ADDITIVELY SEPARABLE FUNCTIONS
FUNCTIONS OF THE FORM $F(x, y) = f(x) + g(y)$

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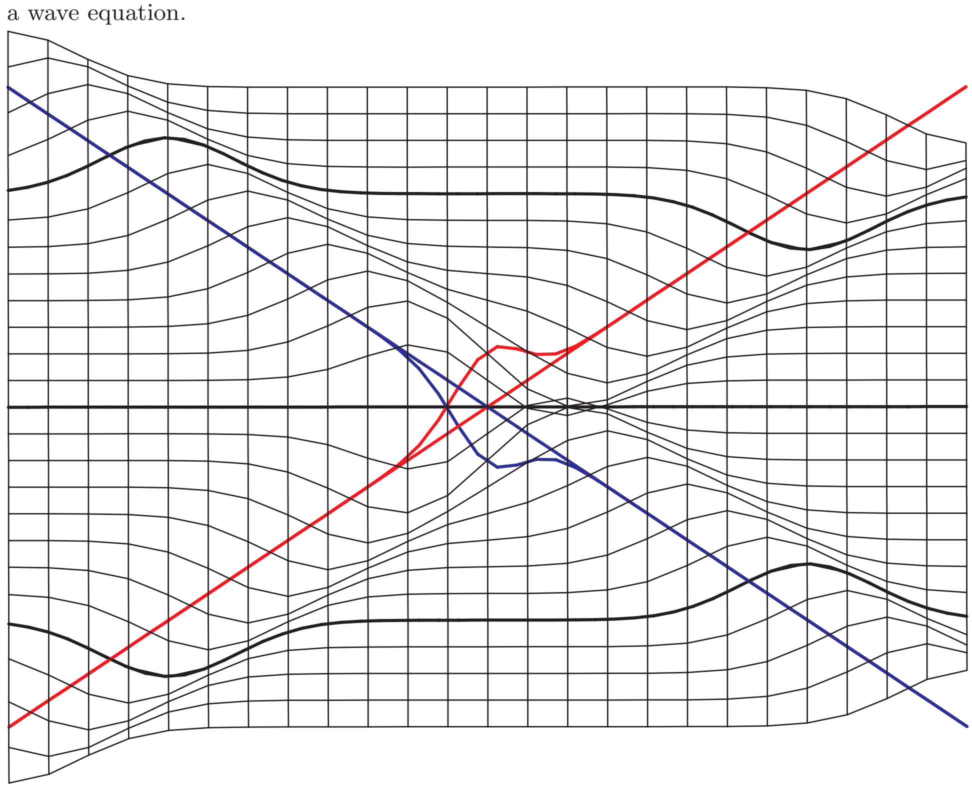
Abstract. A sample project similar to the project for Spring 2005. The picture below has a hortizontal space coordinate s and vertical time coordinate t. We see two waves, a raised hump moving to the left and a depression moving to the right. At $t = 0$, the middle hortizontal bold black line, the two waves cancel each other briefly. From another viewpoint, the red line is the x-axes and the red curve above it is $f(x) = \exp(-x^2)$, the blue line is the y-axes and the blue curve below it is $g(y) = -\exp(-y^2)$. The surface is the graph of $F(x, y) = f(x) + g(y)$, a additively separable function; but also a solution to a wave equation.



Introduction

We consider a simple special class of functions of two variables, those that can be written in the form $F(x,y) = f(x)+g(y)$. (Your project will be about functions that can be written in the form $F(x,y) = f(x)g(y)$.) This is perhaps the simplest collection of functions of two variables that depends on both variables. Our first step is to give a name to the collection, since this collection is not usually studied, we get to make up a name.

