Overview

- “Geometric Structures” on manifolds give rise to conjugacy classes of representations of fundamental groups into various Lie groups.
- Want to understand the space $\mathcal{R}(\Gamma, G) := \text{Hom}(\Gamma, G)/G$, where $\Gamma$ is finitely generated, $G$ is a Lie group, and $G$ acts by conjugation.
- Want to understand the space $\mathcal{R}(\Gamma, G)$ locally near a class of representations $[\rho]$.
- Want to understand the space $\mathcal{R}(\Gamma, G)$ infinitesimally near a class of representations $[\rho]$. 
Broccoli is good for you

= Computation that is good for you.
If $\Gamma$ is finitely generated and $G$ is a “nice” group, then the set, $\mathcal{R}(\Gamma, G) := \text{Hom}(\Gamma, G)$ is an algebraic variety.

More concretely, a presentation for $\Gamma$ gives rise to a polynomial function $f : \mathbb{R}^n \to \mathbb{R}^m$, and $\mathcal{R}(\Gamma, G)$ is $f^{-1}(0)$. 
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If $G = \text{GL}_2(\mathbb{R})$ and $\Gamma = \mathbb{Z}/n\mathbb{Z}$, then $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is given by $f(A) = A^n - I$, where we think of $A \in \mathbb{R}^4$. 
Representation Varieties

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If 0 is a regular value of \( f \) then \( f^{-1}(0) \) is a manifold and the tangent space to \( p \in f^{-1}(0) \) is given by \( \ker(f_*|_p) \).

Even if 0 is not a regular value we can think of these kernels as a tangent spaces for \( \mathcal{R}(\Gamma, G) \).
The way we attempt to realize $\mathfrak{A}(\Gamma, G)$ as a variety by looking at polynomials on $\mathcal{R}(\Gamma, G)$, which are invariant under the action of $G$.

These invariant polynomials are generated by traces of elements of $\Gamma$, and when $G$ is “nice” this construction gives rise to a variety.

However, this variety is not always the same as $\mathfrak{A}(\Gamma, G)$.
We need to exclude representations whose image is like

\[
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix}
\]

because they cannot be distinguished from the trivial representation by looking at traces.

To get a variety we need to restrict to the set $\mathcal{R}'(\Gamma, G)$ of “nice” representations. In this case the quotient $\mathcal{R}'(\Gamma, G) := \mathcal{R}'(\Gamma, G)/G$ is a variety.
Twisted Cohomology

Let $G$ be a group and $M$ a $G$-module. Define a cochain complex $C^n(G; M)$ to be the set of all functions from $G^n$ to $M$ with differential $d_n : C^n(G; M) \to C^{n+1}(G; M)$ by

$$d\phi(g_1, g_2, \ldots, g_{n+1}) = g_1 \cdot \phi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \phi(g_1, \ldots, g_{i-1}, g_ig_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \phi(g_1, \ldots, g_n)$$

Then $H^n(G; M) = Z^n(G; M)/B^n(G; M)$ is the associated cohomology group, where $Z^n(G; M) = \ker d_n$ and $B^n = \text{Im} d_{n-1}$. 
$C^0(G; M)$ is the set of constant functions. If $z \in C^0(G; M)$ then

$$d(z)(g) = g \cdot m_z - m_z.$$ 

Therefore,

$$H^0(G; M) = Z^0(G; M) = \{ m \in M \mid g \cdot m = m \ \forall g \in G \}.$$ 

So $H^0(G; M)$ is the set of elements invariant under the action of $G$. 
If \( z \in Z^1(G; M) \) then

\[
z(g_1 g_2) = z(g_1) + g_1 \cdot z(g_2)
\]

These maps are sometimes called \textit{crossed homomorphisms}.

