HW10

6.5: A maximal planar graph has $q = 3p - 6$ or $2q = 6p - 12$. Now $p = \sum p_i$ and $2q = \sum ip_i$. We have

$$3p_3 + 4p_4 + 5p_5 + 6p_6 + 7p_7 + 8p_8 + \cdots + np_n = 6p_3 + 6p_4 + 6p_5 + 6p_6 + 6p_7 + 6p_8 + \cdots + 6p_n - 12$$

since a maximal planar graph has no vertices of degree less than 3, so by canceling terms we get

$$p_7 + 2p_8 + \cdots (n-6)p_n + 12 = 3p_3 + 2p_4 + p_5.$$

6.13: We need to show T^* is a tree. Let r, q and p be the numbers for G, Euler says $p - q - r = 2$. So T has $p - 1$ edges which leaves $q - (p - 1) = r + (-r + q - p) + 1 = r - 2 + 1 = r - 1$ edges for T^*. Since T^* has r vertices, this is the correct number of edges for a spanning tree of G^*. Since T has only the unbounded region, each vertex of T^* (region of G) is eventually connected to the unbounded region (vertex) in G^*. Thus T^* is connected and hence a spanning tree. [Note: It is also easy to show T^* is acyclic and so you can turn the above on its head and use this to prove Euler's formula.]

6.14: For a planar graph $q \leq 3p - 6$. If both G and $\bar{G}$ are planar we have $\binom{p}{2} \leq 2(3p-6)$. So $p(p-1) \leq 4(3p-6)$ or $f(p) = p^2 - 13p + 24 \leq 0$. This $f(p)$ is an upward pointing parabola which is increasing for $p \geq 13/2$. The last time $f(p)$ is negative is at $p = 10, f(10) = -6$ and $f(11) = 15$ so if $p \geq 11$ one of G or $\bar{G}$ is non-planar.