Complex Homework Summer 2004

These are problems will be due both daily and at the end of classes. This PDF file was created on May 13, 2004.

1. (a) (BC3.1) Reduce each of these 3 expressions to a real number

$$\frac{1+2i}{3-4i}+\frac{2-i}{5i} \qquad \frac{5i}{(1-i)(2-i)(3-i)} \qquad \text{and} \qquad (1-i)^4$$

   (b) (BC4.1) In each case locate z_1+z_2 and z_1-z_2 vectorially
   $z_1=2i$, $z_2=\frac{2}{3}-i$ $\qquad$ $z_1=(-\sqrt{3},0)$, $z_2=(\sqrt{3},0)$
   $z_1=(-3,1)$, $z_2=(1,4)$ $\qquad$ $z_1=x_1+iy_1$, $z_2=x_1-iy_1$

   (c) (BC4.4) Sketch the set of points determined by each equation

$$|z-1+i|=1 \qquad |z+i|\le 3 \qquad \text{and} \qquad |z+4i|\ge 4$$

   (d) (BC5.3,4) Verify $\overline{z_1-z_2}=\overline{z_1}-\overline{z_2}$, $\overline{z_1z_2}=\overline{z}_1\overline{z}_2$, $\overline{z_1z_2z_3}=\overline{z}_1\overline{z}_2\overline{z}_3$ and $\overline{z^4}=\overline{z}^4$.

   (e) (BC5.5) Verify

$$\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}\ (z_2\ne 0)$$

   (f) (BC5.15) Show that the hyperbola $x^2-y^2=1$ can be written $z^2+\overline{z}^2=2$

   (g) (BC7.1) Find the principal argument $\operatorname{Arg} z$ for both

$$z=\frac{i}{-2-2i} \text{ and } z=(\sqrt{3}-i)^6$$

   (h) (BC7.2) Show $|e^{i\theta}|=1$ and $\overline{e^{i\theta}}=e^{-i\theta}$

   (i) (BC7.15) Use de Moivre's formula to derive the following trig identities.

$$\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta=4\cos^3\theta-3\cos\theta$$

$$\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta=3\sin\theta-4\sin^3\theta$$

2. (a) (BC7.7) Show if $\Re z_1>0$ and $\Re z_2>0$ then $\operatorname{Arg}(z_1z_2)=\operatorname{Arg} z_1+\operatorname{Arg} z_2$

   (b) (BC9.1) Find the square roots of $2i$ and $1-\sqrt{3}$ expressed in rectangular form

   (c) (BC9.3) Find all of the roots in rectangle coordinates of $(-1)^{1/3}$ and $8^{1/6}$.

   (d) (BC9.6) Find the 4 roots of $p(z)=z^4+4=0$ and use them to factor $p(z)$ into quadratic factors with real coefficients.

   (e) (BC10.1-3) Sketch the 6 sets and determine which are domains, which are bounded, which are neither open nor closed:
   $|z-2+i|\le 1$ $\qquad$ $|2z+3|>4$ $\qquad$ $\Im z>1$
   $\Im z=1$ $\qquad$ $0\le\arg z\le\pi/4\,(z\ne 0)$ $\qquad$ $|z-4|\le|z|$

   (f) (BC10.4) Find the closure of the 4 sets:

$$-\pi<\arg z<\pi\ (z\ne 0) \qquad |\Re z|<|z| \qquad \Re(\frac{1}{z})\le\frac{1}{2} \qquad \text{and} \qquad \Re(z^2)>0$$

   (g) (BC11.1) For each function, describe the domain that is understood:

$$f(z)=\frac{1}{z^2+1} \qquad f(z)=\operatorname{Arg}(\frac{1}{z}) \qquad f(z)=\frac{z}{z+\overline{z}} \qquad \text{and} \qquad f(z)=\frac{1}{1-|z|^2}$$

(h) (BC11.2) Write $z^3 + z + 1$ as $u(x, y) + iv(x, y)$

(i) (BC11.3) Write and simplify $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$ in terms of z using $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$

(j) (BC11.4) Write $f(z) = z + 1/z \, (z \neq 0)$ in the form $u(r, \theta) + iv(r, \theta)$

