Tree Basis in Banach spaces

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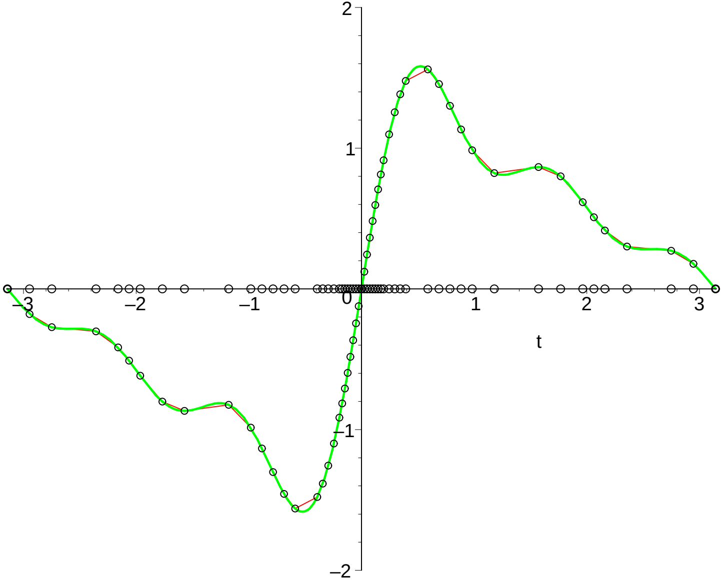
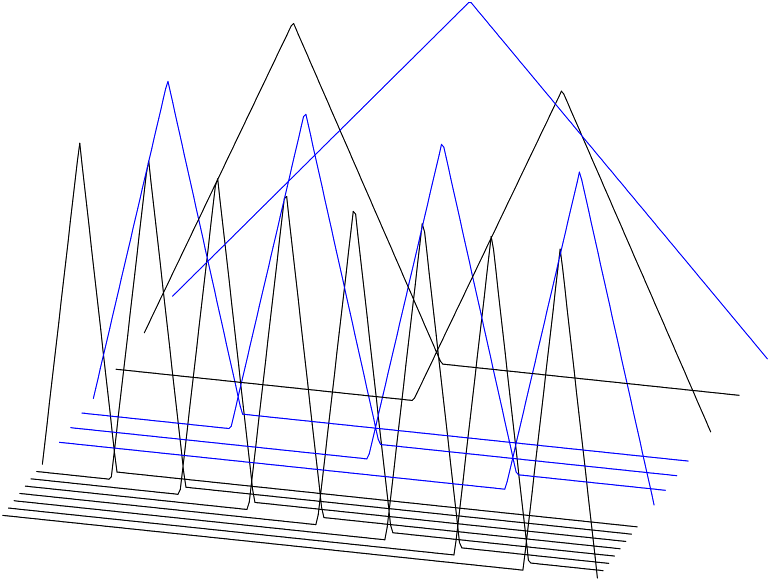
Recursive Adaptation

