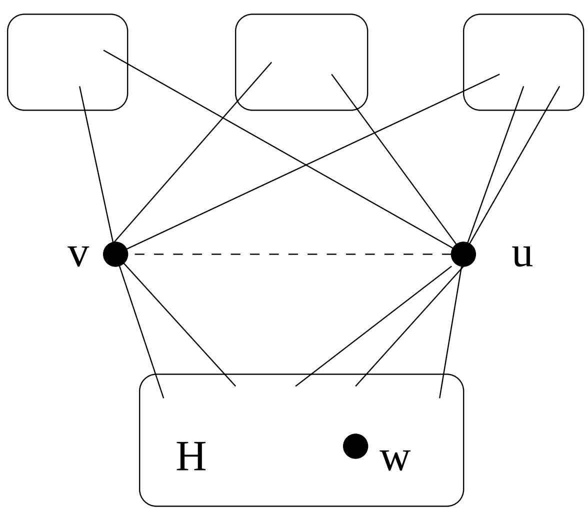
HW4

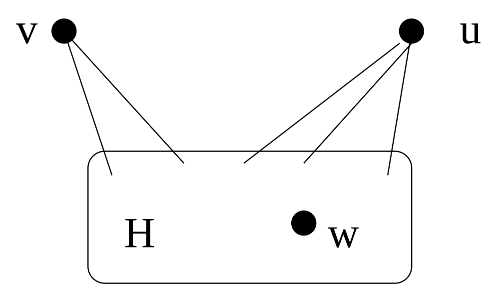
2.23: First we show that it suffices to prove the result for connected graphs. If we know the result for connected graphs, then the result is true for any graph with a non-trivial connected component. If G is non-trivial, but every component is trivial then G has at least two isolated vertices and isolated vertices are not cut-vertices.

So let G be a non-trival connected graph and let u and v be two vertices such that $d(u, v) = \text{diam}(G)$. The claim is that u (and hence v) is not a cut-vertex. If not, then $G - u$ is disconnected. Let w be a vertex in a component of $G - u$ that does not contain v. Since G is connected there are wv-paths, but every wv path must use u. In particular, the shortest wv-path must use u. Thus the distance $d(w, v) > d(u, v)$, which contradicts the definition of $\text{diam}(G)$. Therefore u is a cut-vertex.

2.24: Let G be disconnected and let u and v be vertices of G. If there is no edge uv in G, then uv is an edge and hence a uv-path in $\bar{G}$. If there uv is an edge in G, then u and v are in the same component of G. Let w be a vertex, not in the same component as u. The edges wu and wv are not in G, hence uwv is a uv-path in $\bar{G}$. In either case, we have produced a uv-path, so $\bar{G}$ is connected.

2.26: Let G be a critcial block. Thus for each vertex v, there is at least one vertex u so that $G - v$ is connected, but $G - \{u, v\}$ is disconnected. Let H be a component of $G - \{u, v\}$, and let w be a vertex in H and let x be a vertex of $G - \{u, v\}$ not in H. Since $G - v$ is connected, there is a wx-path P in $G - v$ which is not in $G - \{u, v\}$. The path P must use u, and since w is in H u is adjacent to a vertex in H. Reversing the roles of u and v, we find v is adjacent to a vertex in H. Furthermore, since H is a component of $G - \{u, v\}$, u and v are the only vertices outside $V(H)$ which are adjacent to any vertex of H. So if H is an isolated vertex w, then w is adjacent to exactly u and v in G and thus has degree 2. (See the figure below left.)





Choose v and u so that $G - \{u, v\}$ has a component H which is as small as possible. Suppose H contains more than one vertex and let w be a vertex of H. Consider the subgraph J (see the figure above right) of $\langle H \cup \{u, v\}\rangle - e$, where e is the edge uv which may or may not be in G. If w disconnected u from v in J, then one of the components of $J - w$ contains more that one vertex, say the u side. So $G - \{w, u\}$ is disconnected and one of the component is strictly contained in H. This would contradict H being as small as possible. Thus there is both a tu-path and a tv-path in $J - w$ for every vertex t in H.

Let $x \neq t$ be two vertices of G not in J. Since $G - x$ is connected, there is tw-path in $G - x$. Following this path til it first hits u or v produces a tu or tv-path which misses both x and H. It follows that $G - \{w, x\}$ is connected. Everything not in J is connected to u or v and they are both in the same component of $J - w$. Since G is a critical block there is a vertex x so that $G - \{w, x\}$ is disconnected. We have shown that x must be in J and cannot be u or v (or it would produce a smaller component than H.) Similarly, if H has three or more points one of the conponents of $G - \{w, x\}$ would be strictly contained in H. But if H has only two vertices namely w and x, then $G - \{w, x\}$ is $G - H$ which is connected.