Tripple Trouble

You are the captain of the Starship Exitprize and tripples have invaded the hold of your ship. You are concerned enough to ask Spook, the Vulgar, for an analysis. As usual, Spook takes a superior attitude. Vulgars are experts with infinite series, although they never developed calculus. Hoping to teach you something, he gives his report in a series of riddles, to which you must supply the answers. Supply answers to the questions (a)—(d) below, and then go Spook one better by showing him how Earthlings would use calculus to solve the problem in parts (e) through (h).

Spook speaks

Let y_0 be the population of tripples at some initial time $t = 0$, Captain. Then let y_n be the population after n intervals of length Δt have elapsed, i.e. at $t = n\Delta t$. If you allow one interval to elapse, the population will have increased during the next interval by,

$$\Delta y_n = y_{n+1} - y_n.$$

Suppose Δy_n is proportional to both y_n and Δt:

$$\Delta y_n = k\Delta t y_n,$$

where k is the growth rate. (You may neglect the death rate in your calculations.)

a. Give an expression for y_1, the total number of tripples after one interval Δt, in terms of y_0, the initial population at $t = 0$.
b. Give expressions for y_2, y_3 and y_n, the total after 2, 3 or n intervals respectively, in terms of y_0. Make your expression as compact as possible and factor out y_0. Rewrite your expression for y_n in terms of the total elapsed time t by setting $\Delta t = \frac{t}{n}$.
c. Now expand the expression for y_n you found in part (b) using the Binomial Theorem. In order to get better resolution and accuracy, let the time scale of your analysis become finer and finer; that is, let $n \to \infty$ as as $\Delta t \to 0$. Assume you can take this limit term by term; your result is an infinite series. Please write this series.

This series is your answer for the population at time t, Captain. You may calculate the answer to any desired accuracy by including enough terms from the series.

Captain Kork's Comeback

d. How do you know you can calculate the answer to arbitrary accuracy? For example, how far off would I be if I just added up the first hundred terms? the first N terms? Give me an upper bound for the error after 100 terms, and the error after N terms, Spook, and then prove to me that the error goes to 0 as N goes to ∞. *Hint: The error depends on the time t in question. One can compare to a geometric series here.*

Now wipe that superior smile off of Spook's face by showing him what can be done with Earthling calculus.

e. Start with the same expression for the increment, $\Delta y = (k\Delta t)y$, as Spook used in part (a), and convert it into a differential equation by taking the limit as $\Delta t \to 0$ and using the definition of the derivative. Solve this equation by separation of variables to find the function $y(t)$, which gives the population at time t in terms of the initial population y_0. It is your turn to smile. (Spook had to run to the ship's library to look up separation of variables, but you know it is Section 10.4.)

Spook's Comeback

Captain, I claim that your solution is no different from mine. You have simply renamed my infinite series as your function because Earthlings cannot handle the concept of infinity. Allow me to demonstrate, Captain. In a book in the ship's library, I came across something you Earthlings call the 'derivative'.

f. Differentiate my series term by term with respect to time and see if the answer isn't k times my series. Therefore, my series does satisfy your differential equation, since its derivative is k times itself. Now

compare your function form part (e) with your answer to part (c). Your function and my series must be one in the same!

Kork's Second Comeback

Not necessarily, Spook. The same differential equation may have many different solutions. Allow me to demonstrate ...

At about this time, McCool and Scooter burst into your cabin. Scooter says, "You gentlemen can debate theory 'til you're blue in the face, but that does'na help us with our problem. I'm prepared to use radiation from the ship's engine just one time to exterminate 99% of the little beggars right now. Just give me the word, Captain!"

McCool replies, "I cannot condone such a waste of life, Captain. I've got a drug that will cut their growth rate k in half. I think you'll find this a more effective and less brutal method than the one suggested by our engineer."

g. Decide between Scooter's and McCool's strategies. Which is more effective in the short run? In the long run? Convince them of your answers.
h. In the future, will these two treatments ever result in the same number of tripples again? When?

From the Course Syllabus

PROJECT. You will work on the project in groups of 1–4 students. This project will be a substantial assignment, giving you a chance to earn part of your grade in an environment which simulates the so-called "real world" better than does an in-class exam. It will also give your instructor a chance to base part of your grade on your best work, produced in a setting where time should not be a factor (assuming you start on your project as soon as it is assigned). The results of your work on your project will be presented in a report (one report per group). Each member will also submit a "group evaluation" giving their impression of the relative contribution of each member to the group's effort. These evaluations are due with the project. It is not guaranteed that each member of the group will receive the same grade. **The reports will be graded not only on their mathematical content but also on the quality of the presentation: clarity, neatness, and proper grammar are also important.** Both reports and group evaluations must be typed. The project will be assigned on Thursday, March 8 and due on Thursday, March 29.

Grading

The admirals at Star Feet have decided that each part is worth 10 points which leaves 20 points for clarity, neatness, grammar and general wow value, for a total of 100 points. Roll out the bells and whistles, Star Feet loves a parade.

If you like the statement of this project, you might also like:
http://www.treasure-troves.com/startrek/TheTroubleWithTribbles.html