Generalized Eigenvectors

1. **Example** Consider the 2×2 matrix
$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$
The matrix A has characteristic polynomial λ^2 and hence its only eigenvalue is 0. The eigenvectors for the eigenvalue 0 have the form $[x_2, x_2]^T$ for any $x_2 \neq 0$. Thus the eigenspace for 0 is the one-dimensional $\text{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ which is not enough to span all of $\mathbb{R}^2$. However A^2 is the zero matrix so $A^2\vec{v} = (A - 0I)^2\vec{v} = 0$ for all vectors $\vec{v}$. If we let $\vec{v}_2$ be $[1, -1]^T$ (or any other vector outside the eigenspace), then $A\vec{v}_2$ is in the eigenspace so it is $a\,[1, 1]^T$ for some a (2 in this case). If we let $\vec{v}_1 = [1, 1]^T$, and $P = [\vec{v}_1, \vec{v}_2]$ and we can write $A = PBP^{-1}$ where $B = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix}$
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
(It is more usual in this case to pick $\vec{v}_2$ so that it solves $A\vec{v}_2 = \vec{v}_1$. This will make $a = 1$, and perhaps $\vec{v}_2 = [1, 0]^T$ which yields a slightly different equation)
$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

2. **Theorem** If A is a 2×2 matrix with repeated eigenvalue λ but whose eigenspace is only one-dimension and spanned by the eigenvector $\vec{v}_1$. Let $\vec{v}_2$ be a solution to $(A - \lambda I)\vec{v}_2 = \vec{v}_1$, Let $P = [\vec{v}_1, \vec{v}_2]$ and let $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ then
$$A = PBP^{-1}$$
(and as usual we check the construction using $AP = PB$.)

3. **Problems** Compute the eigenvalues For the given 2×2 matrices A and decide if the theorem above applies. If the theorem applies find B and P, and if the theorem does not find the usual diagonal D and P. Check your answers.
   (a) $A = \begin{bmatrix} 7/2 & -1/2 \\ 1/2 & 5/2 \end{bmatrix}$
   (b) $A = \begin{bmatrix} 11/2 & 1/2 \\ -1/2 & 13/2 \end{bmatrix}$
   (c) $A = \begin{bmatrix} 5 & -1 \\ 0 & 5 \end{bmatrix}$
   (d) $A = \begin{bmatrix} -5/2 & 1/2 \\ -1/2 & -3/2 \end{bmatrix}$

4. **Definition** To handle this problem we generalize notion of an eigenvector to a *generalized eigenvector* we say a non-zero vector $\vec{v}$ is a generalized eigenvector for A corresponding to λ if
$$(A - \lambda I)^k\vec{v} = 0$$
for some positive integer k. The smallest such k is the order of the generalized eigenvector. Note that a regular eigenvector is a generalized eigenvector of order 1. The vector $\vec{v}_2$ in the theorem above is a generalized eigenvector of order 2. Since $(D - I)(te^t) = (e^t + te^t) - te^t = e^t \neq 0$ and $(D - I)e^t = 0$, te^t is a generalized eigenvector of order 2 for D and the eigenvalue 1.

The simplest case is when $\lambda = 0$ then we are looking at the kernels of powers of A. It is easy to see that the chain of subspaces

$$\{0\} = \ker A^0 \subseteq \ker A^1 \subseteq \cdots \subseteq \ker A^k \subseteq \ker A^{k+1} \subseteq \cdots \subseteq \mathbb{R}^n$$

are all subspaces of the big vector space. Because if $A^k\vec{v} = 0$ then $A^{k+1}\vec{v} = A(A^k\vec{v}) = A0 = 0$. Also note that eventually $\ker A^k = \ker A^{k+j}$ for all positive integers j because the dimensions are all less than or equal to n. (This can fail in infinite dimensions.)

But it is easy to show the stronger result that if $\ker A^k = \ker A^{k+1}$ then $\ker A^{k+1} = \ker A^{k+2}$. Let $\vec{v} \in \ker A^{k+2}$, then $A\vec{v} \in \ker A^{k+1} = \ker A^k$ so $A^k(A\vec{v}) = 0$ and hence $A^{k+1}\vec{v} = 0$ so $\vec{v}$ is in $\ker A^{k+1} = \ker A^k$. So for each eigenvalue, there is a largest order.

