Classification of Generalized Cusps

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Outline

1. Cusps of hyperbolic manifolds
   • Description/geometry of cusps
   • Focus on properties to generalize
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   - Description/geometry of cusps
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2. Properly Convex Manifolds
   - What are they?
   - How do they similar/different to hyperbolic manifolds
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   - Description/geometry of cusps
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   - What are they?
   - How do they similar/different to hyperbolic manifolds

3. Generalized Cusps
   - Description/geometry
   - How to classify
Cusps of hyperbolic orbifolds

Let $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ be a lattice and $M = \mathbb{H}^n/\Gamma$ be a complete hyperbolic $n$-orbifold.
Cusps of hyperbolic orbifolds

Let $\Gamma \subset \text{Isom}(\mathbb{H}^n)$ be a lattice and $M = \mathbb{H}^n/\Gamma$ be a complete hyperbolic $n$-orbifold.

Using the “thik-thin” decomposition $M$ can be decomposed into

$$M = M_k \bigsqcup_i C_i,$$

where $C_i$ is finitely covered by $T^{n-1} \times [0, \infty)$. 

\[ \text{Diagram:} \quad M_K \quad \text{and} \quad C \]
Cusps of hyperbolic manifolds

Geometry of cusps

- Let $\mathbb{H}^n = \{(z, v) \in \mathbb{R} \times \mathbb{R}^{n-1} | z > \frac{1}{2} |v|^2 \} \subset \mathbb{RP}^n$
Cusps of hyperbolic manifolds
Geometry of cusps

- Let $\mathbb{H}^n = \{ (z, v) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid z > \frac{1}{2} \|v\|^2 \} \subset \mathbb{RP}^n$
- $\mathbb{H}^n$ is foliated by horospheres
  $S_t = \{ (z, v) \in \mathbb{H}^n \mid x = \frac{1}{2} \|v\|^2 + t \}, \ t > 0$
Cusps of hyperbolic manifolds

Geometry of cusps

Consider the following subgroups of $\text{SL}_{n+1}^\pm(\mathbb{R})$

$$T = \left\{ \begin{pmatrix} 1 & u & \frac{1}{2} |u|^2 \\ 0 & 0 & u \\ 0 & 0 & 1 \end{pmatrix} \mid u \in \mathbb{R}^{n-1} \right\}, \quad O = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid A \in O(n-1) \right\}$$

- $T$ acts simply transitively on each $S_t$
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- $O$ is a point stabilizer
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- $O$ is a point stabilizer
- $G = T \rtimes O$ preserves the foliation leafwise
Let

- $B_T = \bigcup_{t \geq T} S_t$ (horoball)
- $\Delta$ a lattice in $G$. 

Cusps of hyperbolic manifolds

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The cusp \( C \) can be realized as \( B_T/\Delta \)

The \( S_t/\Delta \) give a foliation of \( C \) by Euclidean \((n - 1)\)-orbifolds.
Properly convex manifolds

A subset $\Omega \subset \mathbb{RP}^n$ with non-empty interior is *properly convex* if

1. $\Omega$ is convex in $\mathbb{RP}^n$ (intersections with projective lines are connected)
2. $\overline{\Omega}$ is disjoint from some projective hyperplane.
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$\Omega$ can be realized as a compact, convex subset of $\mathbb{R}^n \subset \mathbb{RP}^n$. 
Properly convex manifolds

Let $\Omega$ be properly convex and let
$\text{PGL}(\Omega) = \{ A \in \text{PGL}_{n+1}(\mathbb{R}) \mid A(\Omega) = \Omega \}$. 
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A manifold $M = \Omega/\Gamma$ where $\Omega$ is properly convex and $\Gamma \subset \text{PGL}(\Omega)$ is a discrete subgroup is called \textit{properly convex}
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$\mathbb{H}^n$ is a properly convex domain (via the Klein model). Therefore complete hyperbolic manifolds are properly convex.
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$\mathbb{H}^n$ is a properly convex domain (via the Klein model). Therefore complete hyperbolic manifolds are properly convex.

In general, properly convex domains can have “flats” in their boundary.
Deforming properly convex manifolds

Let $M \cong \Omega_0/\Gamma_0$ be a complete hyperbolic manifold.
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In many cases one can find \textit{non-trivial} continuous families $\Omega_t/\Gamma_t \cong M$ of properly convex manifolds.
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If $M$ has cusps, what does the geometry of the cusps of $\Omega_t/\Gamma_t$ look like if $t \neq 0$?
Deforming properly convex manifolds

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If $M$ has cusps, what does the geometry of the cusps of $\Omega_t/\Gamma_t$ look like if $t \neq 0$? *They are generalized cusps.*
Generalized cusps

A generalized cusp is a properly convex manifold $C = \Omega/\Gamma$ where

- $C$ is diffeomorphic to $\partial C \times [0, \infty)$, with $\partial C$ compact
- $\Gamma \cong \pi_1 \partial C$ is virtually abelian
- $\partial C$ is strictly convex
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Cusps of finite volume hyperbolic manifolds are generalized cusps
Geometry of generalized cusps

