Fun Exercises II

For maa4402

September 22, 2015

Use the theorem on uniqueness of analytic functions to provide fast proofs of the following identities. Always $z = x + iy$ and same for subscripted variables, so $z_2 = x_2 + iy_2$.

1. $e^z = e^x \cos y + ie^x \sin y$

   Answer The LHS is entire, its derivative is e^z. The RHS is entire by Cauchy Riemann

   $$u_x = (e^x \cos y)_x = e^x \cos y = (e^x \sin y)_y = v_y$$

   $$u_y = (e^x \cos y)_y = -e^x \sin y = -(e^x \sin y)_x = -v_x$$

   And LHS = RHS when z is real, since $z = x$ and $y = 0$ imply $\cos 0 = 1$ and $\sin 0 = 0$. So by the uniqueness theorem LHS = RHS for all z.

2. $\cosh z = \cosh x \cos y + i \sinh x \sin y$

3. $\cos z = \cos x \cosh y - i \sin x \sinh y$

4. Explain why this techniques can't prove $\cos z = \cos x \cosh y + i \sin x \sinh y$

5. $\cos^2 z + \sin^2 z = 1$

6. $\cos 2z = \cos^2 z - \sin^2 z$

7. $\sin(z + 2\pi) = \sin z$

8. $\sin(z + \pi) = -\sin z$

9. $\sin(z + \pi/2) = \cos z$

10. $\cos^2 z = (1 + \cos 2z)/2$

11. $\cosh^2 z - \sinh^2 z = 1$

12. $\sec^2 z = \tan^2 z + 1$ where the LHS or RHS is defined.

13. $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

    Answer First suppose z_2 a fixed real number. Then as a function of z_1, the LHS is entire with derivative $\cos(z_1 + z_2)$ and the RHS is entire with derivative $\cos z_1 \cos z_2 - \sin z_1 \sin z_2$. They agree when z_1 is real, so the two sides are equal if z_2 is real. Now suppose z_1 is a fixed complex number. Then as a function of z_2, the LHS is entire with derivative $\cos(z_1 + z_2)$ and the RHS is entire with derivative $-\sin z_1 \sin z_2 + \cos z_1 \cos z_2$. And the first part of the proof, showed LHS = RHS if z_2 was real. So they are equal for all z_2. Since z_1 was arbitrary, it is true for all complex z_1 and z_2.

14. $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

15. $\cos z_1 \cos z_2 = (\cos(z_1 - z_2) + \cos(z_1 + z_2))/2$

16. $\tan(z_1 + z_2) = (\tan z_1 + \tan z_2)/(1 - \tan z_1 \tan z_2)$ where the LHS or RHS is defined.