There is one more requirement on the dimensions of these spaces which we will illustrate with $k = 1$ and $k = 2$. Let $\{\vec{v}_1, \ldots \vec{v}_s\}$ be a basis for $\ker A \neq \ker A^2$ and we add $\vec{v}_{s+1}, \ldots \vec{v}_{s+t}$ until $\{\vec{v}_1 \ldots \vec{v}_{s+t}\}$ is a basis for $\ker A^2$. So $\ker A$ is s-dimensional and $\ker A^2$ is $s+t$-dimensional. The additional requirement is that $t \leq s$.

If $t > s$ then $\{A\vec{v}_{s+1}, \ldots A\vec{v}_{s+t}\}$ must be linearly dependent in $\ker A$. So there are scalars $c_{s+1}, \ldots c_{s+t}$ not all zero so that

$$\begin{aligned} c_{s+1}A\vec{v}_{s+1} + \cdots c_{s+t}A\vec{v}_{s+t} &= 0 \\ A(c_{s+1}\vec{v}_{s+1} + \cdots c_{s+t}\vec{v}_{s+t}) &= 0 \\ c_{s+1}\vec{v}_{s+1} + \cdots c_{s+t}\vec{v}_{s+t} &\in \ker A \\ c_{s+1}\vec{v}_{s+1} + \cdots c_{s+t}\vec{v}_{s+t} &= c_1\vec{v}_1 + \cdots c_s\vec{v}_s \end{aligned}$$

for some $c_1, \ldots c_s$ since $\{\vec{v}_1, \ldots \vec{v}_s\}$ is a basis for $\ker A$. But this is a contradiction to $\{\vec{v}_1, \ldots \vec{v}_{s+t}\}$ being linearly independent.

Other facts without proof. The proofs are in the down with determinates resource. The dimension of generalized eigenspace for the eigenvalue λ (the span of all all λ generalized eigenvectors) is equal to the number of times λ is a root to the characteristic polynomial. If $\vec{v}_1, \ldots \vec{v}_s$ are generalized eigenvectors for distinct eigenvalues $\lambda_1, \ldots \lambda_s$, then $\{\vec{v}_1, \ldots \vec{v}_s\}$ is linearly independent. Each matrix A is similar to block diagonal matrix where each non-zero block B_j corresponds to the generalized eigenspace of a distinct eigenvalue λ_j.

$$\left[\begin{array}{c|c|c|c} B_1 & 0 & \cdots & 0 \\ \hline 0 & B_2 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & B_s \end{array}\right]$$

We list all a sequence of 4×4 matrices that could be B in a 4-dimensional version of our theorem above.

$$\left[\begin{array}{c|c|c|c} \lambda & 0 & 0 & 0 \\ \hline 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & \lambda \end{array}\right] \left[\begin{array}{cc|c|c} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & \lambda \end{array}\right] \left[\begin{array}{cc|cc} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{array}\right] \left[\begin{array}{ccc|c} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & \lambda \end{array}\right] \left[\begin{array}{cccc} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{array}\right]$$

Note that there are non-zero entries only on the main diagonal and on the diagonal just above the main diagonal. This diagonal is sometimes called the super diagonal.

5. **Bigger Example** Consider the matrix A, eventually A has characteristic polynomial $(\lambda - 5)^5(\lambda + 2)$.

$$A = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

The eigenvalue -2 has a one-dimension eigenspace spanned by $v_6 = [0,0,0,0,0,1]^T$, the eigenvalue 5 has a two-dimensional space spanned by $v_1 = [1,0,0,0,0,0]^T$ and $v_4 = [0,0,0,1,0,0]^T$. The subspace $\ker(A-5I)^2$ is spanned by four vectors v_1, v_4 and the order 2 generalized eigenvectors $v_2 = [0,1,0,0,0,0]^T$ and $v_5 = [0,0,0,0,1,0]^T$. While the subspace $\ker(A-5I)^3$ is spanned by five vectors v_1, v_2, v_4, v_5 and the order 3 generalize eignevector $v_3 = [0,0,1,0,0,0]^T$. Any non-zero vector of the form $[x_1, x_2, x_3, x_4, x_5, 0]^T$ is a generalized eigenvector for the eigenvalue 5.