Definition 1. *A function of two variables $F(x,y)$ will be called* **additively separable** *if it can written as $f(x) + g(y)$ for some single-variable functions $f(x)$ and $g(y)$.*

After a definition one usually adds some simple observations to aid the reader to get a feel for the topic. Note that constant functions like $F(x,y) = 5$ or functions of one variable $F(x,y) = h(y)$ are additively separable. Indeed, let $f(x) = 0$ and $g(y) = 5$ or $g(y) = h(y)$ in the definition. But not all functions are additively separable, later we will see $F(x,y) = xy$ is not additively separable. Finally, the functions $f(x)$ and $g(y)$ in the definition are never unique, since if $F(x,y) = f(x) + g(y)$, then also $F(x,y) = (f(x) + C) + (g(x) - C)$ for any constant C.

This collection is surprisingly rich. We start with a few examples, Figure 1 shows that such functions can look fully three dimensional. This is surprising, since for fixed $x = a$, each cross section $F(a,y)$ is $g(y) + C$ a translate of the function $g(y)$ with $C = f(a)$. That is the sections $x = a$ for different parameters a all wave to the same time steps.

The tangent plane to $z = h(x,y)$ at any point $(a,b,h(a,b))$ is additively separable since is always of the form $z = h_x(a,b)(x-a) + h_y(a,b)(y-b) + h(a,b)$. (So we could take $f(x) = h_x(a,b)(x-a)$ and $g(y) = h_y(a,b)(y-b) + h(a,b)$. Remember a, b, $h(a,b)$, $h_x(a,b)$ and $h_y(a,b)$ are all constants and not variables.)

The three basic extrema, local minimums, local maximums and saddle points, are often drawn like in Figure 2. These are graphs of the functions $F(x,y) = -x^2-y^2$, showing a local maximum at $(0,0)$, $F(x,y) = x^2-y^2$, showing a saddle point at $(0,0)$, and $F(x,y) = x^2+y^2$, showing a local minimum $(0,0)$. All three functions are additively separable. The last function in polar coordinates is just r^2 which is curious. A function of one variable in one coordinate system which is a nice additively separable in a radically different coordinate system. Neither $r = \sqrt{x^2+y^2}$, r^3, nor even $r^4 = x^4 + 2x^2y^2 + y^4$ is additively separable.

Rectangle Conditions

When is a function $F(x,y)$ additively separable? A simple necessary condition is obtained by assuming $F(x,y) = f(x) + g(y)$ and then playing with its values at the four corners of

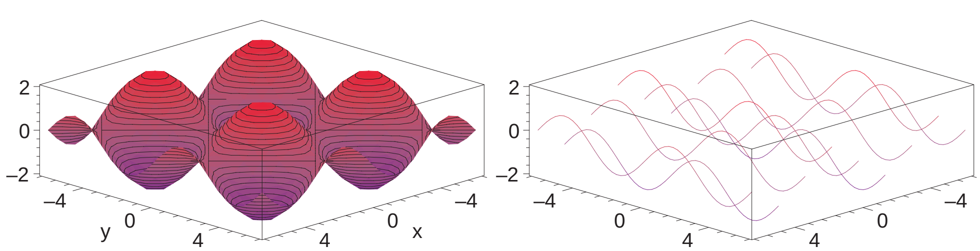


Figure 1. The graph of $\sin x + \sin y$ (left) and selected cross sections (right)

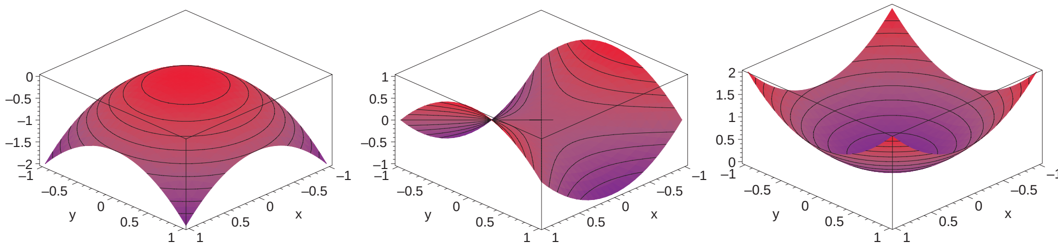


FIGURE 2. Local Max, Saddle and Local Min

a rectangle $\{(a,b),(c,b),(c,d),(a,d)\}$ (see Figure 3). We have

$$F(c,b)+F(a,d)-F(a,b) = (f(c)+g(b))+(f(a)+g(d))-(f(a)+g(b)) = f(c)+g(d) = F(c,d)$$

which we can write as a theorem.

