HW1

1.14: If G is self-complementary, then q, the number of edges in G is the same as the number of edges in $\bar{G}$ which is $\binom{p}{2} - q$. Some arithmetic yields $p(p-1)/2 - q = q$ or $p(p-1) = 4q$. Since the right hand side of this last equation is divisible by 4 so is the left hand side. Thus, either $4|p$, $4|(p-1)$ or both $2|p$ and $2|(p-1)$. The last case is impossible, since if p is even, then $p-1$ is odd and not divisible by 2. If $4|p$, then $p \equiv 0 \bmod 4$ and if $4|(p-1)$, then $p \equiv 1 \bmod 4$.

1.15: Since G itself is k-regular when $k = 0$, we can assume $k > 0$. Consider $J = K_{k+1} - e$, where e is any edge of K_{k+1}. J has two vertices of degree $k-1$ and all the other vertices have degree k. Suppose $n = \sum_{v \in V}(k - \deg v)$ (which is the number of edges with one end in $V(G)$ we need to add to make all these vertices have degree k) is even. Then H is the union of G and $n/2$ copies of J together with n edges with exactly one end in G. Each edge is used to increase the degree of a vertex in G with degree less than k and the other end makes one of the degree $k-1$ vertices have degree k. G is the induced subgraph of H given by $\langle V(G) \rangle$ since we have added no new edges with both ends in G.

If k and p are both odd, then $n = kp - 2q$ is odd. So adding $(n-1)/2$ copies of J as above, leaves one vertex in G (and hence H) with degree $k-1$ and all the others have degree k. Let L be two copies of H, and an edge adjoining the two vertices of degree $k-1$. L is the required supergraph.

1.15 *Method 2:* Consider the graph operation k-mirror. If G is a graph with $\Delta(G) = k$, then the k-mirror of G is two copies of G and corresponding vertices of degree less than k are made adjacent. If K is the k-mirror of G, then $\delta(K) = \delta(G) + 1$ and $\Delta(K) = \Delta(G)$. Furthermore G is an induced subgraph of K because no new edges have both ends in G. By doing the k-mirror, $\Delta(G) - \delta(G)$ times iteratively, the resulting graph has $\Delta = \delta = k$, and it is k-regular

1.16: Suppose G is a k-regular bipartite graph with partite sets V_1 and V_2. Since each edge has one end in V_1 and each vertex of V_1 has degree k, we have $q = k \cdot |V_1|$. Similarly $q = k|V_2|$. Dividing both sides of $k|V_1| = k|V_2|$ by k gives the result.

The error, of course, is that k could be zero. If $k \neq 0$, then the division is valid. So if had the assumption that G is non-empty, the result is valid. The result is false in general. For a counter-example consider $\bar{K}_3$ which is bipartite in any partition of $V(\bar{K}_3)$, but never can $|V_1| = |V_2|$, since $|V|$ is odd.