Using the limit definition of $\int$

Problem A: Use the formula

$$\sum_{1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

and the limit-of-the-left-sum definition

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{0}^{n-1} f(x_i)\Delta x$$

to show that the integral

$$\int_0^1 x^2 dx = \frac{1}{3}$$

In class, we did this for $\int_0^1 x dx$ using $\sum_1^n i = n(n+1)/2$ as follows: $f(x) = x$, $a = 0$, $b = 1$ and $\Delta x = (b-a)/n = (1-0)/n = 1/n$. It follows that $x_i = a + i \cdot \Delta x = i/n$ So

$$\lim_{n\to\infty} \sum_{0}^{n-1} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \sum_{0}^{n-1} x_i \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{0}^{n-1} \frac{i}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{0}^{n-1} i$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \frac{(n-1)((n-1)+1)}{2}$$

$$= \lim_{n\to\infty} \frac{n^2-n}{2n^2}$$

$$= \frac{1}{2}$$