3. (a) (BC13.1) Find a domain in the z-plane whose image under the transformation $w = z^2$ is the square domain in the w-plane bounded by the lines $u = 1, u = 2, v = 1, v = 2$

(b) (BC13.3) Sketch the region onto which the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ is mapped by the 3 transformations $w = z^2, w = z^3$, and $w = z^4$

(c) (BC13.4) Show that lines $ay = x \, (a \neq 0)$ are mapped onto the spirals $\rho = \exp(a\theta)$ under the transformation $w = \exp z$, where $w = \rho \exp(i\phi)$

(d) (BC13.7) Find the image of the semi-infinite strip $x \geq 0, 0 \leq y \leq \pi$ under the transformation $w = \exp z$. Label the corresponding portions of the boundaries.

(e) (BC13.8) Graphically indicate the vector fields represented by $w = iz$ and $w = z/|z|$

4. (a) (BC17.3) Find the limits. n is a positive integer, $P(z)$ and $Q(z)$ are polynomials with $Q(z_0) \neq 0$

$$\lim_{z \to z_0} \frac{1}{z^n} \, (z_0 \neq 0) \qquad \lim_{z \to i} \frac{iz^3 - 1}{z + i} \quad \text{and} \quad \lim_{z \to z_0} \frac{P(z)}{Q(z)}$$

(b) (BC17.5) Show that the following limit does not exist

$$\lim_{z \to 0} \left(\frac{z}{\overline{z}}\right)^2$$

(c) (BC17.10) Use a theorem to show:

$$\lim_{z \to \infty} \frac{4z^2}{(z - 1)^2} = 4 \qquad \lim_{z \to 1} \frac{1}{(z - 1)^3} = \infty \quad \text{and} \quad \lim_{z \to \infty} \frac{z^2 + 1}{z - 1} = \infty$$

(d) (BC17.11) Suppose $ad - bc \neq 0$ and let:

$$T(z) = \frac{az + b}{cz + d}$$

Use a theorem to show

$$\lim_{z \to \infty} T(z) = \infty \, (\text{if } c = 0) \qquad \lim_{z \to \infty} T(z) = \frac{a}{c} \, (\text{if } c \neq 0) \quad \text{and} \quad \lim_{z \to -d/c} T(z) = \infty \, (\text{if } c \neq 0)$$

5. (a) (BC17.13)(Show that a set S is unbounded if and only if every neighborhood of the point at infinity contains at least one point of S.

6. (a) (BC19.1) Find $f'(z)$ when

$$f(z) = 3z^2 - 2z + 4 \quad f(z) = (1 - 4z^2)^3 \quad f(z) = \frac{z - 1}{2z + 1} \, (z \neq -\frac{1}{2}) \quad \text{and} \quad f(z) = \frac{(1 + z^2)^4}{z^2} \, (z \neq 0)$$

(b) (BC19.2) Show if $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ then $P'(z) = a_1 + 2a_2 z + \cdots + na_z z^{n-1}$ and hence

$$a_0 = P(0), \quad a_1 = \frac{P'(0)}{1!}, \quad a_2 = \frac{P''(0)}{2!}, \quad \ldots \quad a_n = \frac{P^{(n)}(0)}{n!}$$

(c) (BC19.9) Let f denote the function whose values are

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

Show that if $z \neq 0$, then $\Delta w/\Delta z = 1$ at each nonzero point on the real and imaginary axes in the Δz or $\Delta x \Delta y$-plane. Then show then $\Delta w/\Delta z = -1$ at each nonzero point along the line $y = x$. Conclude that $f'(0)$ does not exist.

(d) (BC22.6) Let f denote the function above. Show that the Cauchy-Riemann equations are satisfied at the origin $z = (0, 0)$

(e) (BC22.1) Use a theorem to show that $f'(z)$ does not exist at any point for each function:

$$f(z) = \overline{z} \qquad f(z) = z - \overline{z} \qquad f(z) = 2x + ixy^2 \qquad \text{and} \qquad f(x) = e^x e^{-iy}$$

(f) (BC22.2) Use a theorem to show that $f'(z)$ and its derivative $f''(z)$ exist everywhere and find $f''(z)$.

$$f(z) = iz + 2 \qquad f(z) = e^{-x}e^{-iy} \qquad f(z) = z^3 \quad \text{and} \quad f(z) = \cos x \cosh y - i \sin x \sinh y$$

(g) Extra Credit (BC22.10) Recall $z = x + iy$ implies $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$. Use the formal chain rule to show

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y})$$

Define the operator

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$$

and apply it to $u(x, y) + iv(x, y)$ to obtain the complex form of the Cauchy-Reimann equations $\partial f/\partial \overline{z} = 0$.

