The Structure of Properly Convex Manifolds

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• What sort of structure do convex projective manifolds have?
  *Deformations of finite volume strictly convex manifolds are structurally similar to complete finite volume hyperbolic manifolds*
Projective Space

- $\mathbb{R}P^n$ is the space of lines through origin in $\mathbb{R}^{n+1}$.
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- The automorphism group of $\mathbb{R}P^n$ is $\text{PGL}_{n+1}(\mathbb{R}) := \text{GL}_{n+1}(\mathbb{R})/\mathbb{R}^\times$. 

- A codimension $k$ projective plane is the projectivization of a codimension $k$ plane in $\mathbb{R}^n$.
- A projective line is the projectivization of a 2-plane in $\mathbb{R}^n$.
- A projective hyperplane is the projectivization of an $n$-plane in $\mathbb{R}^n$. 
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A Decomposition of $\mathbb{R}P^n$

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- $\mathbb{R}P^n \setminus P(H)$ is called an *affine patch*.
What is convex projective geometry?
Motivation from hyperbolic geometry

- Let $\langle x, y \rangle = x_1 y_1 + \ldots x_n y_n - x_{n+1} y_{n+1}$ be the standard bilinear form of signature $(n, 1)$ on $\mathbb{R}^{n+1}$
- Let $C = \{x \in \mathbb{R}^{n+1} | \langle x, x \rangle < 0\}$
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- Let $C = \{ x \in \mathbb{R}^{n+1} | \langle x, x \rangle < 0 \}$
- $P(C)$ is the *Klein model* of hyperbolic space.
- $P(C)$ has isometry group $\text{PSO}(n, 1) \leq \text{PGL}_{n+1}(\mathbb{R})$
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Nice Properties of Hyperbolic Space

- Convex: Intersection with projective lines is connected.
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Nice Properties of Hyperbolic Space

- **Convex**: Intersection with projective lines is connected.
- **Properly Convex**: Convex and closure is contained in an affine patch $\iff$ Disjoint from some projective hyperplane.
- **Strictly Convex**: Properly convex and boundary contains no non-trivial projective line segments.
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Motivation from hyperbolic geometry

Convex projective geometry focuses on the geometry of manifolds that are locally modeled on properly (strictly) convex domains.
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**Hyperbolic Geometry**

\[ \mathbb{H}^n / \Gamma \]

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\( \Gamma \) discrete + torsion free
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Convex Projective Geometry

\[ \Omega / \Gamma \]
\[ \Omega \text{ properly (strictly) convex} \]
\[ \Gamma \leq \text{PGL}(\Omega) \]
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What is Convex Projective Geometry

Examples

1. Hyperbolic manifolds
What is Convex Projective Geometry

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2. Let $T$ be the interior of a triangle in $\mathbb{R}P^2$ and let $\Gamma \leq \text{Diag}^+$ be a suitable lattice inside the group of $3 \times 3$ diagonal matrices with determinant 1 and distinct positive eigenvalues. $T/\Gamma$ is a properly convex torus.
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These are extreme examples of properly convex manifolds. Generic examples interpolate between these extreme cases.
Hilbert Metric

Let $\Omega$ be a properly convex set and $\text{PGL}(\Omega)$ be the projective automorphisms preserving $\Omega$.

When $\Omega$ is an ellipsoid $d_{\Omega}$ is twice the hyperbolic metric.

$\text{PGL}(\Omega)$ isomorphic to $\text{Isom}(\Omega)$ and equal when $\Omega$ is strictly convex.

Discrete subgroups of $\text{PGL}(\Omega)$ act properly discontinuously on $\Omega$. 

![Diagram of a properly convex set $\Omega$ with points $x$, $y$, and $z$, and the Hilbert metric $d_{\Omega}(x,y)$ calculated as the logarithm of the ratio of distances between points.](diagram.png)
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$$d_\Omega(x, y) = \log[a, x; y, b] = \log \left( \frac{|x - b| |y - a|}{|x - a| |y - b|} \right)$$

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Classification of Isometries
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If $\Omega$ is open and properly convex then $\text{PGL}(\Omega)$ embeds in $\text{SL}^\pm_{n+1}(\mathbb{R})$ which allows us to talk about eigenvalues.

