of the thin part called *cusps*. As we will see, the Margulis lemma tells us that the $C_i$ have relatively simple geometry.
Geometry of the Cusps

Let $C$ be a cusp of a finite volume hyperbolic manifold and let

$$P = \left\{ \begin{pmatrix} 1 & v^T & |v|^2 \\ 0 & I_{n-1} & v \\ 0 & 0 & 1 \end{pmatrix} \bigg| v \in \mathbb{R}^{n-1} \right\}$$

be the group of parabolic translations fixing $\infty$. Let $x_0 \in \mathbb{H}^n$, then $C \cong B/\Delta$ where $B$ is horoball bounded by $Px_0$ and $\Delta$ is a finite extension of a lattice in $P$. 

![Diagram illustrating the relationship between $B$, $ Px_0 $, and $ B/\Delta $]
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The Finite Volume Case

• \( \Gamma \) no longer acts cocompactly on \( \mathbb{H}^n \) and \( \Gamma \) is no longer \( \delta \)-hyperbolic
Structure of Hyperbolic Manifolds
The Finite Volume Case

- $\Gamma$ no longer acts cocompactly on $\mathbb{H}^n$ and $\Gamma$ is no longer $\delta$-hyperbolic
- Instead $\Gamma$ is $\delta$-hyperbolic \textit{relative to the cusps}
• $\Gamma$ no longer acts cocompactly on $\mathbb{H}^n$ and $\Gamma$ is no longer $\delta$-hyperbolic
• Instead $\Gamma$ is $\delta$-hyperbolic relative to the cusps
• If $1 \neq \gamma \in \Gamma$ is freely homotopic into a cusp then $\gamma$ is parabolic, otherwise $\gamma$ is hyperbolic (positive proximal)
Let $\Omega/\Gamma$ be a finite volume (Hausdorff measure of Hilbert metric) strictly convex manifold.

**Theorem 4 (Cooper, Long, Tillmann ‘11)**

Let $M = \Omega/\Gamma$ be as above then

- $M = M_K \bigsqcup_i C_i$, where $M_K$ is compact and $C_i$ is projectively equivalent to the cusp of a finite volume hyperbolic manifold,
- $\Gamma$ is $\delta$-hyperbolic relative to its cusps, and
- If $1 \neq \gamma \in \Gamma$ is freely homotopic into a cusp then $\gamma$ is parabolic. Otherwise $\gamma$ is hyperbolic (positive proximal).
Consider the following example.

Let $K$ be the figure-8 knot, let $M = S^3 \setminus K$, and let $G = \pi_1(M)$.
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**Theorem 5 (B)**

There exists $\varepsilon > 0$ such that for each $t \in (-\varepsilon, \varepsilon)$ there is a properly convex domain $\Omega_t$ and a discrete group $\Gamma_t \leq \text{PGL}(\Omega_t)$ such that

- $\Omega_t / \Gamma_t \cong M$,
- $\Omega_0 / \Gamma_0$ is the complete hyperbolic structure on $M$, and
- If $t \neq 0$ then $\Omega_t$ is not strictly convex.
Figure-8 Example

Theorem 6 (B)

For each $t \in (-\varepsilon, \varepsilon)$ we can decompose $\Omega_t/\Gamma_t$ as $M_t^t \sqcup C_t$, where $M_t^t$ is compact and $C_t \cong T^2 \times [1, \infty)$.

- For each $t$, $C_t \cong B_t/\Delta_t$, where $\Delta_t$ is a lattice an Abelian group $P_t$ of “translations,” and $B_t$ is a “horoball” bounded by an orbit of $P_t$. 
For each $t \neq 0$ there is $\gamma_t \in \Gamma_t$ such that $\gamma_t$ is hyperbolic, freely homotopic into $C_t$, but not positive proximal.

$\Omega_t$ contains non-trivial line segments in $\partial \Omega_t$ that are preserved by conjugates of $\Delta_t$. In particular, $\Omega_t$ is not $\delta$-hyperbolic.
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Theorem 7 (B, Long)

$1 \neq \gamma \in \Gamma_t$ is positive proximal if and only if it cannot be freely homotoped into $C^t$. 
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Proof.

$\iff$ Let $1 \neq \gamma \in \Gamma_t$. No elements of $P_t$ are positive proximal, so if $\gamma$ is freely homotopic to $C^t$ then it is not positive proximal.
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\implies If \gamma is not freely homotopic to C^t then \gamma has positive translation length and is thus hyperbolic. Furthermore, this translation length is realized by points on an axis.
Proof (Continued).

Use Margulis lemma to construct a disjoint and $\Gamma_t$ invariant collection $\mathcal{H}_t$ of horoballs in $\Omega_t$. 
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Let $\hat{\Omega}_t$ be the *electric space* obtained by collapsing the horospherical boundary components of $\Omega_t \setminus \mathcal{H}_t$. 

**Lemma 8 (B, Long)**

$\hat{\Omega}_t$ is $\delta$-hyperbolic.
Proof (Continued).
Use Margulis lemma to construct a disjoint and $\Gamma_t$ invariant collection $\mathcal{H}_t$ of horoballs in $\Omega_t$. Let $\hat{\Omega}_t$ be the electric space obtained by collapsing the horospherical boundary components of $\Omega_t \setminus \mathcal{H}_t$.

Lemma 8 (B, Long)
$\hat{\Omega}_t$ is $\delta$-hyperbolic
Figure-8 Example

Proof (Continued).

• Since $\gamma$ is hyperbolic and preserves $\Omega_t$ we know that $\gamma$ has real eigenvalues of largest and smallest modulus and that these eigenvalues have the same sign.
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- This gives rise to arbitrarily fat triangles in $\hat{\Omega}_t$. 

\[ \text{Diagram: Figure-8 Example} \]
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What can we say for deformations of deformations of infinite volume hyperbolic manifolds?