Blue and Red's Excellent Adventure

Florida State University coeds Blue and Red live a relatively happy life but their dreams of a garage nanotechnology start up are threatened by the fact that they're flunking history of science. Red's advisor, a hard-nosed engineer, has threaten to ship Red off to a beauty school in Gainesville if she doesn't pass.

Cool dude Roofless arrives in a time traveling phone booth and tells the young engineers that one day their nanotechnology will be the foundation of a peaceful, prosperous civilization, but none of that can happen if they don't pass. Roofless loans them the phone booth and they are off to do a little hands-on research. Fortunately the phone booth has Maple installed.

Archimedes I

The first stop is ancient Greece around 250 BC, the place is the city of Syracuse on the south east corner of Sicily, where Archimedes is running through the streets yelling 'Eureka'. Your first job is to learn how Archimedes established the first exact expression for the volume of a sphere. But first some calculus.

Just Calculus

a. Show how a modern day Calculus student would derive the formula $\frac{4}{3}\pi r^3$ by rotating the circle $x^2+y^2=r^2$ about the x-axis and integrating.

b. While you are at, derive the formulas for the volume of a cylinder, $\pi r^2 h$, by rotating $y=r$ and for the volume of a cone, $\frac{1}{3}\pi r^2 h$, by rotating $y=\frac{r}{h}x$.

The Method

Archimedes says we need to know the excellent trick of knowing that two figures with the same total height and equal areas of horizontal cross- sections at every intermediate height most have the same volume. [This is similar to noting that an equilateral triangle and a right triangle with same height and base have the same area because they have the same total height, and the equal lengths of horizontal cross-sections at every intermediate height.] Second we need to that the volume of a pyramid is $\frac{1}{3}Bh$ where B is the area of the base. Archimedes said this was known by the excellent ancient Egyptians, which, of course, were known for their pyramids.

c. Show how to construct a cone and a square pyramid (a pyramid with a square base). with the same height h and area of the base B by computing the dimensions (radius $r=r(y)$ for the cone, side $s=s(y)$ for the square pyramid) at any height y and show that the horizontal cross-sections have the same area. [This is slightly easier if you stand the figures on their head, or equivalently start $y=0$ at their "points".] Thus the volume of a cone, by the excellent trick is the same as the volume of the pyramid, namely $\frac{1}{3}Bh$.

Archimedes II

Archimedes lived almost two millennnium before the invention of calculus yet he seemed to know how to integrate. The trick Archimedes used was so excellent he wanted it placed on his grave — and it was. The sphere obtained by rotating $x^2+y^2=r^2$ about the x-axis is inscribed in the cylinder obtained by rotating $y=r$, for $-r\le x\le r$, about the x-axis. For a fixed $x=k$ the plane cross section hits the cylinder in a circle of radius r and the sphere in a circle of smaller radius.

d. Show the area of the part $x=k$ cross section that is not in the sphere is exactly the same as the area of the $x=k$ cross section of 'two cones' obtained by rotating $y=x$ for $-r\le x\le r$ about the x-axis. Thus by "the method" and addition, the volume of the sphere = volume of the cylinder minus the volume of the cones.

"Hey," says Red, "if we think of the sphere as a ball, we have shown that the ball-not is a pointy little conical thing."

Archimedes' excellent trick also computed the surface area of the sphere. Indeed, he noted the following, but you can use the calculus based formula for surface area for when a curve $y=f(x)$ is rotated about the x-axis.

$$S=\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\;dx$$

e. Show that the surface area between $x=k_1$ and $x=k_2$ is the same for the cylinder as for the inscribed sphere. Thus the surface area of the sphere = the surface area of the side of the whole cylinder (without the top and bottom) $=4\pi r^2$.

Expanding Spheres

f. Another way to get the surface area of a sphere, is to notice that if we increase the radius of a sphere by Δr, then the increase in volume ΔV is roughly a skin with area equal to the surface area S of the sphere and height Δr. Divide by Δr and take limits. Show how this yields the same $4\pi r^2$ formula for the surface area.

Archimedes waved the two coeds off after telling them about his excellent lever. Just give me a lever long enough and a place to stand and I can move the earth. “What a way to shake up a party,” says Blue, “Yes, party on dude,” says Red as both return to the phone booth for their second stop.

Einstein

The second stop is Switzerland at the early part of the 20^{th} century where Einstein is riding a trolley to work in a relative way. Your job is to learn how to compute the 4-dimensional “hyper-volume” of some standard objects. Einstein explains the excellent trick of how you can think of time as the 4^{th} dimension. “If a cube exists from time $t = t_0$ to $t = t_1$, then the cube has hyper-volume or time-space volume which is equal to $t_1 - t_0$ times the volume of the cube. Now if the $V = V(t)$ is the volume at time t varies as a function of time, the hyper-volume between $t = t_0$ and t_1 will be

$$H = \int_{t_0}^{t_1} V(t)\ dt$$

which is same form as the way volume can be computed from area, or area can be computed from length. The 4^{th} dimension could be just another space dimension and it would still work the same. You know ladies, I didn’t invent the idea of four dimensions. Indeed, much of the mathematics I needed for my theories of relativity had already been developed by the most excellent Riemann.”

Hyper-volume

g. A hyper-pyramid has hyper-height h (in the time direction if you like) And as t goes from $t = 0$ to $t = h$, the 3-dimension volume of the cross-section is a cube with changing size. The size of the side of the cube, $s(t)$ changes linearity from 0 to S as t goes from 0 to h. Show the hyper-volume is given by the formula $H = \frac{1}{4}Vh$ Where $V = S^3$ is the volume of the base cube (at $t = h$).

h. A hyper-sphere of radius r has equation $x^2 + y^2 + z^2 + t^2 = r^2$. Show that the cross section at $t = a$ is a regular 3-dimensional sphere of radius $\sqrt{r^2 - a^2}$ and that the hyper-volume is given by the formula $H = \frac{1}{2}\pi^2 r^4$. [Maple will do the integral faster if you do ‘assume($r > 0$)’ first.]

Einstein waved the two coeds off after telling them stories about red-shifted and blue-shifted light. “Cool” said Blue, “it is nice to know if you are coming or going.” As they return to the phone booth, Red says “this has been a most excellent adventure”.

From the Course Syllabus

PROJECT. You will work on the project in groups of 1–4 students. This project will be a substantial assignment, giving you a chance to earn part of your grade in an environment which simulates the so-called “real world” better than does an in-class exam. It will also give your instructor a chance to base part of your grade on your best work, produced in a setting where time should not be a factor (assuming you start on your project as soon as it is assigned). The results of your work on your project will be presented in a report (one report per group). Each member will also submit a “group evaluation” giving their impression of the relative contribution of each member to the group’s effort. These evaluations are due with the project. It is not guaranteed that each member of the group will receive the same grade. The reports will be graded not only on their mathematical content but also on the quality of the presentation: clarity, neatness, and proper grammar are also important. **Both reports and group evaluations must be typed.** The project will be assigned on Tuesday, October 23 and due on Tuesday, November 6. [Dates change to assigned on Thursday October 25 and due on Thursday November 8.]

Grading

Each part is worth 10 points which leaves 20 points for clarity, neatness, grammar and general wow value, for a total of 100 points. You are free to use Maple on all the integrals, but for clarity you should do the easy ones by hand. There are a very small number of bonus points.

If you like the statement of this project, you might also like:

http://www.yesterdayland.com/popopedia/shows/movies/mv1525.php

Also some maple drawn graphics illustrating some of objects in the project is available at:

http://www.math.fsu.edu/~bellenot/class/f01/cal2/project.html