If $\gamma \in \text{PGL}(\Omega)$ then $\gamma$ is

1. *elliptic* if $\gamma$ fixes a point in $\Omega$ (zero translation length + realized),

2. *parabolic* if $\gamma$ acts freely on $\Omega$ and has all eigenvalues of modulus 1 (zero translation length + not realized), and

3. *hyperbolic* otherwise (positive translation length)
Similarities to Hyperbolic Isometries

Strictly Convex Case

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2. When $\Omega$ is \textit{strictly} convex, parabolic isometries have a unique fixed point on $\partial \Omega$.

3. When $\Omega$ is \textit{strictly} convex, hyperbolic isometries have 2 fixed points on $\partial \Omega$ and act by translation along the line connecting them.
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3. When $\Omega$ is strictly convex, hyperbolic isometries have 2 fixed points on $\partial \Omega$ and act by translation along the line connecting them.

4. In particular, when $\Omega$ is strictly convex, hyperbolic isometries are positive proximal (eigenvalues of minimum and maximum modulus are unique, real, and positive)
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- Parabolic elements have a connected fixed set in $\partial \Omega$.
- Hyperbolic elements have an attracting and repelling subspaces $A_+$ and $A_-$ in $\partial \Omega$. The action on these sets is orthogonal and their dimension is determined by the number of “powerful” Jordan blocks of $\gamma$. 
Margulis Lemma

Let $\Omega \subset \mathbb{R}P^n$ is an open properly convex domain and let $\Gamma \leq \text{PGL}(\Omega)$ be a discrete group. Then there exists a number $\mu_n$ (depending only on $n$) such that if $x \in \Omega$ then the group

$$\Gamma_x = \langle \gamma \in \Gamma \mid d_{\Omega}(x, \gamma x) < \mu_n \rangle$$

is virtually nilpotent.
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Result due to Gromov-Margulis-Thurston for $\mathbb{H}^n$ and Cooper-Long-Tillmann in general.
Rigidity and Flexibility

When \( n \geq 3 \) Mostow-Prasad rigidity tells us that complete finite volume hyperbolic structures are very rigid

**Theorem 1 (Mostow ’70, Prasad ’73)**

Let \( n \geq 3 \) and suppose that \( \mathbb{H}^n/\Gamma_1 \) and \( \mathbb{H}^n/\Gamma_2 \) both have finite volume. If \( \Gamma_1 \) and \( \Gamma_2 \) are isomorphic then \( \mathbb{H}^n/\Gamma_1 \) and \( \mathbb{H}^n/\Gamma_2 \) are isometric.
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There is no Mostow-Prasad rigidity for properly (strictly) convex domains.

There are examples of finite volume hyperbolic manifolds whose complete hyperbolic structure can be “deformed” to a non-hyperbolic convex projective structure.
Deformations

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- We say that $M_1 = \Omega_1/\Gamma_1$ is a deformation of $M_0$. 

Ex: Let $\Omega_0 \sim = H_n$, $\Gamma_0 \leq \text{PSO}(n, 1)$, such that $\Omega_0/\Gamma_0$ is finite volume and contains an embedded totally geodesic hypersurface $\Sigma$. Let $\Gamma_1$ be obtained by “bending” along $\Sigma$. 
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Structure of Hyperbolic Manifolds
The Closed Case

Let $\mathbb{H}^n/\Gamma$ be a closed hyperbolic manifold.

- Since $\Gamma$ acts cocompactly by isometries on $\mathbb{H}^n$ we see that $\Gamma$ is $\delta$-hyperbolic group (Švarc-Milnor)
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- By compactness, we see that if $1 \neq \gamma \in \Gamma$ then $\gamma$ is hyperbolic
- In particular, if $1 \neq \gamma \in \Gamma$ then $\gamma$ is positive proximal
Let $M = \Omega/\Gamma$ be a closed properly convex manifold that is a deformation of a closed strictly convex manifold $M_0 = \Omega_0/\Gamma_0$. 