7. (a) (BC28.1) Show that $\exp(2 \pm 3\pi i) = -e^2$, $\exp((2x+\pi i)/4) = (1+i)\sqrt{e/2}$ and $\exp(z+\pi i) = -\exp z$.

(b) (BC28.2) State why the function $2z^2 - 3 - ze^z + e^{-z}$ is entire.

(c) (BC28.3) Show $f(z) = \exp \overline{z}$ is not analytic anywhere.

(d) (BC28.7) Prove $|\exp(-2z)| < 1$ if and only if $\Re z > 0$.

(e) (BC28.8) Find all values of z such that $e^z = -2$, or $e^z = 1 + \sqrt{3}i$ or $\exp(2z - 1) = 1$

(f) (BC28.10) Show that if e^z is real, then $\Im z = nz \qquad (n = 0, \pm1, \pm2, \dots)$. If e^z is pure imaginary, what restriction is placed on z?

(g) (BC30.1) Show that $\text{Log}(-ei) = 1 - \frac{\pi}{2}i$ and $\text{Log}(1 - i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

8. (a) (BC30.2) Verify for $n = 0, \pm1, \pm2, \dots$:

$$\log e = 1 + 2n\pi i \qquad \log i = (2n + \frac{1}{2})\pi i \quad \text{and} \quad \log(-1 + \sqrt{3}i) = \ln 2 + 2(n + \frac{1}{3})\pi i$$

.

(b) (BC30.3) Show that $\text{Log}(1 + i)^2 = 2\,\text{Log}(1 + i)$ and $\text{Log}(-1 + i)^2 \neq 2\,\text{Log}(-1 + i)$.

(c) (BC30.5) Show that the set of values of $\log(i^{1/2})$ is $\{(n + \frac{1}{4})\pi i : n = 0, \pm1, \pm2, \dots\}$ and that the same is true of $(1/2)\log i$.

(d) (BC30.6) Given that the branch $\log z = \ln r + i\theta\,(r > 0, \alpha < \theta < \alpha + 2\pi)$ of the logarithmic function is analytic at each point z in the stated domain, obtain its derivative by differentiating each side of the identity $\exp(\log z) = z$ and using the chain rule.

(e) (BC30.7) Find all the roots of the equation $\log z = i\pi/2$.

(f) (BC30.9) Show that $\text{Log}(z-i)$ is analytic everywhere except on the half line $y = 1\,(x \leq 0)$. Show

$$\frac{\text{Log}(z+4)}{z^2+i}$$

is analytic everywhere except at the points $\pm(1-i)/\sqrt{2}$ and on the portion $x \leq -4$ of the real axis.

9. (a) (BC31.1) Show if $\Re z_1 > 0$ and $\Re z_2 > 0$ then $\text{Log}(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2$.

   (b) (BC31.2) Show that for any two complex numbers z_1 and z_2, $\text{Log}(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2 + 2N\pi i$ where N has one of the values $0, \pm 1$.

   (c) (BC32.1) Show that when $n = 0, \pm 1, \pm 2 \ldots$

$$(1+i)^i = \exp(-\frac{\pi}{4} + 2n\pi)\exp(\frac{i}{2}\ln 2) \qquad \text{and} \qquad (-1)^{1/\pi} = e^{(2n+1)i}$$

   (d) (BC32.2) Find the principal values of each expression:

$$i^i \qquad [\frac{e}{2}(-1-\sqrt{3}i)]^{3\pi i} \qquad \text{and} \qquad (1-i)^{4i}$$

   (e) (BC32.5) Show that the principal n-th root of a nonzero complex number z_0 is the same as the principal value of $z_0^{1/n}$ that was previously defined.