We have already seen that \( B^1(G; M) \) consists of maps where \( z(g) = g \cdot m_z - m_z \) for some \( m \in M \).
A Simple Example

Let $\mathbb{Z}$ act by conjugation (i.e. trivially) on $\mathbb{R}$, then if $B^1(\mathbb{Z}, \mathbb{R}) = 0$ and if $z \in Z^1(\mathbb{Z}, \mathbb{R})$ then

$$z(mn) = z(m) + z(n),$$

and so $H^1(\mathbb{Z}, \mathbb{R}) = \text{Hom}(\mathbb{Z}, \mathbb{R}) = \mathbb{R}$ is the tangent space to $\text{Hom}(\mathbb{Z}, \mathbb{R}) = \mathcal{N}(\mathbb{Z}, \mathbb{R}) = \mathbb{R}$

In general, $H^1$ can be thought of as a “tangent space” to $\mathcal{N}(\Gamma, G)$. 
Let $\rho_0 : \Gamma \rightarrow G$ be a representation, let $\mathfrak{g}$ be the lie algebra of $G$, and let $\Gamma$ act on $\mathfrak{g}$, by $\gamma \cdot x = \text{Ad}_{\rho_0}(\gamma) \cdot x$.

Denote the resulting cohomology groups $H^* (\Gamma, \mathfrak{g}_{\rho_0})$.

Let $\rho_t$ be a curve of representations passing through $\rho_0$.

For $\gamma \in \Gamma$ we can use a series expansion to write

$$
\rho_t(\gamma) = (I + z_\gamma t + O(t^2))\rho_0(\gamma),
$$

where $z_\gamma \in \mathfrak{g}$.

In this way we can think of $z$ as an element of $C^1 (\Gamma, \mathfrak{g}_{\rho_0})$. 

**$H^1$ as a Tangent Space**
Repeatedly using this expansion again we see that

\[ \rho_t(\gamma_1 \gamma_2) = (I + z_{\gamma_1 \gamma_2} t + O(t))\rho_0(\gamma_1 \gamma_2) \quad \text{and} \]

\[ \rho_t(\gamma_1) \rho_t(\gamma_2) = (I + (z_{\gamma_1 \gamma_2} + \gamma_1 \cdot z_{\gamma_2}) t + O(t^2))\rho_0(\gamma_1 \gamma_2) \quad \text{and} \]

Therefore

\[ z_{\gamma_1 \gamma_2} = z_{\gamma_1} + \gamma_1 \cdot z_{\gamma_2} \]

and so \( \rho_t \) gives rise to an element of \( \mathbb{Z}_1^1(\Gamma; g \rho_0) \). In this way \( \mathbb{Z}_1^1(\Gamma, G \rho_0) \) is the tangent space to \( \mathbb{R}(\Gamma, G) \).
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\[ \rho_t(\gamma_1)\rho_t(\gamma_2) = (I + z_{\gamma_1} t + O(t^2))\rho_0(\gamma_1)(I + z_{\gamma_2} t + O(t^2))\rho_0(\gamma_2) = (I + (z_{\gamma_1} + \gamma_1 \cdot z_{\gamma_2}) t + O(t^2))\rho_0(\gamma_1 \gamma_2) \]
Repeatedly using this expansion again we see that

\[ \rho_t(\gamma_1 \gamma_2) = (I + z_{\gamma_1 \gamma_2} t + O(t)) \rho_0(\gamma_1 \gamma_2) \]

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\[ \rho_t(\gamma_1) \rho_t(\gamma_2) = (I + z_{\gamma_1} t + O(t^2)) \rho_0(\gamma_1)(I + z_{\gamma_2} t + O(t^2)) \rho_0(\gamma_2) \]

\[ = (I + (z_{\gamma_1} + \gamma_1 \cdot z_{\gamma_2}) t + O(t^2)) \rho_0(\gamma_1 \gamma_2) \]

Therefore \( z_{\gamma_1 \gamma_2} = z_{\gamma_1} + \gamma_1 \cdot z_{\gamma_2} \), and so \( \rho_t \) gives rise to an element of \( Z^1(\Gamma; g_{\rho_0}) \).

In this way \( Z^1(\Gamma, g_{\rho_0}) \) is the tangent space to \( \mathcal{R}(\Gamma, G) \).
If $\rho_t(\gamma) = g_t^{-1}\rho_0 g_t$, where $g_t \in G$ and $g_0 = I$, then

$$\rho_t(\gamma) = (I - ct + O(t^2))\rho_0(\gamma)(I + ct + O(t^2))$$

So for deformations of this type, $z_\gamma = \gamma \cdot c - c$, and so $z \in B^1(\Gamma; g_{\rho_0})$

In this way trivial curves of deformations give rise to 1-coboundaries, and so $H^1(\Gamma, g_{\rho_0})$ is the tangent space to $\mathcal{R}'(\Gamma, G)$ at $\rho_0$. 
Another Simple Example

