Swimming the Cold Cube

Imagine a swimming pool in the shape of a cube with sides of length 2, in suitable units. Two opposite vertices are at a temperature of 1°, again, in suitable units, while the other vertices are at 0°. A swimmer named Frank is going to swim on the diagonal between the two vertices at the higher temperature. He is invigorated by differences in temperature–he likes cold water; in fact his acceleration is proportional to the negative of the gradient of the function $T(x, y, z)$ giving the temperature as a function of location within the pool. We wish to find his position as a function of time. To make the discussion easier, assume that the center of the cube is the origin and that the vertices at 1° are at $(-1, -1, -1)$ and $(1, 1, 1)$. Our first task will be to find a suitable function T to describe the temperature (which is constant in time) of the pool. It turns out that any such T must satisfy Laplace's equation $T_{xx} + T_{yy} + T_{zz} = 0$. Of course our T must also give the correct temperatures at the vertices.

(a) Show that the function $T_1(x, y, z) = \frac{1}{8}(x + 1)(y + 1)(z + 1)$ satisfies Laplace's equation. Moreover, evaluate T_1 at the vertices.

(b) Prove that the sum of any two solutions to Laplace's equation is again a solution. We describe this by saying that two solutions can be *superimposed*. (In proving this, it is not enough only to use examples.)

(c) Find another solution T_2 that is 1° at the opposite vertex. What happens when you superimpose T_1 and T_2?

(d) Find the gradient of $T = T_1 + T_2$ and evaluate it on the diagonal D of the cube. Explain why it is possible for Frank to swim on the diagonal with acceleration proportional to $\mathbf{grad}\, T$.

(e) Assuming that $(1, 1, 1)$ is on the positive side of the origin let $s(t)$ be the signed distance from Frank to the origin at time t. Express the fact that Frank's acceleration is proportional to $\mathbf{grad}\, T$ as a differential equation. Solve this differential equation assuming that Frank starts at $(-1, -1, -1)$ at time $t = 0$ with velocity 0.

Report Requirements

(i) The report should be a Maple worksheet with no pen or pencil additions. Use either Maple text or Maple comments to explain what and why you are doing. (As well as doing it.) Your report should be self contained. It should be understandable without reference to this assignment.

(ii) Give you report an different title. Feel free to personalize the report.

(iii) This is a group project and should include the standard group project stuff.

(iv) This project will be graded on English and clarity as well as mathematical correctness.

(v) Start early and good luck.