   (f) (BC32.8) Let c, d, z be complex numbers with $z \neq 0$. Prove that if all the powers involved are principal values, then

$$\frac{1}{z^c} = z^{-c} \qquad (z^c)^n = z^{cn}\,(n = 1, 2, \ldots) \qquad z^c z^d = z^{c+d} \qquad \text{and} \qquad \frac{z^c}{z^d} = z^{c-d}$$

   (g) (BC37.2) Evaluate

$$\int_1^2 (\frac{1}{t} - i)^2\, dt \qquad \int_0^{\pi/6} e^{i2t}\, dt \qquad \text{and} \qquad \int_0^{\infty} e^{-zt}\, dt\,(\Re z > 0)$$

.

   (h) (BC37.5) Let $w(t)$ be a continuous complex-valued funtion of t defined on an interval $a \leq t \leq b$. By considering the special case $w(t) = e^{it}$ on the interval $0 \leq t \leq 2\pi$, show that it is not always true that there is a number c in the interval $a < t < b$ such that

$$\int_a^b w(t)\, dt = w(c)(b-a)$$

10. (a) (BC38.2) Let C denote the right-hand half of the circle $|z| = 2$, in the counterclockwise direction and note that two parametric representations for C are

$$z = z(\theta) = 2e^{i\theta} \quad (-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2})$$

and

$$z = Z(y) = \sqrt{4-y^2} + iy \quad (-2 \leq y \leq 2)$$

Verify that $Z(y) = z[\phi(y)]$, where

$$\phi(y) = \arctan \frac{y}{\sqrt{4-y^2}} \qquad (-\frac{\pi}{2} \leq \arctan t \leq \frac{\pi}{2})$$

Also, show that this function ϕ has a positive derivative, as required in the conditions following (9) Sec 38.

(b) (BC40.1,2,3,5,6) Evaluate

$$\int_C f(z)\,dz$$

for the given $f(z)$ and contour C

$f(z) = (z+2)/z$	C is $z = 2e^{i\theta}\,(0 \le \theta \le \pi)$
$f(z) = (z+2)/z$	C is $z = 2e^{i\theta}\,(\pi \le \theta \le 2\pi)$
$f(z) = (z+2)/z$	C is $z = 2e^{i\theta}\,(0 \le \theta \le 2\pi)$
$f(z) = z+1$	C is $z = 1 + e^{i\theta}\,(\pi \le \theta \le 2\pi)$
$f(z) = z+1$	C is $z = t\,(0 \le t \le 2)$
$f(z) = \pi\exp(\pi\overline{z})$	C is square from $0, 1, 1+i, i$
$f(z) = 1$	C is arbitrary curve from z_1 to z_2
$f(z) = z^{-1+i}$	C is $\lvert z\rvert = 1$ positively oriented
	use branch $\exp[(-1+i)\log z]\,(\lvert z\rvert > 0, 0 < \arg z < 2\pi)$

(c) (BC40.10) Let C_0 denote the circle $|z - z_0| = R$ taken counterclockwise. Use the parametric representation $z = z_0 + Re^{i\theta}\,(-\pi \le \theta \le \pi)$ for C_0 to derive the following integration formula's:

$$\int_{C_0} \frac{dz}{z - z_0} = 2\pi i \qquad \text{and} \qquad \int_{C_0} (z - z_0)^{n-1}\,dz = 0\,(n = \pm 1, \pm 2, \dots)$$

11. (a) (BC41.4) Let C_R denote the upper half of the circle $|z| = R\,(R > 2)$, taken in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z + 4}\,dz\right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}$$

(b) (BC43.1) Use an antiderivative to show that, for every contour C extending from a point z_1 to a point z_2,

$$\int_C z^n\,dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1})\,(n = 0, 1, \dots)$$

(c) (BC43.2) By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration.

$$\int_i^{i/2} e^{\pi z}\,dz \qquad \int_0^{\pi + 2i} \cos(\frac{z}{2})\,dz \qquad \text{and} \qquad \int_1^3 (z-2)^3\,dz$$

12. (a) (BC43.3) Use a theorem to show

$$\int_{C_0} (z - z_0)^{n-1}\,dz = 0\,(n = \pm 1, \pm 2, \dots)$$

when C_0 is any closed contour which does not pass through the point z_0.