Let's compute the dimension of $H^1(\mathbb{Z}^2, \mathfrak{sl}_2(\mathbb{C})_{\rho_0})$ where

$$\rho_0(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \rho_0(b) = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \omega \neq 0$$
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Using “implicit differentiation” at $t = 0$ on the relation

$$\rho_t(a)\rho_t(b) = \rho_t(b)\rho_t(a)$$

we get a $2 \times 2$ matrix equation that is equivalent to 2 real valued equations.
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Using the exact sequence

$$0 \rightarrow \mathbb{Z}^0(\mathbb{Z}^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_0}) \rightarrow C^0(\mathbb{Z}^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_0}) \rightarrow B^1(\mathbb{Z}^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_0}) \rightarrow 0$$

We see that $B^1(\mathbb{Z}^2; \mathfrak{sl}_2(\mathbb{C})_{\rho_0})$ has dimension 2.
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Therefore,

$$\dim H^1(\mathbb{Z}^2, \mathfrak{sl}_2(\mathbb{C})_{\rho_0}) = 6 - 2 - 2 = 2$$
Consequences

Theorem (Weil 64)

If $\rho_0$ is infinitesimally rigid (i.e. $H^1(\Gamma, \mathfrak{g}_{\rho_0}) = 0$), then $\rho_0$ is locally rigid (i.e. representations sufficiently close to $\rho_0$ are all conjugate)

• More generally, the dimension of $H^1(\Gamma, \mathfrak{g}_{\rho_0})$ is an upper bound for the dimension of $R'(\Gamma, G)$ near $\rho_0$.

• One problem is that this bound is not always sharp (there can be infinitesimal deformations that do not come from actual deformations).

Example $f(x) = x^2$ gives rise to a variety that is a single point, but whose tangent space is 1-dimensional.
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Rigidity Results

There are various situations where rigidity results are known to hold.

Theorem (Weil)

If $M$ is a compact, hyperbolic manifold of dimension $n \geq 3$ and $\rho_0$ is a discrete, faithful representation of $\Gamma = \pi_1(M)$ then $H^1(\Gamma, \mathfrak{so}(n, 1)_{\rho_0}) = 0$

Similar results hold for cocompact lattices in most other semi-simple Lie groups.

However when $\Gamma$ is no longer cocompact then interesting flexibility phenomena can occur.
Rigidity and Flexibility

The previous result tells us that $\rho_0$ cannot be deformed in $\text{PSO}(n, 1)$.

However, we can embed $\text{PSO}(n, 1)$ into other Lie groups (e.g. $\text{PSO}(n + 1, 1)$, $\text{PSU}(n, 1)$, or $\text{PGL}_{n+1}(\mathbb{R})$), and ask if it is possible to deform $\rho_0$ inside of this larger Lie group.
Rigidity and Flexibility

Examples

Quasi-Fuchsian Deformations

When $n = 2$ then quasi-Fuchsian deformations are an example of a deformation from $\text{SO}(2, 1)$ into $\text{SO}(3, 1)$. 
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Projective Deformations

Cooper, Long, and Thistlethwaite examined deformations into $\text{PGL}_4(\mathbb{R})$ by computing $H^1(\Gamma, \mathfrak{sl}_4 \rho_0)$ for all closed, hyperbolic, two generator manifolds in the SnapPea census.

A majority of these two generator manifolds were rigid, however about 1.4 percent were infinitesimally deformable, and of those several have been rigorously shown to deform.
Rigidity and Flexibility
Non-compact Case
When $M$ is a non-compact, finite volume, hyperbolic manifold of dimension 3 there are always non-trivial, hyperbolic deformations near $\rho_0$, but only one whose peripheral elements map to parabolics.

In this case, we can still ask if $\rho_0$ is locally rigid relative $\partial M$ (i.e. peripheral elements of $\pi_1(M)$ are sent to “parabolic” elements of $\text{PGL}_4(\mathbb{R})$).
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Theorem (Heusener-Porti, B)

For the two-bridge links with rational number $5/2$, $8/3$, $7/3$, and $9/5$ are locally rigid relative $\partial M$ at $\rho_0$. 
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There is strong numerical evidence that the knots $11/3$, $13/3$, and $13/5$ are also rigid in this sense.
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**Question**

Are all two-bridge knots and links rigid in this sense?