Fourier Transform Examples

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1 Formula Sheet

(1) $$\mathscr{F}[f(x)] = \widehat{f}(w) \text{ or simply } \mathscr{F}[f] = \widehat{f}$$

(2) $$\mathscr{F}^{-1}[\widehat{f}(w)] = f(x) \text{ or simply } \mathscr{F}^{-1}[\widehat{f}] = f$$

(3) $$\mathscr{F}[f(x)](w) = \widehat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\, dx$$

(4) $$\mathscr{F}^{-1}[\widehat{f}(w)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(w) e^{iwx}\, dw$$

(5) $$\mathscr{F}[u(x,t)](w,t) = \widehat{u}(w,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-iwx}\, dx$$

(6) $$\mathscr{F}^{-1}[\widehat{u}(w,t)](x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{u}(w,t) e^{iwx}\, dw$$

(7) $$\mathscr{F}[af(x) + bg(x)](w) = a\widehat{f}(w) + b\widehat{g}(w)$$

(8) $$\mathscr{F}[f'(x)](w) = iw\widehat{f}(w)$$

(9) $$\mathscr{F}[f''(x)](w) = -w^2\widehat{f}(w)$$

(10) $$\mathscr{F}[\frac{\partial}{\partial x} u(x,t)](w,t) = iw\widehat{u}(w,t)$$

(11) $$\mathscr{F}[\frac{\partial^2}{\partial x^2} u(x,t)](w,t) = -w^2\widehat{u}(w,t)$$

(12) $$\mathscr{F}[\frac{\partial}{\partial t} u(x,t)](w,t) = \frac{\partial}{\partial t}\widehat{u}(w,t)$$

(13) $$\mathscr{F}[\frac{\partial^2}{\partial t^2} u(x,t)](w,t) = \frac{\partial^2}{\partial t^2}\widehat{u}(w,t)$$

(14) $$[f * g](x) = \int_{-\infty}^{\infty} f(w)g(x-w)\, dw = [g * f](x) = \int_{-\infty}^{\infty} f(x-w)g(w)\, dw$$

(15) $$\mathscr{F}[f * g] = \sqrt{2\pi}\widehat{f}\widehat{g}$$

(16) $$f(x-a) = \mathscr{F}^{-1}[e^{-iwa}\widehat{f}(w)]$$

(17) $$\mathscr{F}[\exp(-ax^2)] = \frac{1}{\sqrt{2a}} \exp(\frac{-w^2}{4a})$$

(18) $$\sin wa = \frac{e^{iwa} - e^{-iwa}}{2i}$$

(19) $$\cos wa = \frac{e^{iwa} + e^{-iwa}}{2}$$

2 Formula Justifications

Equations (1), (3) and (5) readly say the same thing, (3) being the usual definition. (Warning, not all textbooks define the these transforms the same way.) Equations (2), (4) and (6) are the respective inverse transforms.

What kind of functions is the Fourier transform defined for? Clearly if $f(x)$ is real, continuous and zero outside an interval of the form $[-M, M]$, then $\widehat{f}$ is defined as the improper integral $\int_{-\infty}^{\infty}$ reduces to the proper integral $\int_{-M}^{M}$. If $f(x)$ decays fast enough as $x \to \infty$ and $x \to -\infty$, then $\widehat{f}(w)$ is also defined. However there are much larger collections of objects for which the transform can be defined. For example, if $\delta(x)$ is the Dirac delta function, then $\widehat{\delta}(w) = 1/\sqrt{2\pi}$ the constant function. Also one can see that the inverse transform of $\delta(w)$ is the constant function $1/\sqrt{2\pi}$.

Equation (7) follows because the integral is linear, the inverse transform is also linear.

Equation (8) follows from integrating by parts, using $u = e^{-iwx}$ and $dv = f'(x)\,dx$ and the fact that $f(x)$ decays as $x \to \infty$ and $x \to -\infty$.

$$\int_{-\infty}^{\infty} f'(x)e^{-iwx}\,dx = f(x)e^{-iwx}\big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} -f(x)iwe^{iwx}\,dx = (0-0) + iw\widehat{f}(w)$$

Equation (9) is just (8) applied twice. And (10) and (11) are just restatements with more variables.

Equation (12) requires going back to the definition of the limit.

$$\mathscr{F}\left[\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}\right] = \int_{-\infty}^{\infty} \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} e^{-iwx}\,dx$$

$$= \frac{\widehat{u}(w,t+\Delta t) - \widehat{u}(w,t)}{\Delta t} \to \frac{\partial}{\partial t}\widehat{u}(w,t)$$

One now takes limits of both sides. We need to know that the fourier transform is continuous with this kind of limit, which is true, but beyond our scope to show. Equation (13) is (12) done twice.