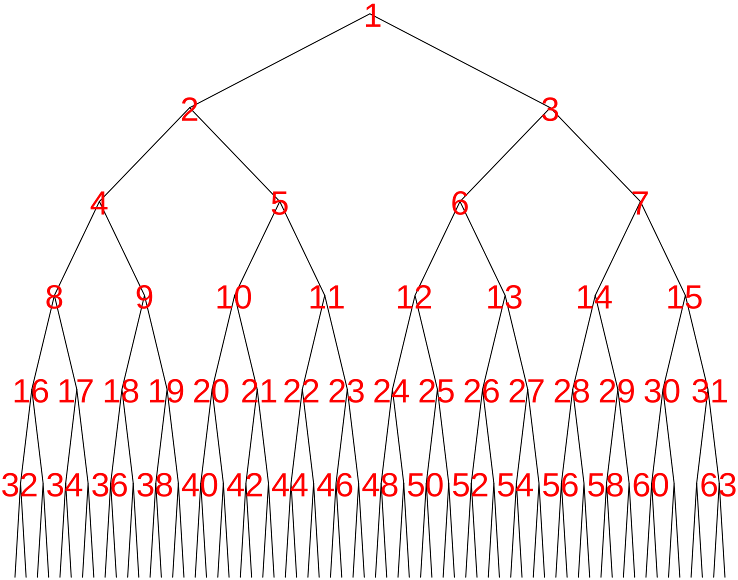


Function in Green, Approximation (connecting circles) in Red

Schauder's basis for $C[0,1]$

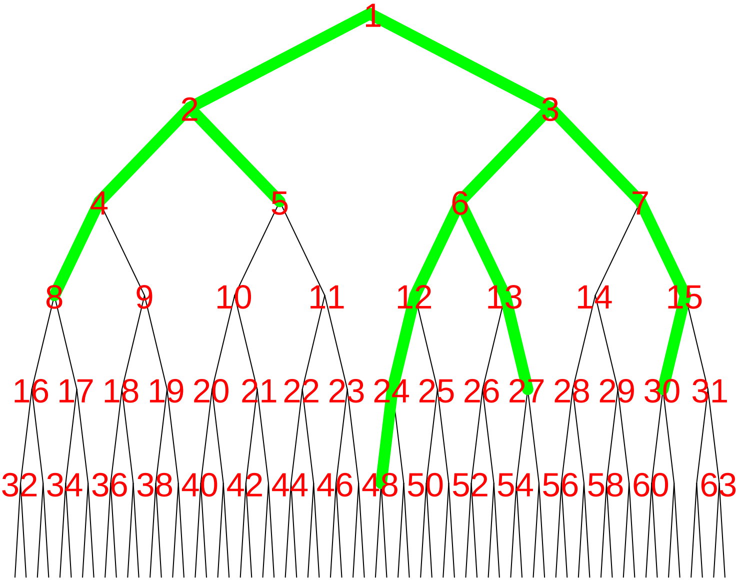


Actually for codim 2 subspace of $\{f : f(0) = f(1) = 0\}$.



Predecessor function $\phi(n) = \lfloor n/2 \rfloor$

Tree Subset



$n \in F \implies \phi(n) \in F$

Conditional, Tree, Unconditional

The sequence $\{e_n\}$ is basic when

$$\|\sum_{n\in F} a_n e_n\| \leq M\|\sum_n a_n e_n\|$$

- Conditional for all initial $F = \{1, \ldots N\}$
- Tree basis: for all tree subsets F.
- Unconditional for all finite F

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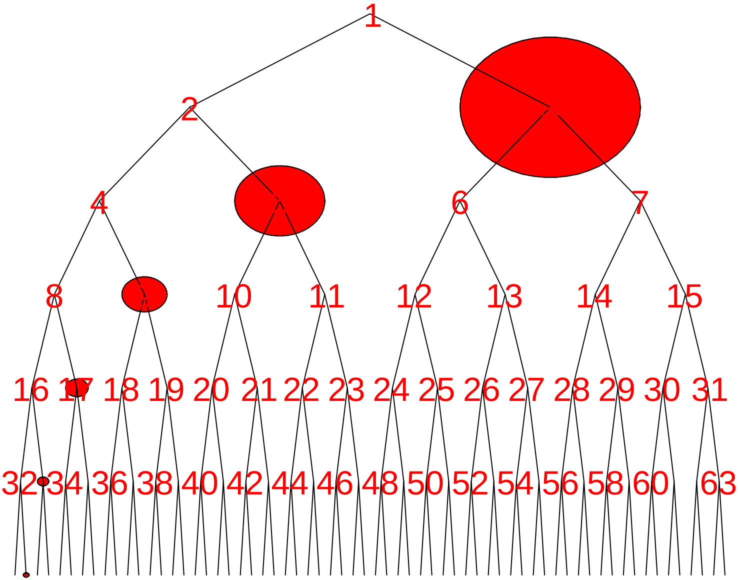
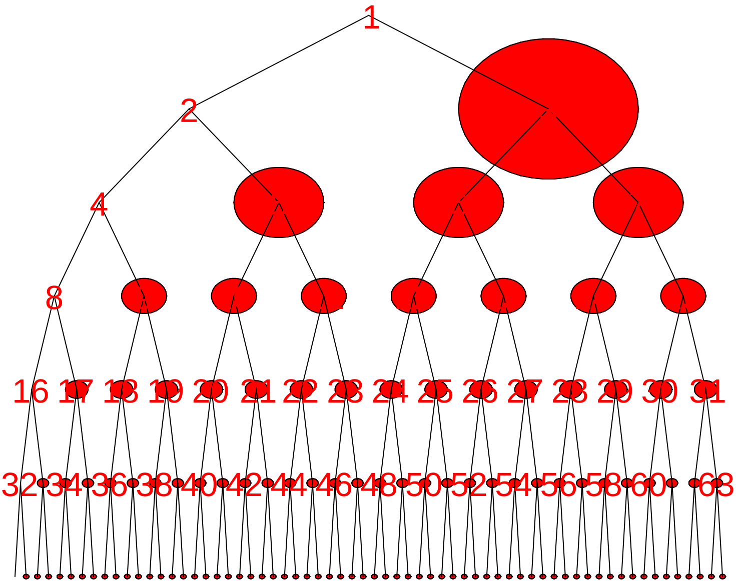
Tree = $B \oplus U$ part I