Theorem 2. *If $F(x,y)$ is additively separable, then for all a,b,c,d, $F(c,b)+F(a,d)-F(a,b) = F(c,d)$.*

Using $a=b=0$ and $c=d=1$ we see $F(x,y)=xy$ is not additively separable. as the left hand side is zero, while the right hand side is one.

Theorem 2 has a converse which is also true. Let $f(x) = F(x,0)$ and $g(y) = F(0,y) - F(0,0)$ and use the theorem with $x=c$, $y=d$ and $a=b=0$. We have

$$F(x,y) = F(x,0) + F(0,y) - F(0,0) = f(x) + g(y)$$

which we can write as another theorem.

Theorem 3. *If for all a,b,c,d, $F(c,b)+F(a,d)-F(a,b) = F(c,d)$, then $F(x,y)$ is additively separable.*

DERIVATIVE CONDITIONS

Obviously if $F(x,y) = f(x)+g(y)$ then the partial derivatives of F are easy to compute. $F_x = f'(x)$ since $g(y)$ is constant as a function of x. Similarly $F_y = g'(y)$, $F_{xx} = f''(x)$, $F_{yy} = g''(y)$ and more importantly $F_{xy} = F_{xy} = 0$. Again lets make this a theorem.

Theorem 4. *If $F(x,y)$ is additively separable, then*

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

Again we see $F(x,y) = xy$ is not additively separable since $F_{xy} = 1 \neq 0$.

Theorem 4 says all additively separable functions satisfy a simple PDE (Partial Differential Equation). The converse is also true. First remember from one variable that $f'(x) = 0$

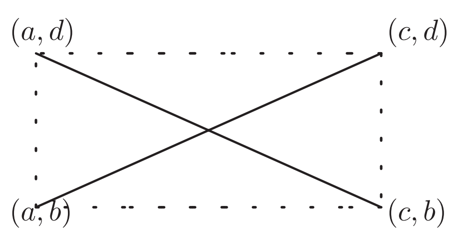


FIGURE 3. The sum of the values at the diagonally opposite corners are equal

means that $f(x)$ is a constant. For a function of two variables $G_y(x, y) = 0$ means that $G(x, y)$ is a constant function of y but can be an arbitrary function of x. So $F_{xy} = (F_x)_y = 0$ means $F_x = h(x)$ some arbitrary function of x and hence $F(x, y) = f(x) + g(y)$ where $g(y)$ is some arbitrary function of y and $f'(x) = h(x)$ (so $f(x) = \int_a^x h(t)\, dt$). Lets state this as another theorem

Theorem 5. *If $F(x, y)$ satisfys the PDE*

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

then $F(x, y)$ is additively separable,

The Wave Equation

It turns out this PDE is really a famous PDE in disguise. With a a change of coordinates, this PDE becomes the one dimensional wave equation. The solutions to the one dimension wave equation can have two components, a wave function moving to the right (the $g(y)$) and another wave function moving to the left (the $f(x)$). Figure 4 shows the function $f(x) = \exp(-x^2), g(y) = -\exp(-y^2), F(x, y) = f(x) + g(y)$ along with the curve at several times. The $t = 0$ graph is the bold straight line in the middle. Times before zero are below and in front of $t = 0$ and times after zero are above and behind. So time increase as we go up Figure 4 The space dimension shows the leftward motion of the wave. This is too important a fact to not show the connection, and it is a good application of the chain rule for several variables, but this section is not used in the sequel. The graph in Figure 4 (also on the cover) page shows raised wave moving to the left and depression moving to the right. The x-axis is red, the y-axis is blue and the function is $F(x, y) = \exp(-x^2) - \exp(-y^2)$. Time increases as one goes up the page and the one space dimension is hortizontal.