Equation (14) says $f * g = g * f$ and this is done by substitution; use $u = x - w$; $du = -dw$; $w = x - u$; $u = \infty$ when $w = -\infty$ and $u = -\infty$ when $w = \infty$ to obtain

$$\int_{w=-\infty}^{\infty} f(w)g(x-w)\,dw = \int_{u=\infty}^{-\infty} f(x-u)g(u)\,(-du) = \int_{u=-\infty}^{\infty} f(x-u)g(u)\,du$$

which used the negative sign to change the order of integration.

Equation (15) uses

$$\int_{x=-\infty}^{\infty} (f*g)e^{-iwx}\,dx = \int_{x=-\infty}^{\infty}\int_{s=-\infty}^{\infty} f(s)g(x-s)\,ds e^{-iwx}\,dx$$

$$= \int_{x=-\infty}^{\infty}\int_{s=-\infty}^{\infty} f(s)g(x-s)e^{-iwx}\,ds\,dx$$

$$= \int_{s=-\infty}^{\infty}\int_{x=-\infty}^{\infty} f(s)g(x-s)e^{-iwx}\,dx\,ds$$

Note that we have interchanged the order of integration, now we let $u = x - s$, $x = u + s$, $du = ds$, $u = \pm\infty$ when $x = \pm\infty$

$$= \int_{s=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f(s)g(u)e^{-iw(u+s)}\,du\,ds$$

$$= \int_{s=-\infty}^{\infty} f(s)e^{-iws}\,ds \int_{u=-\infty}^{\infty} g(u)e^{-iwu}\,du$$

since the u terms are constant as the integral with respect to ds is concerned. So the inital expression is $\sqrt{2\pi}\mathscr{F}[f*g]$ and the end epression is $\sqrt{2\pi}\widehat{f}\sqrt{2\pi}\widehat{g}$ which is where the $\sqrt{2\pi}$ factor comes from.

To show equation (16) we compute $\mathscr{F}[f(x-a)]$ and substitute $u = x - a$; $x = u + a$; $dx = du$:

$$\int_{x=-\infty}^{\infty} f(x-a)e^{-iwx}\,dx = \int_{u=-\infty}^{\infty} f(u)e^{-iw(u+a)}\,du = e^{-iwa}\int_{u=-\infty}^{\infty} f(u)e^{-iwu}\,du = e^{-iwa}\widehat{f}(w)$$

For the bell shaped curves, equation (17) is done in earlier editions of the textbook. We repeat the calculation for reference only

$$\mathscr{F}[\exp(-ax^2)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ax^2 - iwx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(-\left(\sqrt{a}x + \frac{iw}{2\sqrt{a}}\right)^2 + \left(\frac{iw}{2\sqrt{a}}\right)^2\right)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{w^2}{4a}\right)\int_{-\infty}^{\infty}\exp\left(-\left(\sqrt{a}x + \frac{iw}{2\sqrt{a}}\right)^2\right)\,dx$$

We claim that the integral above has value $I = \sqrt{\frac{\pi}{a}}$. First we do the substitution

$$v = \sqrt{a}x + \frac{iw}{2\sqrt{a}}$$

so that $dv = \sqrt{a}dx$ and hence

$$I = \int_{-\infty}^{\infty}\exp(-v^2)\,\frac{dv}{\sqrt{a}}$$

The result follows since

$$\int_{-\infty}^{\infty}\exp(-v^2)\,dv = \sqrt{\pi}$$

comes from Calculus 3.

Finally (18) and (19) are from Euler's $e^{i\theta} = \cos\theta + i\sin\theta$.

3 Solution Examples

- Solve $2u_x + 3u_t = 0; \quad u(x,0) = f(x)$ using Fourier Transforms.

  Take the Fourier Transform of both equations. The initial condition gives

  $$\widehat{u}(w,0) = \widehat{f}(w)$$

  and the PDE gives

  $$2(iw\widehat{u}(w,t)) + 3\frac{\partial}{\partial t}\widehat{u}(w,t) = 0$$

  Which is basically an ODE in t, we can write it as

  $$\frac{\partial}{\partial t}\widehat{u}(w,t) = -\frac{2}{3}iw\widehat{u}(w,t)$$

  and which has the solution

  $$\widehat{u}(w,t) = A(w)e^{-2iwt/3}$$

  and the initial condition above implies $A(w) = \widehat{f}(w)$

  $$\widehat{u}(w,t) = \widehat{f}(w)e^{-2iwt/3}$$

  We are now ready to inverse Fourier Transform and equation (16) above, with $a = 2t/3$, says that