Red dots are an unconditional basis sequence.

Tree = $B \oplus U$ part II



From $B \oplus U$ to Tree basis.

Tree Basis Examples/Properties

- Any unconditional basis is a tree basis.
- Tree like spaces $C[0,1]$, JT, Haar basis in rearrangement invariant.
- The quasi-reflexive space $J \approx J \oplus \ell_2$.
- Cor: There are spaces with tree basis that don't have an unconditional basis
- Cor: Tree basis space has c_0, ℓ_1 or reflexive subspace.
- Cor: There are spaces with basis that don't have a tree basis.

Tree Translation – limited subsymmetry

The tree translation Φ_M is defined recursively. It moves the whole tree to the subtree rooted at M.

- $\Phi_M(1) = M$
- $\Phi_M(2n) = 2\Phi_M(n)$
- $\Phi_M(2n+1) = 2\Phi_M(n) + 1$
- $\Phi_M(\sum a_n e_n) = \sum a_n e_{\Phi_M(n)}$

If Φ_M is always an isometry, then the space is *Tree Translation Invariant,* and if Φ_M is always an isomorphism, the the space is *Tree Translation Equivalent.*

Invariance vs Equivalence

- $C[0,1]$, JT are tree translation invariant
- Tsirelson's space T is tree translation equivalent, not invariant – even when re-normed
- rearrangement invariant spaces are (in general only) tree translation equivalent
- There are Tsirelson superspaces which are not even tree translation equivalent

Properties of Tree Translation Equivalent

- A Tree Translation Equivalent space X is
- $\approx$ hyperplanes
- $\approx X \oplus X$
- $\approx$ unconditional sum (X_n) with each $X_n \approx X$

Primary

The space X is primary, if $X \approx Y \oplus Z$ implies $X \approx Y$ or $X \approx Z$.

- Most Primary spaces are Tree Translation Equivalent (JT, $C[0,1]$, certain Rearrangement Invariant spaces)
- Tsirelson's T is not primary
- Does Tree Translation Invariance imply primary?
- Subsymmetric bases are Tree Translation Invariant.

Branch Invariant Tree Spaces

A generalization of symmetric bases.

- Permutation π is *Branch Invariant* if
- $\ell(i) = \ell(\pi(i))$ preserves level and
- $\phi(\pi(i)) = \pi(\phi(i))$ preserves branches
- A tree basis $\{e_n\}$ is *Branch Invariant* if for branch invariant permutations π

$$\|\sum a_n e_n\| = \|\sum a_n e_{\pi(n)}\|$$

Examples Branch Invariant

- JT, rearrangement invariant spaces
- Not $C[0,1]$, some branches c_0, some the summing basis.
- T can be renormed to be Branch Invariant

Rademachers complemented

In a branch invariant tree basis, the projection

$$P\left(\sum a_i e_i\right) = \sum_{n=0}^{\infty}\left(\sum_{i=0}^{2^n-1} a_{2^n+i}\right)\left(\sum_{i=0}^{2^n-1} e_{2^n+i}\right)/2^n$$

has norm one.
For the Haar basis, this is the projection onto the Rademachers.

Summary

- The Tree basis is equivalent to $B \oplus U$
- The Tree Translation Equivalence $\approx$ hyperplanes, squares, UD($X_n \approx X$); but not always primary, nor Tree Translaton Invariant.
- Question, is there a tree translation invariant space that is not primary?