HW5

2.12: Let G have $\delta \geq p/2$. Let W be $k = \kappa_1(G)$ edges of G so that $G - W$ is disconnected and let K be the smallest component of $G - W$. Let K have x vertices. Since $x \leq p/2 \leq \delta$, each vertex v of K has at least $\delta - x + 1$ edges in W. Thus W has at least $f(x) = x(\delta + 1 - x)$ edges. The equation shows $f(x)$ is a downward pointing parabola. So the minimum value of $f(x)$ over the interval $[1, \delta] \supset [1, p/2]$ occurs at one of the endpoints $x = 1$ or $x = \delta$ but $f(1) = f(\delta) = \delta$ and so $k \geq \delta$. We already know the reverse inequality, so $\kappa_1(G) = \delta(G)$.

3.3: Let the tree T have p vertices and $q = p - 1$ edges. Let d_i be the degrees of the vertices suppose $d_p = k = \Delta(T)$ and suppose the n vertices $d_1 \ldots d_n$ are the leaves of T. By the handshaking lemma, $2q = \sum d_i = \sum_1^n d_i + \sum_{n+1}^{p-1} d_i + d_p \geq n + 2(p - n - 1) + k$ or $2p - 2 \geq 2p - n - 2 + k$ or $n \geq k$.

3.10: If G has a cycle C, then the induced graph $H = \langle V(C) \rangle$ has $\delta(H) \geq 2$. Thus we have shown the contrapositive, if every induced subgraph has $\delta \leq 1$, then G is acyclic.

Conversely, if G is acyclic so is every subgraph, and hence every component of any subgraph. These acycle components are trees, which have $\delta = 0$ if they are trivial or $\delta = 1$ if they are non-trivial.