  $$u(x,t) = f(x - 2t/3)$$

- Solve $2tu_x + 3u_t = 0; \quad u(x,0) = f(x)$ using Fourier Transforms.
  Take the Fourier Transform of both equations. The initial condition gives

$$\widehat{u}(w,0) = \widehat{f}(w)$$

  and the PDE gives

$$2t(iw\widehat{u}(w,t)) + 3\frac{\partial}{\partial t}\widehat{u}(w,t) = 0$$

  Which is basically an ODE in t, we can write it as

$$\frac{\partial}{\partial t}\widehat{u}(w,t) = -\frac{2}{3}iwt\widehat{u}(w,t)$$

  and which has the solution (separate variables)

$$\widehat{u}(w,t) = A(w)e^{-iwt^2/3}$$

  and the initial condition above implies $A(w) = \widehat{f}(w)$

$$\widehat{u}(w,t) = \widehat{f}(w)e^{-iwt^2/3}$$

  We are now ready to inverse Fourier Transform and equation (16) above, with $a = t^2/3$, says that

$$u(x,t) = f(x - t^2/3)$$

- Solve the heat equation $c^2 u_{xx} = u_t; \quad u(x,0) = f(x)$
  Take the Fourier Transform of both equations. The initial condition gives

$$\widehat{u}(w,0) = \widehat{f}(w)$$

  and the PDE gives

$$c^2(-w^2\widehat{u}(w,t)) = \frac{\partial}{\partial t}\widehat{u}(w,t)$$

  Which is basically an ODE in t, we can write it as

$$\frac{\partial}{\partial t}\widehat{u}(w,t) = -c^2w^2\widehat{u}(w,t)$$

  Which has the solution

$$\widehat{u}(w,t) = A(w)e^{-c^2w^2t}$$

  and the initial condition above implies $A(w) = \widehat{f}(w)$

$$\widehat{u}(w,t) = \widehat{f}(w)e^{-c^2w^2t}$$

  We are now ready to inverse Fourier Transform: First use (17) with

$$\frac{1}{4a} = c^2t \text{ or } a = \frac{1}{4c^2t}$$

  to note that

$$\frac{\sqrt{2}}{2c\sqrt{t}}\mathscr{F}\left[\exp(-\frac{x^2}{4c^2t}\right] = e^{-c^2w^2t}$$

  So that by the convolution equation (15)

$$u(x,t) = f(x) * \left(\frac{1}{2c\sqrt{\pi t}}\right)\exp\left(-\frac{x^2}{4c^2t}\right)$$

- Solve the wave equation $c^2 u_{xx} = u_{tt}; \quad u(x,0) = f(x)$ and $u_t(x,0) = g(x)$

  Take the Fourier Transform of both equations. The initial condition gives

$$\widehat{u}(w,0) = \widehat{f}(w)$$

$$\widehat{u_t}(w,0) = \frac{\partial}{\partial t}\widehat{u}(x,t)\bigg|_{t=0} = \widehat{g}(w)$$

  and the PDE gives

$$c^2(-w^2\widehat{u}(w,t)) = \frac{\partial^2}{\partial t^2}\widehat{u}(w,t)$$

  Which is basically an ODE in t, we can write it as

$$\frac{\partial^2}{\partial t^2}\widehat{u}(w,t) + c^2 w^2 \widehat{u}(w,t) = 0$$

  Which has the solution, and derivative

$$\widehat{u}(w,t) = A(w)\cos cwt + B(w)\sin cwt$$

$$\frac{\partial}{\partial t}\widehat{u}(w,t) = -cwA(w)\sin cwt + cwB(w)\cos cwt$$

  so the first initial condition gives $A(w) = \widehat{f}(w)$ and the second gives $cwB(w) = \widehat{g}(w)$ make the solution

$$\widehat{u}(w,t) = \widehat{f}(w)\cos cwt + \frac{\widehat{g}(w)}{w}\frac{\sin cwt}{c}$$

  Lets first look at

$$\widehat{f}(w)\cos cwt = \widehat{f}(w)\left(\frac{e^{iwct} + e^{-iwct}}{2}\right)$$

$$= \frac{1}{2}\left(\widehat{f}(w)e^{icwt} + \widehat{f}(w)e^{-icwt}\right)$$

  Applying equation (16) with $a = -ct$ and with $a = ct$ yields

$$\mathscr{F}^{-1}[\widehat{f}(w)\cos cwt] = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$$

