Use the recursive definitions of ordinal arithemitic:

$$\alpha + 0 = \alpha, \qquad \alpha \cdot 0 = 0 \qquad \alpha^0 = 1$$

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1 \qquad \alpha \cdot (\beta + 1) = \alpha \cdot \beta + \alpha \qquad \alpha^{\beta+1} = \alpha^\beta \cdot \alpha$$

And if β is a non-zero limit ordinal

$$\alpha + \beta = \cup\{\alpha + \gamma : \gamma < \beta\} \qquad \alpha \cdot \beta = \cup\{\alpha \cdot \gamma : \gamma < \beta\} \qquad \alpha^\beta = \cup\{\alpha^\gamma : \gamma < \beta\}$$

to show the following:

1. For all ordinals $\alpha, \beta, \gamma, \delta$:

$$0 + \alpha = \alpha, \text{ and } \omega \leq \alpha \Rightarrow 1 + \alpha = \alpha$$

$$0 < \beta \Rightarrow \alpha < \alpha + \beta$$

$$\alpha \leq \beta \,\&\, \gamma \leq \delta \Rightarrow \alpha + \gamma \leq \beta + \delta$$

$$\alpha \leq \beta \,\&\, \gamma < \delta \Rightarrow \alpha + \gamma < \beta + \delta$$

   Show also, that, in general

$$\alpha < \beta \text{ does not imply } \alpha + \delta < \beta + \gamma$$

2. For all ordinals $\alpha, \beta, \gamma, \delta$:

$$0 \cdot \alpha = 0$$

$$0 < \alpha \,\&\, 1 < \beta \Rightarrow \alpha < \alpha \cdot \beta$$

$$\alpha \leq \beta \,\&\, \gamma \leq \delta \Rightarrow \alpha \cdot \gamma \leq \beta \cdot \delta$$

$$0 < \alpha \leq \beta \,\&\, \gamma < \delta \Rightarrow \alpha \cdot \gamma < \beta \cdot \delta$$

   Show also that even when $\gamma > 0$ in general,

$$\alpha < \beta \text{ does not imply } \alpha \cdot \gamma < \beta \dot{\gamma}$$

3. (Cancellation laws) For all ordinals α, β, γ :

$$\alpha + \beta < \alpha + \gamma \Rightarrow \beta < \gamma$$

$$\alpha + \beta = \alpha + \gamma \Rightarrow \beta = \gamma$$

$$\alpha \cdot \beta < \alpha \cdot \gamma \Rightarrow \beta < \gamma$$

$$0 < \alpha \,\&\, \alpha \cdot \beta = \alpha \cdot \gamma \Rightarrow \beta = \gamma$$

   Show also that, in general

$$0 < \alpha \,\&\, \beta \cdot \alpha = \gamma \cdot \alpha \text{ does not imply } \beta = \gamma$$

4. For all $\alpha \geq \omega$ and $n < \omega$:

$$n + \alpha = \alpha$$

$$(\alpha + 1) \cdot n = \alpha \cdot n + 1 \qquad (n > 0)$$

$$(\alpha + 1) \cdot \omega = \alpha \cdot \omega$$

   Give an example:

$$(\alpha + \beta) \cdot \gamma \neq \alpha \cdot \gamma + \beta \cdot \gamma$$

5. If $\alpha \leq \gamma$ then there exists exactly one β such that $\gamma = \alpha + \beta$.

6. For all ordinals α, β, γ with $\alpha > 1$

$$\beta < \gamma \Rightarrow \alpha^{\beta} < \alpha^{\gamma}$$

$$\alpha^{(\beta+\gamma)} = \alpha^{\beta} \cdot \alpha^{\gamma}$$

$$\left(\alpha^{\beta}\right)^{\gamma} = \alpha^{\beta \cdot \gamma}$$

7. If $\alpha > 0$, then there is a largest β such that $\omega^{\beta} \leq \alpha$ and for that β, $\alpha = \omega^{\beta} + \gamma$ for some $\gamma < \alpha$.

8. If $\beta < \gamma$, then $\omega^{\beta} + \omega^{\gamma} = \omega^{\gamma}$. Every ordinal $\alpha > 0$ can be uniquely written as $\alpha = \omega^{\beta} + \gamma$ with $\gamma < \alpha$

9. Every ordinal $\alpha > 0$ can be uniquely written

$$\alpha = \omega^{\beta_1} \cdot n_1 + \omega^{\beta_2} \cdot n_2 \cdots + \omega^{\beta_k} \cdot n_k$$

$$(\beta_1 > \beta_2 > \cdots > \beta_k, 0 < n_i < \omega)$$

   This is called the Cantor normal form.

10. Find the Cantor normal form of

$$\omega \cdot (\omega^{\omega} + 1) + (\omega^{\omega} + 1) \cdot \omega$$