HW3

2.9: Find the size of a maximal disconnected graph on p-vertices and add 1. If G has p vertices and is disconnected, then V can be divided into two non-empty sets V_1, V_2 such that no edge joins a vertex in V_1 to a vertex in V_2. Let $x = |V_1|$ and $y = |V_2|$. The size of G is no bigger then the size of $K_x \cup K_y$ which is $\binom{x}{2} + \binom{y}{2}$. One maximizes the function $f(x) = (x(x+1) + (p-x)(p-x+1))/2 = (2x^2 - 2px + p^2 + p)/2$ either by calculus or directly (its an upward pointing parabola) and obtain the maximum occuring at the end points ($x = 1$ or $x = p-1$). The largest size of a disconnected graph is thus $\binom{p-1}{2}$. So if $q \geq \binom{p-1}{2} + 1$ the graph is connected. [There are several inductive proofs too.]

2.10: Every induced subgraph of G is connected $\iff$ G is complete.
Proof: If G is complete so is every induced subgraph and hence every induced subgraph is connected. Conversely, if every induced subgraph of G is connected, let u and v be vertices in G. The induced graph $\langle\{u, v\}\rangle$ is connected, so it must have an edge uv. Thus uv is a edge in G. Therefore G is complete.

2.22: Let G be a graph with p vertices and $\delta \geq (p-1)/2$. Let u and v be vertices of G we want to show there is a uv path in G. This is certainly the case if $u = v$ or if uv is an edge of G. Assume uv is not an edge, and let $W = V - \{u, v\}$. Since $|W| = p - 2$ and both u and v are adjacent to at least $(p-1)/2$ vertices, there is a $w \in W$ adjacent to both u and v. uwv is the required path. [The sharpness of this result comes from considering $K_{p/2} \cup K_{p/2}$ which has $\delta = (p-2)/2$.]