Problems on continuous linear functionals (part of HW#4)

1. Let $E = \mathbb{R}^2$ with the norm $\|(x, y)\| = \max\{|x|, |y|, |y - x|\}$
   (a) Draw the unit sphere $S = \{(x, y) : \|(x, y)\| = 1\}$
   (b) Compute the norms of the functionals $f(x, y) = x$, $g(x, y) = y$, $h(x, y) = x+y$ and $k(x, y) = x-y$.
   (c) Derive a formula for the norm of the function $f(x, y) = ax + by$ for any reals a and b.

2. Let $c = \{(x_n) : \lim x_n \text{ exists}\}$ with the sup norm so that $c_0 \subset c \subset m = \ell_\infty$.
   (a) Show $\phi : c \to \mathbb{R}$ given by $\phi((x_n)) = \lim x_n$ is a continuous linear function on c and compute its norm.
   (b) show ϕ is not defined on m and ϕ is the 0 functional on c_0.
   (c) Let η be the sequence (1) which is constantly one. Show for $x = (x_n) \in c$, $x - \phi(x)\eta \in c_0$.

3. Let c and ϕ be as in 2, Show the dual of c is given by

$$c^* = \{\alpha\phi + x : \alpha \in \mathbb{R}, x \in \ell_1\}$$

   Part of this problem is coming up with a formula for $\|\alpha\phi+x\|$. Hint: If $y = (y_n) \in \ell_1$ and $x = (x_n) \in c$ then $\langle y, x\rangle = \sum_n y_n x_n$ is the linear functional on c associated with this y in ℓ_1.