Theorem 2 (Benoist)

Suppose $\Omega/\Gamma$ is closed. $\Omega/\Gamma$ is strictly convex if and only if $\Gamma$ is $\delta$-hyperbolic. 

Proof sketch.

If $\Omega$ is not strictly convex then it will contain arbitrarily fat triangles and is thus not $\delta$-hyperbolic. Since $\Gamma$ acts cocompactly by isometries on $\Omega$, Švarc-Milnor tells us that $\Omega$ is q.i. to $\Gamma$ and is thus $\delta$-hyperbolic.
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Theorem 3 (Benoist)

Let \( 1 \neq \gamma \in \Gamma \) then \( \gamma \) is positive proximal.

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- Again by compactness we have that if $1 \neq \gamma \in \Gamma$ then $\gamma$ is hyperbolic.
- Since $\Omega$ is strictly convex and $\gamma$ is hyperbolic we see that $\gamma$ has exactly 2 fixed points in $\partial \Omega$ and acts as translation along the geodesic connecting them. $\gamma$ is thus positive proximal.
Let $M = \mathbb{H}^n / \Gamma$ be a finite volume hyperbolic manifold. We can decompose $M$ as

$$M = M_K \bigsqcup_i C_i,$$

where $M_K$ is a compact and $\pi_1(M_K) = \Gamma$ and $C_i$ are components of the thin part called *cusps*. 
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where $M_K$ is a compact and $\pi_1(M_K) = \Gamma$ and $C_i$ are components of the thin part called *cusps*. As we will see, the Margulis lemma tells us that the $C_i$ have relatively simple geometry.
Geometry of the Cusps

Let $C$ be a cusp of a finite volume hyperbolic manifold and let

$$P = \left\{ \begin{pmatrix} 1 & v^T & |v|^2 \\ 0 & I_{n-1} & v \\ 0 & 0 & 1 \end{pmatrix} \mid v \in \mathbb{R}^{n-1} \right\}$$

be the group of parabolic translations fixing $\infty$. Let $x_0 \in \mathbb{H}^n$, then $C \cong B/\Delta$ where $B$ is horoball bounded by $Px_0$ and $\Delta$ is a finite extension of a lattice in $P$. 
Structure of Hyperbolic Manifolds
The Finite Volume Case

- $\Gamma$ no longer acts cocompactly on $\mathbb{H}^n$ and $\Gamma$ is no longer $\delta$-hyperbolic
Structure of Hyperbolic Manifolds
The Finite Volume Case

- Γ no longer acts cocompactly on $\mathbb{H}^n$ and Γ is no longer $\delta$-hyperbolic
- Instead Γ is $\delta$-hyperbolic \textit{relative to the cusps}
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• If $1 \neq \gamma \in \Gamma$ is freely homotopic into a cusp then $\gamma$ is parabolic, otherwise $\gamma$ is hyperbolic (positive proximal)
Let $\Omega / \Gamma$ be a finite volume (Hausdorff measure of Hilbert metric) strictly convex manifold.

**Theorem 4 (Cooper, Long, Tillmann ‘11)**

Let $M = \Omega / \Gamma$ be as above then

- $M = M_K \bigsqcup_i C_i$, where $M_K$ is compact and $C_i$ is projectively equivalent to the cusp of a finite volume hyperbolic manifold,
- $\Gamma$ is $\delta$-hyperbolic relative to its cusps, and
- If $1 \neq \gamma \in \Gamma$ is freely homotopic into a cusp then $\gamma$ is parabolic. Otherwise $\gamma$ is hyperbolic (positive proximal).
Consider the following example.

