HW8

5.10: Let G be Hamiltonian, let C be the Hamiltonian cycle and let S be a proper (and non-empty) subset of $V(G)$. Since the number of components of $G - S$ is no more than the number of components of $C - S$, it suffices to prove the theorem for C. We claim $C - S$ has at most $|S|$ components each of which is a path graph. The proof is by induction. The first vertex creates a single path graph. While each addition vertex, is either a vertex at the end of a path (and the number of components does not increase) or it divides the path into two components, increasing the number of path components by one.

5.12: Note $K_{1,1}$ is not Hamiltonian so assume $p \geq 3$. (Hence $K_{m,n}$ is Hamiltonian if and only if $m = n \geq 2$ by the result below.)

Suppose $p_1 \geq p_2 \geq \ldots p_n$ and

$$p_1 > \sum_2^n p_i,$$

then $K_{p_1,p_2,\ldots p_n}$ can not be Hamiltonian. Indeed, at most every other vertex in a cycle can be in the p_1-element partite set, and there are not enough other vertices in the graph to complete the cycle.

If

$$p_1 \leq \sum_2^n p_i,$$

then $K_{p_1,p_2,\ldots p_n}$ is Hamiltonian. We show the closure is the complete graph. First consider the $p1$-element partite set. The degree of each vertex in this set is $\sum_2^n p_i$ which is at least $p/2$, hence each pair vertices in this partite of would be adjacent in the closure. This is also true of every vertex in another partite set, since the degree of a vertex in the p_j-element partite set is

$$\sum_1^n p_i - p_j \geq \sum_2^n p_i \geq p/2.$$

So if

$$p_1 \geq p_2 \geq \cdots p_n$$

then $K_{p_1,p_2,\ldots p_n}$ is Hamiltonian if and only if

$$p_1 \leq \sum_2^n p_i.$$

5.19: Since K_1, K_2, K_3 are Hamiltionian connected but not 3-connected we need to add the hypothesis $p \geq 4$. Let u and v be two vertices of the Hamiltionian connected graph G. Let P be a uv-Hamiltonal path in G, Since P is a spanning tree of G and u and v are leaves of P, $P - u - v$ is a spanning tree of $G - u - v$ and hence the later graph is still connected. Therefore G is 3-connected.