The last problem set

1. Let E be a norm space with dual E^*, if $A \subset E$ (resp $B \subset E^*$) define $A^\perp = \{f \in E^* : f(a) = 0\ \forall a \in A\}$ (resp $B^\top = \{x \in E : f(x) = 0\ \forall f \in B\}$)

   (a) Show both $A^\perp$ and $B^\top$ are closed subspaces.

   (b) Show $A \subset A^{\perp\top}$ and $B \subset B^{\top\perp}$

   (c) Show $A^{\perp\top}$ is the closure of the linear span of A.

   (d) If $E = \ell_1$ and $B = c_0 \subset m = \ell_\infty = E^*$ is a closed subspace of E^*, but $B \neq B^{\top\perp}$

2. Show if $T : E \to E$ is bounded linear operator, then let $N_n = \{x \in E : T^n x = 0\}$ be the kernel of T^n. Show $N_0 \subset N_1 \subset N_2 \cdots \subset N_n \subset N_{n+1} \ldots$

   (a) Show if A is compact then for $T = I - A$ there is an m so that for all k, $N_m = N_{m+k}$.

   (b) Show that for the shift to the left $T : \ell_2 \to \ell_2$ given by $T((a_1, a_2, a_3, \ldots)) = (a_2, a_3, a_4, \ldots)$ the subspaces satisfy $N_n \neq N_{n+1}$

3. Show if (x_n) are norm one elements of the norm space E, (f_n) are norm one functionals in E^*, and (λ_n) are scalars with $\sum_{n=1}^{\infty} |\lambda_n| < \infty$ then the operator $A(x) = \sum_{n=1}^{\infty} \lambda_n f_n(x) x_n$ is a well-defined compact operator on E.

   Suppose further that $f_i(x_j) = \delta_{i,j}$, show that x_n is an eigenvector of T with eigenvalue λ_n.

4. If $A : E \to E$ compact then for every x_n with $x_n \to x$ weakly, then $Ax_n \to Ax$ in norm.

   If E is reflexive, then the converse is also true.