Overview

Let \( W_n = \{ (\lambda_1, \ldots, \lambda_n) \mid 0 \leq \lambda_1 \leq \ldots \leq \lambda_n \} \)

Given an \( n \)-dimensional generalized cusp \( C \cong \Omega/\Gamma \) we get

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Given an $n$-dimensional generalized cusp $C \cong \Omega/\Gamma$ we get

- $\lambda \in \mathcal{W}_n$, unique up to scaling.
- A Lie group $\text{PGL}_{n+1}(\mathbb{R}) \supset G_\lambda \cong \underbrace{T_\lambda}_{\text{translations}} \rtimes \underbrace{O_\lambda}_{\text{point stabilizer}}$ that contains a conjugate of $\Gamma$ as a lattice.
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- A $G_\lambda$-invariant properly convex domain $\Omega_\lambda \subset \Omega$ (e.g. $B_T \subset \mathbb{H}^n$)
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- A \( G_\lambda \)-invariant properly convex domain \( \Omega_\lambda \subset \Omega \) (e.g. \( B_T \subset \mathbb{H}^n \))
- A foliation of \( \Omega_\lambda \) by strictly convex hypersurfaces (horospheres)
A quasi-hyperbolic cusp

- Let $\Omega_{(0,1)} = \{(z, y) \in \mathbb{R} \times \mathbb{R}_+ \mid z > -\log(y)\}$
- $\Omega_{(0,1)}$ is foliated by $S_t = \{(z, y) \in \Omega \mid z = -\log(y) + t\}$ (horospheres)
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Let $\Gamma$ be a lattice in the Lie group $G_{(0,1)} = \left\{ \begin{pmatrix} 1 & 0 & -u \\ 0 & e^u & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid u \in \mathbb{R} \right\}$
Mixed cusps

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- $\lambda \in \mathcal{W}_n$ such that $\lambda_1 = 0$
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- Let $f_\lambda : \mathbb{R}^{p-1}_s := \mathbb{R}^{p-1} \times \mathbb{R}_+^s \rightarrow \mathbb{R}$ given by

$$
(x_1, \ldots, x_{p-1}, y_1, \ldots, y_s) \mapsto \frac{1}{2} \sum_{i=1}^{p-1} x_i^2 - \sum_{i=1}^s \lambda_{p+i}^{-1} \log(y_i)
$$

hyperbolic part

quasi-hyperbolic part
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- hyperbolic part
- quasi-hyperbolic part

- Let $\Omega_\lambda = \{(z, (x, y)) \in \mathbb{R} \times \mathbb{R}^{p-1}_s \mid z \geq f_\lambda(x, y)\}$ foliated by $f_\lambda$ level sets
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**Figure:** On the left \( \Omega_{(0,0,1)} \) and on the right \( \Omega_{(0,1,1)} \)
Mixed cusps
Symmetry group

\[ T_\lambda = \left\{ \begin{pmatrix} 1 & x & 0 & f(x, y) \\ 0 & I_{p-1} & 0 & x \\ 0 & 0 & D_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in PGL_{n+1}(\mathbb{R}) \mid (x, y) \in \mathbb{R}^{p-1}_s \right\} \]

\[ O_\lambda = \underbrace{O_x}_{\text{Orthogonal}} \times \underbrace{P_{y,\lambda}}_{\text{Permutations}} \]
Diagonalizable cusps

Let $\lambda \in W_n$ with $\lambda_1 > 0$ and let

$$O_\lambda = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \mid \sum_{i=1}^n \lambda_i^{-1} \log(x_i) > 0\}$$

$O_\lambda$ is foliated by $S_t = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n \mid \sum_{i=1}^n \lambda_i^{-1} \log(x_i) = t\}$
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Let $\Gamma$ be a lattice in the Lie group

$$T_\lambda = \left\{ \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \mid \sum_{i=1}^n \lambda^{-1}_i \log(u_i) = 0 \right\}$$

$O_\lambda$ = Coordinate permutation where $\lambda_i = \lambda_j$
Main Theorem

Theorem 1

(B–Cooper–Leitner) Let $C = \Omega/\Gamma$ be an $n$-dimensional generalized cusp. Then there is a is a $\lambda \in W_n$, unique up to scaling, such that

- \( \Gamma \) is conjugate to a lattice \( \Gamma' \subset G_\lambda \)
- \( C \) deformation retracts onto a submanifold \( C' = \Omega'/\Gamma \) that is projectively equivalent to \( \Omega_\lambda/\Gamma' \).
Remaining questions

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- What is the moduli space of generalized cusps? Is it an orbifold?
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- Realization Problem: given a generalized cusp $C$, can you find an interesting properly convex manifold $M$ with a cusp projectively equivalent to $C$?
- Can we use the geometry of generalized cusps to give coordinates on the space of convex projective structures on a fixed manifold? (Fenchel-Nielsen coordinates)
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Thank you