The change of coordinates is given by $t = x + y$ (time) and $s = x - y$ is the one space dimension. Inverting this transformation gives $x = (s + t)/2$ and $y = (t - s)/2$. Using the chain rule

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s} = \left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)\frac{1}{2}$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\right)\frac{1}{2}$$

To take second partials F_{ss}, the equation becomes messy since we have to apply the chain rule to F_x and F_y since they are functions of both x and y as well as applying the product rule.

$$\frac{\partial^2 F}{\partial s^2} = \left(\frac{\partial^2 F}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 F}{\partial y \partial x}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial s} + \frac{\partial F}{\partial x}\frac{\partial^2 x}{\partial s^2} + \left(\frac{\partial^2 F}{\partial x \partial y}\frac{\partial x}{\partial s} + \frac{\partial^2 F}{\partial y^2}\frac{\partial y}{\partial s}\right)\frac{\partial y}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial^2 y}{\partial s^2}$$

Since $F_{xy} = x_{ss} = y_{ss} = 0$ This simplifys to

$$\frac{\partial^2 F}{\partial s^2} = \frac{\partial^2 F}{\partial x^2}\left(\frac{\partial x}{\partial s}\right)^2 + \frac{\partial^2 F}{\partial y^2}\left(\frac{\partial y}{\partial s}\right)^2 = \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)\frac{1}{4}$$

Similarly

$$\frac{\partial^2 F}{\partial t^2} == \left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)\frac{1}{4}$$

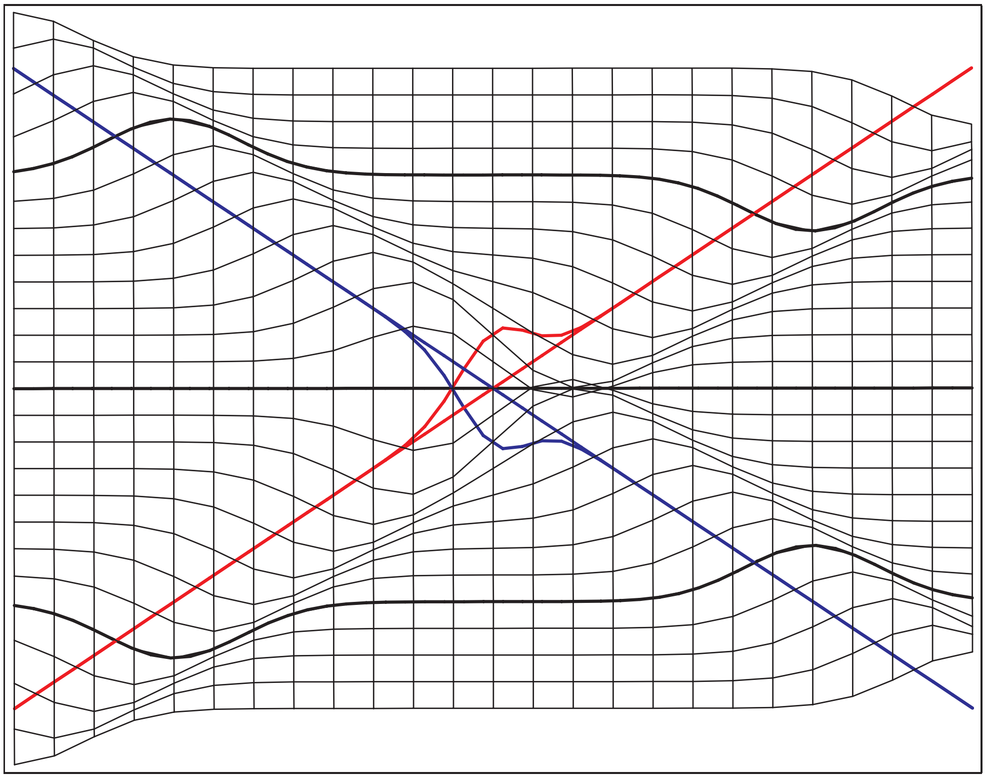


Figure 4. The space dimension is hortitontal, the time vertical, the red line is the x-axes and the blue line is the y-axes

and we have the one dimensional wave equation

$$\frac{\partial^2 F}{\partial s^2} = \frac{\partial^2 F}{\partial t^2}$$

The two forms of the wave equation are equivalent, although we have shown only one direction. Usually the wave equation has a constant c which represents the speed of the waves, in our case this $c = 1$. It is more common to see the wave equation written as

$$\frac{\partial^2 F}{\partial s^2} = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$$

