The Rental Car Problem

Suppose that you rent a car in Tallahassee (point A) with a full tank of fuel in order to drive to Tampa (point B). Tampa is close enough so that it can be reached without stopping for additional fuel. The rental company requires that the tank be full when the car is returned. Otherwise they will charge an outrageous amount to fill the tank. Suppose that the price of fuel varies along the route in some known way, say that the price per gallon x miles from A towards B is $P(x)$ dollars per gallon. What strategy should you follow in purchasing fuel on the trip in order to minimize total fuel costs?

If the price of fuel is lowest at B, then, obviously, the best strategy is to wait until reaching B and then to fill up. But in other cases filling up at the place along the way where fuel is cheapest is usually not the optimal strategy.

We will assume that the car's fuel consumption rate is a constant R miles/gallon and that you fill up completely at each stop. In order to simplify the model we will also assume that x can take on any value in the interval $[0, b]$, where b is the number of miles from A to B.

The simplest interesting case is the *two-fill problem.* In this case the motorist stops at most once along the way to fill up x miles from A and then tops off the tank at B.

(a) Show that the total cost $C(x)$ of the trip in the *two-fill problem* is given by

$$C(x) = \frac{x}{R}P(x) + \frac{b-x}{R}P(b), \qquad 0 \le x \le b. \tag{$*$}$$

Assume in what follows that you start with a full tank of gas in a car that gets 25 miles/gallon and has a 20−gallon tank. City B is 500 miles away. Further assume that gas prices in city A are \$1.00 per gallon and that gas prices in city B are \$1.50 per gallon.

(b) If the gas prices increase linearly from A to B, where should you stop for gas to minimize your total cost? (Remember that you have to fill your tank at B and that you are allowed just one stop between A and B.)

(c) Assume next that $P(x)$ increases (nonlinearly) according to the rule $P(x) = p + e^{kx}$, where p and k are constants. Now where should you stop? (First determine p and k and then minimize $C(x)$.)

(d) Graph the function $(1/25)P$, where P is the price function in part (b). Since the car gets 25 miles/gallon, $(1/25)P(x)$ is the cost per mile during the first x miles if you fill the tank x miles from A. (Why?) Find a geometric interpretation of the total cost of gasoline as the sum of areas of rectangles overlaid on this graph. Repeat this procedure for the function P in part (c).

(e) Suppose now that you are allowed two intermediate stops, at x_1 and x_2 with $0 \le x_1 \le x_2 \le b$. Write down a cost function $C(x_1, x_2)$ analogous to $(*)$ and do an interpretation analogous to the one in part (d). Use the cost function P from part (c). Using this geometric interpretation, what is the limiting (lowest) cost if there is no limit on the number of gas stops?