(b) (BC43.4) Let C_1, (resp. C_2), be any contour from $z = -3$ to $z = 3$ that except for its end points, lies above (resp. below) the x-axis. Find an antiderivative $F_2(z)$ of the branch $f_2(z)$ of

$$z^{1/2} = \sqrt{r}e^{i\theta/2} \quad (r > 0, \frac{pi}{2} < \theta < \frac{5pi}{2})$$

to show that the integral

$$\int_{C_2} z^{1/2}\,dz$$

has value $2\sqrt{3}(-1+i)$. Note that the value of the integral of the function

$$z^{1/2} = \sqrt{r}e^{i\theta/2}$$

around the closed contour $C_2 - C_1$ in that example is, therefore $-4\sqrt{3}$ given that

$$\int_{C_1} z^{1/2}\,dz = 2\sqrt{3}(1+i)$$

. (Lots of parts from example 43.4.)

13. (a) (BC46.1) Apply the Cauchy-Goursat theorem to show that

$$\int_C f(z)\,dz = 0$$

when the contour C is the circle $|z| = 1$, in either direction and when

$$f(z) = \frac{z^2}{z-3} \qquad f(z) = ze^{-z} \qquad f(z) = \frac{1}{z^2+2z+2}$$
$$f(z) = \operatorname{sech} z \qquad f(z) = \tan z \qquad f(z) = \operatorname{Log}(z+2)$$

(b) (BC46.2) Let C_1 be the positively oriented circle $|z| = 4$ and let C_2 be the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1, y = \pm 1$. Point out why

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

when

$$f(z) = \frac{1}{3z^2+1} \qquad f(z) = \frac{z+2}{\sin(z/2)} \qquad \text{and} \qquad f(z) = \frac{z}{1-e^z}$$

(c) (BC46.3) If C is the boundary of the rectangle $0 \le x \le 3, 0 \le y \le 2$, described in the positive sense, then

$$\int_C (z-2-i)^{n-1} = 2\pi i \text{ when } n = 0 \text{ and } 0 \text{ when } n = \pm 1, \pm 2, \ldots$$

(d) (BC46.4) Extra Credit ????

14. (a) (BC48.1abc) Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2, y = \pm 2$. Evaluate the integrals

$$\int_C \frac{e^{-z}\,dz}{z-(\pi i/2)} \qquad \int_C \frac{\cos z\,dz}{z(z^2+8)} \qquad \text{and} \qquad \int_C \frac{z\,dz}{2z+1}$$

(b) (BC48.2) Find the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when $g(z) = 1/(z^2+4)$ and when $g(z) = 1/(z^2+4)^2$.

(c) (BC48.3) Let C be the circle $|z| = 3$ decribed in the positive sense. Show that if

$$g(w) = \int_C \frac{2z^2-z-2}{z-w}\,dz \qquad (|w| \ne 3)$$

then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

(d) (BC48.7) Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$. First show that for any real constant a,

$$\int_C \frac{e^{az}}{z}\,dz = 2\pi i$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^\pi e^{a\cos\theta}\cos(a\sin\theta)\,d\theta = \pi$$

(e) (BC48.6) Extra Credit ???? Let f denote a function that is *continuous* on a simple closed contour C. Prove the function

$$g(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)\,d\xi}{\xi - z}$$

is analytic as each point z interior to C and and that

$$g'(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)\,d\xi}{(\xi - z)^2}$$

at such a point.

15. (a) (BC50.1) Let f be an entire function such that $|f(z)| \le A|z|$ for all z, where A is a fixed positive number. Show that $f(z) = a_1 z$, where a_1 is a complex constant. [Hint: use Cauchy's inequality to show $f''(z)$ is zero.]

   (b) (BC50.1) Suppose $f(z)$ is entire and that the harmonic function $u(x,y) = \Re f(z)$ has an upper bound u_0: that is, $u(x,y) \le u_0$ for all points (x,y) in the xy-plane. Show that $u(x,y)$ must be constant throughout the plane. [Hint: use Liouville's theorem on $\exp(f(z))$.]

   (c) (BC50.4,5) Let a function f be continuous in a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming $f(z) \ne 0$ anywhere in R, prove that $|f(z)|$ has a *minimum value* m in R which occurs on the boundary of R and never in the interior. [Hint: look at $1/f(z)$.]