There are a couple of ways to work on these problems. One is to backtract from generalized eigenvector of order k to generalize eigenvector of order $k+1$ as needed. Sometimes, trial and error is the fastest method.

6. **Problems part 2** For each $n \times n$ matrix, find a basis of $\mathbb{R}^n$ consisting of generalized eigenvectors.

   (a) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   (e) $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

   (f) $A = \begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

7. **True or False Problems**

   (a) An eigenvector is a generalized eigenvector.

   (b) If $A^2\vec{v} = 0$ and $\vec{v} \neq 0$ then $\vec{v}$ is a generalized eigenvector of order 2 for the eigenvalue 0 of A.

   (c) Sometimes $\dim\ker A^2 > \dim\ker A$.

   (d) Sometimes $\dim\ker A^2 > 2\dim\ker A$.

   (e) te^{-2t} is a generalized eigenvector of order 2 for the eigenvalue -2 of the derivative operator D.

   (f) If $\vec{v}$ is a generalized eigenvector of order k for the eigenvalue λ then λ is a root at least k times to the characteristic polynomial.

   (g) If λ is a root k times of the characteristic polynomial of the matrix A, then A has a generalized eigenvalue of order k.

   (h) If $\vec{v}$ is a generalize eigenvector of order k for λ and A then $(A - \lambda I)\vec{v}$ is a generalized eigenvector of order $k+1$.

   (i) The matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable.

8. **Answers** These answers are not unique.

   (a) Characteristic polynomial $p(\lambda) = (\lambda - 3)^2$ which has a one dimensional eigenspace spanned by $v_1 = [1,1]^T$. There are many choices possible for v_2, how about $v_2 = [2,0]^T$. then $B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Checking $AP = \begin{bmatrix} 3 & 7 \\ 3 & 1 \end{bmatrix}$ and $PB = \begin{bmatrix} 3 & 7 \\ 3 & 1 \end{bmatrix}$ ✓

   (b) Repeated eigenvalue $\lambda = 6$, $v_1 = [1,1]^T$, $v_2 = [0,2]^T$ $B = \begin{bmatrix} 6 & 1 \\ 0 & 6 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

   (c) Repeated eigenvalue $\lambda = 5$, $v_1 = [1,0]^T$, $v_2 = [0,-1]^T$ $B = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

   (d) Repeated eigenvalue $\lambda = -2$, $v_1 = [1/2, 1/2]^T$, $v_2 = [1,2]^T$ $B = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} 1/2 & 1 \\ 1/2 & 2 \end{bmatrix}$

9. **Answers part 2** The answers are not unique, but there is a logic to answers choosen, it is so the super-diagonal entries of something would be one. Careful, these answers were machine generated and not yet checked.

   (a) $\lambda = 2$ order $k = 1$ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\lambda = 3$ order $k = 1$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and order $k = 2$ $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

   (b) $\lambda = 3$ order $k = 1$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, order $k = 2$ $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and order $k = 3$ $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

   (c) $\lambda = 3$ order $k = 1$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, order $k = 2$ $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and order $k = 1$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

   (d) $\lambda = 2$ order $k = 1$ $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\lambda = 1$ order $k = 1$ $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, order $k = 2$ $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and order $k = 3$ $\begin{bmatrix} 0 \\ -1/2 \\ -2 \\ 1 \end{bmatrix}$

10. **True or False answers**

   (a) True, an eigenvector is a generalized eigenvector of order 1.

   (b) False, it could be order 1.

   (c) True.

   (d) False, otherwise $t > s$.

   (e) True.

   (f) True.

   (g) False, consider the identity matrix for example.

   (h) False, it goes the wrong way $(A - \lambda I)\vec{v}$ has order $k - 1$.

   (i) False.