Let $K$ be the figure-8 knot, let $M = S^3 \setminus K$, and let $G = \pi_1(M)$.
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**Theorem 5 (B)**

There exists $\varepsilon > 0$ such that for each $t \in (-\varepsilon, \varepsilon)$ there is a properly convex domain $\Omega_t$ and a discrete group $\Gamma_t \leq \text{PGL}(\Omega_t)$ such that

- $\Omega_t/\Gamma_t \cong M$,
- $\Omega_0/\Gamma_0$ is the complete hyperbolic structure on $M$, and
- If $t \neq 0$ then $\Omega_t$ is not strictly convex.
Theorem 6 (B)

For each $t \in (-\epsilon, \epsilon)$ we can decompose $\Omega_t/\Gamma_t$ as $M^t_K \cup C^t$, where $M^t_K$ is compact and $C^t \cong T^2 \times [1, \infty)$.

- For each $t$, $C^t \cong B_t/\Delta_t$, where $\Delta_t$ is a lattice an Abelian group $P_t$ of “translations,” and $B_t$ is a “horoball” bounded by an orbit of $P_t$. 
For each $t \neq 0$ there is $\gamma_t \in \Gamma_t$ such that $\gamma_t$ is hyperbolic, freely homotopic into $C_t$, but not positive proximal.

$\Omega_t$ contains non-trivial line segments in $\partial \Omega_t$ that are preserved by conjugates of $\Delta_t$. In particular, $\Omega_t$ is not $\delta$-hyperbolic.
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Figure-8 Example

For each $t \neq 0$ there is $\gamma \in \Gamma_t$ such that $\gamma$ is hyperbolic, freely homotopic into $C_t$, but not positive proximal.

$\Omega_t$ contains non-trivial line segments in $\partial \Omega_t$ that are preserved by conjugates of $\Delta_t$. In particular, $\Omega_t$ is not $\delta$-hyperbolic.
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Theorem 7 (B, Long)

\[ 1 \neq \gamma \in \Gamma_t \text{ is positive proximal if and only if it cannot be freely homotoped into } C^t. \]
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$1 \neq \gamma \in \Gamma_t$ is positive proximal if and only if it cannot be freely homotoped into $C^t$.

Proof.

$\Leftarrow$ Let $1 \neq \gamma \in \Gamma_t$. No elements of $P_t$ are positive proximal, so if $\gamma$ is freely homotopic to $C^t$ then it is not positive proximal.
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\rightarrow If \( \gamma \) is not freely homotopic to \( C^t \) then \( \gamma \) has positive translation length and is thus hyperbolic. Furthermore, this translation length is realized by points on an axis.
Figure-8 Example

Proof (Continued).

Use Margulis lemma to construct a disjoint and $\Gamma_t$ invariant collection $\mathcal{H}_t$ of horoballs in $\Omega_t$. 

$\hat{\Omega}_t$ is $\delta$-hyperbolic
Proof (Continued).

Use Margulis lemma to construct a disjoint and $\Gamma_t$ invariant collection $\mathcal{H}_t$ of horoballs in $\Omega_t$. Let $\hat{\Omega}_t$ be the electric space obtained by collapsing the horospherical boundary components of $\Omega_t \setminus \mathcal{H}_t$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure8_example.png}
\caption{Figure-8 Example}
\end{figure}
Proof (Continued).

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**Lemma 8 (B, Long)**

$\hat{\Omega}_t$ is $\delta$-hyperbolic
Figure-8 Example

Proof (Continued).

• Since $\gamma$ is hyperbolic and preserves $\Omega_t$ we know that $\gamma$ has real eigenvalues of largest and smallest modulus and that these eigenvalues have the same sign.
Figure-8 Example

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- This gives rise to arbitrarily fat triangles in $\hat{\Omega}_t$. 

\[ \begin{array}{c}
\text{Diagram:}
\end{array} \]
Summary and Questions

- The structure of a finite volume strictly convex manifold is well understood.

- As you deform the structure, the "coarse" geometry of the compact part doesn't change.

- The geometry of the cusps may change as we deform, but can be understood using the Margulis lemma.

- Theorem 7 holds for all properly convex deformations of finite volume strictly convex manifolds in dimension 3.

- Theorem 7 should hold for higher dimensions.

- What can we say for deformations of deformations of infinite volume hyperbolic manifolds?
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