  The second piece

$$\frac{\widehat{g}(w)}{w}\frac{\sin cwt}{c} = \frac{\widehat{g}(w)}{iw}\frac{\sin cwt}{-ic}$$

  and now the first factor looks like an integral, as a derivative with respect to x would cancel the iw in bottom. Define

$$h(x) = \int_{s=0}^{x} g(s)\,ds$$

  By fundamental theorem of calculus

$$h'(x) = g(x)$$

  and by (8)

$$\widehat{g}(w) = iw\widehat{h}(w)$$

  So

$$\frac{\widehat{g}(w)}{w}\frac{\sin cwt}{c} = \widehat{h}(w)\left(\frac{e^{icwt} - e^{-icwt}}{2i}\right)\frac{1}{-ic}$$

$$= \frac{1}{2c}\left(\widehat{h}(w)e^{icwt} - \widehat{h}(w)e^{-icwt}\right)$$

Applying equation (16) with $a = -ct$ and with $a = ct$ yields

$$\mathscr{F}^{-1}[\frac{1}{wc}\widehat{g}(w)\sin cwt] = \frac{1}{2c}\left(h(x+ct) - h(x-ct)\right)$$
$$= \frac{1}{2c}\left(\int_0^{x+ct} g(s)\,ds - \int_0^{x+ct} g(s)\,ds\right) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

Putting both pieces together we get D'Alembert's solution

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

(The careful reader will notice that there might be a problem finding the fourier transform of $h(x)$ due to likelyhood of $\lim_{x\to\pm\infty} h(x) \neq 0$. But that is a story for another day.)

- Solve $u_{xx} + u_{yy} = 0$ on infinite strip $(-\infty,\infty)\times[0,1]$ with boundary conditions $u(x,0) = 0$ and $u(x,1) = f(x)$.

  Take the Fourier Transform of all equations. The boundary conditons yield

$$\widehat{u}(w,0) = 0$$
$$\widehat{u}(w,1) = \widehat{f}(w)$$

  and the PDE gives

$$-w^2\widehat{u}(w,y) + \frac{\partial^2}{\partial y^2}\widehat{u}(w,y) = 0$$

  Which is basically an ODE in y, with a solution of the form

$$\widehat{u}(w,y) = A(w)\cosh wy + B(w)\sinh wy$$

  The $y = 0$ condition implies $A(w) = 0$ and the $y = 1$ implies

$$B(w) = \frac{\widehat{f}(w)}{\sinh w}$$
$$\widehat{u}(w,y) = \widehat{f}(w)\frac{\sinh wy}{\sinh w}$$

  We get the solution

$$u(x,y) = \frac{1}{\sqrt{2\pi}}\int_{w=-\infty}^{\infty} \widehat{f}(w)\frac{\sinh wy}{\sinh w}e^{iwx}\,dw$$

- Solve $u_x + u_t = 0; \quad u(x,0) = f(x)$ (Old Homework Problem) Take the Fourier Transform of both equations. The initial condition gives

$$\widehat{u}(w,0) = \widehat{f}(w)$$

  and the PDE gives

$$iw\widehat{u}(w,t) + \frac{\partial}{\partial t}\widehat{u}(w,t) = 0$$

  Which is basically an ODE in t, we can write it as

$$\frac{\partial}{\partial t}\widehat{u}(w,t) = -iw\widehat{u}(w,t)$$

  and which has the solution

$$\widehat{u}(w,t) = A(w)e^{-iwt}$$

  and the initial condition above implies $A(w) = \widehat{f}(w)$

$$\widehat{u}(w,t) = \widehat{f}(w)e^{-iwt}$$

  We are now ready to inverse Fourier Transform and equation (16) above, with $a = t$, says that

$$u(x,t) = f(x-t)$$

- Solve $u_x + u_t + u = 0; \quad u(x,0) = f(x)$ (Old Homework Problem)

  Take the Fourier Transform of both equations. The initial condition gives

$$\widehat{u}(w,0) = \widehat{f}(w)$$

  and the PDE gives

$$iw\widehat{u}(w,t)) + \frac{\partial}{\partial t}\widehat{u}(w,t) + \widehat{u}(w,t) = 0$$

  Which is basically an ODE in t, we can write it as

$$\frac{\partial}{\partial t}\widehat{u}(w,t) = (-iw-1)\widehat{u}(w,t)$$

  and which has the solution

$$\widehat{u}(w,t) = A(w)e^{(-iw-1)t}$$

  and the initial condition above implies $A(w) = \widehat{f}(w)$

$$\widehat{u}(w,t) = e^{-t}\widehat{f}(w)e^{-iwt}$$

  We are now ready to inverse Fourier Transform and equation (16) above, with $a = t$, says that

$$u(x,t) = e^{-t}f(x-t)$$