Critical Points and local extrema

Here is one of sections that you will need to do for your project on functions of the form $f(x)g(y)$. We want to characterize the local extrema of additively separable functions $F(x, y) = f(x) + g(y)$ in terms of the critical points of $f(x)$ and $g(y)$. From the derivatives section we have

Theorem 6. *The point (a, b) is a critical point of $F(x, y)$, if and only if both $x = a$ is a critical point of $f(x)$ and $y = b$ is a critical point of $g(y)$.*

To use the several variables classification function, big $D = F_{xx}F_{yy} - F_{xy}^2$, we use $F_{xy} = 0$ and the other second partials from the derivatives section to compute big $D = f''(x)g''(y)$. As long as big D is not zero, namely when both $f''(a) \neq 0 \neq g''(b)$. We cover all cases in the table below which is only for critical points (a, b).

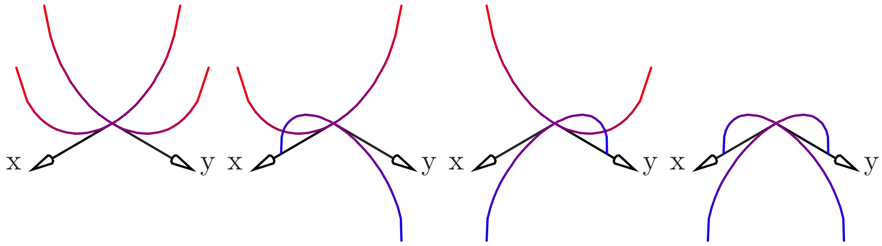


FIGURE 5. The four basic cases: when big $D \neq 0$

$f''(a)$	$g''(b)$	big D	Classification
$+$	$+$	$+$	Local Min
$+$	$-$	$-$	Saddle
$-$	$+$	$-$	Saddle
$-$	$-$	$+$	Local Max

Clearly these crition are nicely illustrated by $\sin x + \sin y$ in Figure 1 at the points $(\pi/2, \pi/2)$, $(\pi/2, 3\pi/2)$, $(3\pi/2, \pi/2)$ and $(3\pi/2, 3\pi/2)$.

The additively separable functions are simple enough we can handle the general case with Taylor series. If a is a critical point for $f(x)$ then $f'(a) = 0$, let $f^{(n)}(a) = a_n \neq 0$ be the first non-zero higher derivative of f at a. (Similarly let m be the first non-zero higher derivative of g at b and let $g^{(m)}(b) = b_m \neq 0$.) Taylor says $f(x) \approx f(a) + a_n(x-a)^n$. So if n is odd this is like x^3 at 0, the critical point $x = a$ is neither a local min nor a a local max. This is because $a_n(x-a)^n$ will be positive on one side of a and negative on the other side of a. If n is even, the point is a local min if $a_n > 0$ and a local max if $a_n < 0$. Our more complete collection is summarized in Table below.

n	a_n	m	b_m	Classification
even	$+$	even	$+$	Local Min
even	$+$	even	$-$	Saddle
even	$+$	odd	$\pm$	Saddle
even	$-$	even	$+$	Saddle
even	$-$	even	$-$	Local Max
even	$-$	odd	$\pm$	Saddle
odd	$\pm$	even	$+$	Saddle
odd	$\pm$	even	$-$	Saddle
odd	$\pm$	odd	$\pm$	Saddle

