Open, Closed, Interior, Exterior, Boundary, Connected

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These are a collection of definitions from point set topology. We give some examples based on the sets collected below.

- $D = \{z \in \mathbb{C} : |z| \leq 1\}$, the closed unit disc.
- $B = \{z \in \mathbb{C} : |z| < 1\}$, the open unit disc.
- $S = \{z \in \mathbb{C} : |z| = 1\}$, the unit circle.
- $H = \{z \in \mathbb{C} : |z| > 1\}$, the complement of D.
- $R = \{z \in \mathbb{C} : \Im z = 0\}$, the real line.
- For any set A, the complement of A, written A^c or $\mathbb{C}\backslash A = \{z : z \notin A\}$

Definitions:

1. *ε-Neighborhood* of a point z_0,
$$D_\varepsilon(z_0) = \{z : |z - z_0| < \varepsilon\}$$

2. The point w is an *interior point* of the set A, if for some $\varepsilon > 0$, the ε-neighborhood of w, $D_\varepsilon(w) \subset A$. The *interior of* A, $\operatorname{int} A$ is the collection of interior points of A. The set A is *open*, if and only if, $\operatorname{int} A = A$. Note B is open and $B = \operatorname{int} D$. Both S and R have empty interiors. H is open and its own interior.

3. The point w is an *exterior point* of the set A, if for some $\varepsilon > 0$, the ε-neighborhood of w, $D_\varepsilon(w) \subset A^c$. The *exterior of* A, $\operatorname{ext} A$ is the collection of exterior points of A. The set A is *closed*, if and only if, $\operatorname{ext} A = A^c$. Note D and S are both closed. The exterior of either D or B is H. The exterior of S is $B \cup H$.

4. The point w is an *boundary point* of the set A, if every ε neighborhood of w meets both A and A^c, (has a non-empty intersection),
$$D_\varepsilon(w) \cap A \neq \emptyset \neq D_\varepsilon(w) \cap A^c.$$
The *boundary of* A, ∂A is the collection of boundary points. The set A is closed, if and only if, it contains its boundary, and is open, if and only if $A \cap \partial A = \emptyset$. Note S is the boundary of all four of B, D, H and itself.

5. A set A is said to be *bounded* if it is contained in $B_r(0)$ for some $r < \infty$, otherwise the set is *unbounded*. The sets H and R are unbounded, while B, D and S are bounded.

6. An open set A is said to be *connected* if there is a polygonal line between any two points in A. A connected open set is called a *domain*. A domain together with part of its boundary is called a *region*. So B and H are domains and D is a region. Note that since S is not open, we haven't defined connected for S. The points of S are not connected with polygonal lines, but it is obviously "connected" in some sense which is called *pathwise connected*. The set $\mathbb{C}\backslash S = B \cup H$ is open but not connected.

7. A point w is said to be an *accumulation point* of the set A, if every ε-neighborhood $D_\varepsilon(w)$ contains a point of A different from w. That is

$$(D_\varepsilon(w)\backslash\{w\}) \cap A \neq \emptyset.$$

This is called a deleted ε-neighborhood of w, it is the set of points $\{z \in \mathbb{C} : 0 < |z - w| < \varepsilon\}$. The set $\{1/n : n = 1, 2, \dots\}$ has only one accumulation point, namely 0.

8. The *closure* of the set A, (written $\mathrm{cl}\, A$), is the smallest close set containing A which is $A \cup \partial A$. A set is closed, if and only if, $A = \mathrm{cl}\, A$.

A	$\mathrm{cl}\, A$	$\mathrm{int}\, A$	$\mathrm{ext}\, A$	∂A
$\{z : \|z\| < 1\}$	$\{z : \|z\| \leq 1\}$	$\{z : \|z\| < 1\}$	$\{z : \|z\| > 1\}$	$\{z : \|z\| = 1\}$
$\{z : \|z\| \leq 1\}$	$\{z : \|z\| \leq 1\}$	$\{z : \|z\| < 1\}$	$\{z : \|z\| > 1\}$	$\{z : \|z\| = 1\}$
$\{z : \|z\| = 1\}$	$\{z : \|z\| = 1\}$	$\emptyset$	$\{z : \|z\| \neq 1\}$	$\{z : \|z\| = 1\}$
$\{z : \|z\| > 1\}$	$\{z : \|z\| \geq 1\}$	$\{z : \|z\| > 1\}$	$\{z : \|z\| < 1\}$	$\{z : \|z\| = 1\}$
$\{z : \Im z = 0\}$	$\{z : \Im z = 0\}$	$\emptyset$	$\{z : \Im z \neq 0\}$	$\{z : \Im z = 0\}$
$\{z : \Im z > 0\}$	$\{z : \Im z \geq 0\}$	$\{z : \Im z > 0\}$	$\{z : \Im z < 0\}$	$\{z : \Im z = 0\}$
$\{z : \Im z \geq 0\}$	$\{z : \Im z \geq 0\}$	$\{z : \Im z > 0\}$	$\{z : \Im z < 0\}$	$\{z : \Im z = 0\}$