Indeterminate Forms

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0\cdot\infty, \quad 0^0, \quad \infty^0, \quad 1^\infty, \quad \infty-\infty$$

These are the so called indeterminate forms. One can apply L'Hopital's rule directly to the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. It is simple to translate $0\cdot\infty$ into $\frac{0}{1/\infty}$ or into $\frac{\infty}{1/0}$, for example one can write $\lim_{x\to\infty} xe^{-x}$ as $\lim_{x\to\infty} x/e^x$ or as $\lim_{x\to\infty} e^{-x}/(1/x)$. To see that the other forms are indeterminate note that

$$\ln 0^0 = 0\ln 0 = 0(-\infty) = 0\cdot\infty, \quad \ln\infty^0 = 0\ln\infty = 0\cdot\infty, \quad \ln 1^\infty = \infty\ln 1 = \infty\cdot 0 = 0\cdot\infty$$

$$e^{\infty-\infty} = \frac{e^\infty}{e^\infty} = \frac{\infty}{\infty}$$

These formula's also suggest ways to compute these limits using L'Hopital's rule. Basically we use two things, that e^x and $\ln x$ are inverse functions of each other, and that they are continuous functions. If g(x) is a continuous function then $g(\lim_{x\to a} f(x)) = \lim_{x\to a} g(f(x))$.

For example let's figure out $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$. This is of the indeterminate form 1^∞. We write $\exp(x)$ for e^x so to reduce the amount of exponents on exponents.

$$\lim_{x\to\infty}(1+\frac{1}{x})^x = \exp(\ln(\lim_{x\to\infty}(1+\frac{1}{x})^x)) = \exp(\lim_{x\to\infty}\ln((1+\frac{1}{x})^x))$$

$$= \exp(\lim_{x\to\infty} x\ln(1+\frac{1}{x})) = \exp(\lim_{x\to\infty}\frac{\ln(1+\frac{1}{x})}{1/x})$$

We can now apply L'Hopital's since the limit is of the form $\frac{0}{0}$.

$$= \exp(\lim_{x\to\infty}\frac{(1/(1+\frac{1}{x}))(-1/x^2)}{-1/x^2}) = \exp(\lim_{x\to\infty} 1/(1+\frac{1}{x})) = \exp(1) = e.$$

The indeterminate form $\infty-\infty$ does not work with L'Hopital even after the conversion to e^∞/e^∞. These must be done via algebraic or other calculus tricks. For example let's figure out $\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x}) = 0$.

$$\lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x}) = \lim_{x\to\infty}(\sqrt{x+1}-\sqrt{x})(\frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}})$$

$$= \lim_{x\to\infty}\frac{(x+1)-x}{\sqrt{x+1}+\sqrt{x}} = \lim_{x\to\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}} = \frac{1}{\infty+\infty} = \frac{1}{\infty} = 0$$

Exercises:

(A) Show $\lim_{x\to\infty}(1+\frac{k}{x})^x = e^k$.

(B) Show $\lim_{x\to 0^+} x^x = 1$.

(C) Show $\lim_{x\to\infty}(1/x)^x = 0$.

(D) Show $\lim_{x\to 0^+} x^{\tan x} = 1$.

(E) Show $\lim_{x\to 0^+} x^{(1/\ln x)} = e$.

(F) Show $\lim_{x\to\infty}((x+1)^2 - x^2) = \infty$.

(G) Show $\lim_{x\to\infty}((x+2)^1 - x^1) = 2$.