   Use the function $f(z) = z$ to show that the condition $f(z) \ne 0$ anywhere is necessary for this conclusion.

16. (a) (BC52.6) Show if $\sum_{n=1}^{\infty} z_n = S$, then $\sum_{n=1}^{\infty} \overline{z}_n = \overline{S}$.

   (b) (BC52.7) Show for any complex number c Show if $\sum_{n=1}^{\infty} z_n = S$, then $\sum_{n=1}^{\infty} c z_n = cS$.

   (c) (BC52.8) Show if $\sum_{n=1}^{\infty} z_n = S$ and $\sum_{n=1}^{\infty} w_n = T$, then $\sum_{n=1}^{\infty}(z_n + w_n) = S + T$.

17. (a) (BC54.2) Obtain the Taylor

$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!} \qquad (|z-1| < \infty)$$

   two ways. First using $f^{(n)}(1)$ and second by using $e^z = e e^{z-1}$.

   (b) (BC54.3) Find the Maclaurin series expansion for the function

$$f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+z^4/9}$$

   (c) (BC54.5) Derive the Maclaurin series for $\cos z$ by showing $f^{(2n)}(0) = (-1)^n$ and $f^{(2n+1)}(0) = 0$ and by using $\cos z = (e^{iz} + e^{-iz})/2$.

   (d) (BC54.11) Show when $z \ne 0$,

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots$$
$$\frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots$$

   (e) (BC54.13) Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

18. (a) (BC56.1) Find the Laurent series that represents the function $f(z) = z^2 \sin(1/z^2)$ in the domain $0 < z < \infty$.

   (b) (BC56.2) Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right]$$

   (c) (BC56.3) Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} \cdot \frac{1}{1+(1/z)}$$

   in negative powers of z that is valid for $1 < |z| < \infty$.

(d) (BC56.4) Give two Laurent series expansions in powers of z for the function $f(z) = 1/[z^2(1-z)]$ and specify the regions in which the expansions are valid. [Hint: about 0 and ∞]

(e) (BC56.5) Represent the function

$$f(z) = \frac{z+1}{z-1}$$

by both its Maclaurin series (stating where it is valid) and by a Laurent series in the domain $1 < |z| < \infty$

(f) (BC56.6) Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}$$

19. (a) (BC60.1) By differentiating the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad (|z| < 1)$$

obtain the expressions

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n \qquad (|z| < 1)$$

and

$$\frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} (n+1)(n+2)z^n \qquad (|z| < 1)$$

(b) (BC60.2) By substituting $1/(1-z)$ for z in the expansion

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n \qquad (|z| < 1)$$

found above, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n(n-1)}{(z-1)^n} \qquad (1 < |z-1| < \infty)$$

(c) (BC60.3) Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2} \cdot \frac{1}{1+(z-2)/2}$$

about the point $z_0 = 2$. Then by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n(n+1)(\frac{z-2}{2})^n \qquad (|z-2| < 2)$$

(d) (BC61.1) Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots \qquad (0 < |z| < 1)$$

(e) (BC61.3) Use division to obtain the Laurent series representation

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots \qquad (0 < |z| < 2\pi)$$

(f) (BC64.1) Find the residue at $z = 0$ of the functions

$$\frac{1}{z+z^2} \qquad z\cos(\frac{1}{z}) \qquad \frac{z-\sin z}{z} \qquad \frac{\cot z}{z^4} \qquad \text{and} \qquad \frac{\sinh z}{z^4(1-z^2)}$$

(g) (BC64.2) Use Cauchy's residue theorem to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

$$\frac{\exp(-z)}{z^2} \qquad \frac{\exp(-z)}{(z-1)^2} \qquad z^2\exp(\frac{1}{z}) \qquad \text{and} \qquad \frac{z+1}{z^2-2z}$$

(h) (BC64.3) Use a theorem involving a single residue to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense.

$$\frac{z^5}{1-z^3} \qquad \frac{1}{1+z^2} \qquad \text{and} \qquad \frac{1}{z}$$

20. (a) (BC65.1) In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point or an essential singular pont.