But a better statement is the following theorem

Theorem 7. *The critical point (a, b) of the additive separable function $F(x, y) = f(x)+g(y)$ is a local max [respectively local min], if and only if, both a is a local max [respectively local min] for $f(x)$ and b is a local max[respectively local min] for $g(y)$. Otherwise the point (a, b) is a saddle point for $F(x, y)$.*

Basically this says if $F(x, y) \leq F(a, b)$ for (x, y) nearby (a, b) then $f(x)+g(b) = F(x, b) \leq F(a, b) = f(a) + g(b)$ so $f(x) \leq f(a)$ and similarly $g(y) \leq g(b)$. Conversely if $f(x) \leq f(a)$ and $g(y) \leq g(b)$ then $f(x) + g(y) \leq f(a) + g(b)$ or $F(x, y) \leq F(a, b)$.

Theorem 7 is false for functions in general. The function $F(x, y) = xy$ has both $F(x, 0) = 0$ and $F(0, y) = 0$ so that $x = 0$ is both a local min and local max for $F(x, 0) = 0$ and $y = 0$

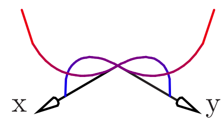


FIGURE 6. Some of the odd power cases: note big $D = 0$

is both a local min and local max for $F(0, x) = 0$ and yet $(0, 0)$ is clearly a saddle point for F.

Closure and Uniqueness Properties

This is a bonus section and is off the topic. Closure properties say what can you do a class or collection and still remain in the collection. For example, if $F(x, y)$ and $G(x, y)$ are both additively separable then so is $F(x, y) + G(x, y)$, but not every product $F(x, y)G(x, y)$ is additively separate. Multiplication by a constant C is ok, since $CF(x, y)$ is additively separable whenever F is additively separable. Also partial derivatives and integrals of additively separable functions are additively separable. We summarize in the next theorem.

Theorem 8. *The collection of additively separable functions is closed under the following operations. (The first two make the collection a Linear space.)*

(1) *addition*
(2) *multiplication by constants*
(3) *partial derivatives with respect to x or y*
(4) *integrals with respect to dx or dy*

Here is a uniqueness question, let L be any line in the xy-plane which is not parallel to the x or y axis. If the additively separable function $F(x, y)$ is zero on all points on L and the directional derivative of F in the direction normal to L is also zero on L, then must $F(x, y)$ be identically zero everywhere? The picture on the cover page, shows it is not enough for F to be zero on one line.

Your optional group assignment

This is a group project. Groups are 3-4 people except in unusual cases. Individual projects are not accepted. The work on the project must be a true collaboration in which each member of the team will carry their own weight. It is **NOT** acceptable for the team members to split the project between them and work on them independently. Instead all members must actively work together on all parts of the assignment.

The fact that group members have to work together means that you need to carefully consider a potential group member's schedule before forming the group. You cannot be a group if you cannot find large chunks of time to spend together.

Your assignment is to write in prose, in understandable English, a typed report about separation of variables, that is functions of the form $F(x, y) = f(x)g(y)$. There are three parts to the assignment. First you must do the local extrema for functions of the form $f(x)g(y)$, note that this is harder than the case for functions of the form $f(x) + g(y)$ done above. Second, you must find something interesting about about such functions, unfortunately one cannot just take logs and translate a additively separable result via $\ln(f(x)g(y)) = \ln f(x) + \ln g(y)$ since logs are only defined for positive values. This fact must be strong enough to show that $x + y$ cannot be written in the form $f(x)g(y)$. Third, you must research and find a place where separation of variables is used outside the ordinary differential equation case you might have learned in Calculus 2.

The project has two due dates: A hardcopy outline which lists who is in your group and what your project is going to contain is due on Friday 25 March at 3pm and the completed hardcopy project itself is due on Friday 8 April also at 3pm. (The project must be bound or stapled and not loose, nor dog-eared. Part of the grade will be based on presentation and clarity. There are a few bonus points available for wow value.)

In addition to the project, **EACH** team member will write a separate evaluation of the relative contributions of all the team members.