$$z\exp(\frac{1}{z}) \qquad \frac{z^2}{1+z} \qquad \frac{\sin z}{z} \qquad \frac{\cos z}{z} \qquad \text{and} \qquad \frac{1}{(2-z)^3}$$

(b) (BC65.2) Show that the singular point of each of the following functions is a pole. Determine the order m of the pole and the corresponding residue B.

$$\frac{1-\cosh z}{z^3} \qquad \frac{1-\exp(2z)}{z^4} \qquad \text{and} \qquad \frac{\exp(2z)}{(z-1)^2}$$

(c) (BC65.3) Suppose f is analytic at z_0 and write $g(z) = f(z)/(z - z_0)$. Show that:

i. If $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$.

ii. If$f(z_0) = 0$, then z_0 is a removable singular point of g.

21. (a) (BC65.4) Write the function

$$f(z) = \frac{8a^3z^2}{(z^2+a^2)^3} \qquad (a > 0)$$

as

$$f(z) = \frac{\phi(z)}{(z-ai)^3} \text{ where } \phi(z) = \frac{8a^3z^2}{(z+ai)^3}$$

Point out why $\phi(z)$ has a Taylor series representation about $z = ai$, and then use it to show that the principal part of f at that point is

$$\frac{\phi''(ai)/2}{z-ai} + \frac{\phi'(ai)}{(z-ai)^2} + \frac{\phi(ai)}{(z-ai)^3} = -\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2i}{(z-ai)^3}$$

(b) (BC67.1) In each case, show that any singular point of the function is a pole. Determine the order m of the pole and find the corresponding residue B

$$\frac{z^2+2}{z-1} \qquad (\frac{z}{2z+1})^3 \qquad \text{and} \qquad \frac{\exp z}{z^2+\pi^2}$$

(c) (BC67.2) Show that

$$\operatorname*{Res}_{z=-1} \frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

$$\operatorname*{Res}_{z=i} \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\pi+2i}{8}$$

$$\operatorname*{Res}_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

(d) (BC67.3) Find the value of the integral

$$\int_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz$$

taken counterclockwise around both circles $|z-2|=2$ and $|z|=4$

22. (a) (BC67.4) Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)}$$

taken counterclockwise around both circles $|z|=2$ and $|z+2|=3$

(b) (BC69.1) Show that the point $z=0$ is a simple pole of the function $f(z)=\csc z=1/\sin z$ by a theorem and by computing the Laurent series.

(c) (BC69.3a) Show that

$$\operatorname*{Res}_{z=z_n}(z\sec z)=(-1)^{n+1}z_n,\ \text{where } z_n=\frac{\pi}{2}+n\pi \qquad (n=0,\pm1,\pm2,\ldots$$

(d) (BC69.4a) Let C denote the positively oriented circle $|z|=2$ and evaluate the integral

$$\int_C \tan z\,dz$$

(e) (BC69.5) Let C_N denote the positive oriented boundary of the square whose edges lie along the lines

$$x=\pm(N+\frac{1}{2})\pi \text{ and } y=\pm(N+\frac{1}{2})\pi$$

where N is a positive integer. Show that

$$\int_{C_N}\frac{dz}{z^2\sin z}=2\pi i\left[\frac{1}{6}+2\sum_{n=1}^{N}\frac{(-1)^n}{n^2\pi^2}\right]$$

then using the fact that the value of this integral tends to zero as N tends to infinity, point out how it follows that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi^2}{12}$$

23. (a) (BC69.9) Let p and q denote functions that are analytic at a point z_0 where $p(z_0)\neq 0$ and $q(z_0)=0$. Show that if the quotient $p(z)/q(z)$ has a pole of order m at z_0, then z_0 is a zero of order m of q.

24. (a) (BC72.1,2,4) Use residues to evaluate the following integrals

$$\int_0^\infty \frac{dx}{x^2+1} \qquad \int_0^\infty \frac{dx}{(x^2+1)^2} \qquad \text{and} \qquad \int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)}$$

(b) (BC74.1,2) Use residues to evaluate the following integrals

$$\int_{-\infty}^{\infty}\frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} \quad (a>b>0) \qquad \text{and} \qquad \int_0^\infty \frac{\cos ax\,dx}{x^2+1} \quad (a>0)$$