TREE BASIS IN BANACH SPACES

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Abstract. Tree basis in Banachs spaces, which are Schauder basis spaces with nice "tree projections", which is a property strictly between conditional and unconditional basis, are classified. Stronger basis properties like symmetric, and subsymmetric have weaker tree versions as well. These bases are motivated by well known adaptive approximation algoritms.

1. Introduction

It is well known that there is a huge different between a Banach space having a (conditional) Schauder basis and it having the more restrictive unconditional basis. The existence of subspaces with Schauder basis was known to Banach, while examples of Banach spaces with no unconditional basic sequence [8] are more recent and harder. A tree basis is defined to be an intermediate property, strictly stronger than Schauder and strictly weaker than unconditional. There are many tree based spaces like JT and wavelet bases, like the Haar system, which are tree spaces by construction and have a natural tree basis. The spaces with a tree basis includes most interesting Banach spaces that have a basis. Even James quasi-reflexive space J has a tree basis (Proposition 3.10). Most properties of a Banach space with tree basis can be obtained from the classification as the direct sum of a space with unconditional basis and another space with Schauder basis (Proposition 3.8). So tree basis are stronger than Schauder basis as they imply a complemented subspace with an unconditional basis.

We briefly consider scrub basis as a generalization of a tree basis and show the existence of non-trivial scrub basis that implied the existence of a tree basis (Proposition 3.11). Even higher dimensional "trees" thus reduce to the usual one-dimensional binary tree.

Most of the tree based constructions have the stronger property that each rooted subtree is isometric to the original space. We call these spaces tree translation invariant. Such spaces have many nice properties, including being isomorphic to their square. (So J's tree basis is not tree transition invariant.) Such spaces are often primary and many rearrangement invariant spaces are tree translation invariant. Another classical example of a tree translation invariance is the classic Schrauder basis for C.

Although these properties are amusing in their own right, the motivation was originally to abstract the adoptive approximations commonly used in many numerical algorithms, which subdivide an interval only where the variation is relatively large and do not subdivide the relatively flat spots.

2. Preliminaries

Our notation about bases in Banach spaces follows [14] or [11]. The sequence (e_n) is Schauder (respectively unconditional) basis for its closed linear span $X = [e_n]$ if there is a constant M to that $\|\sum_{n\in F}\alpha_n e_n\| \le M\|\sum \alpha_n e_n\|$ for all $x = \sum \alpha_n e_n \in X$ and all initial finite subsets $F = \{1, 2, 3 \dots k\} \subset \mathbb{N}$ (respectively all finite subsets $F \subset \mathbb{N}$).

A Banach space X is isomorphic to its hyperplanes (respectively its square) if $X \approx X \oplus \mathbb{K}$ where $\mathbb{K}$ is the scalar field (respectively if $X \approx X \oplus X$). A space X is said to be primary if $X \approx Y \oplus Z$ implies that either $X \approx Y$ or $X \approx Z$.

There are many common ways of descripting binary trees in analysis. We use the (binary tree) predessor function $\phi : \mathbb{N}\backslash 1 \to \mathbb{N}$ given by $\phi(n) = \lfloor n/2 \rfloor$, Where $\lfloor\cdot\rfloor$ is the floor or greatest integer function. If $\phi(n) = m$, then we say m is the *parent* of n and n is a *child* of m. Each integer m has two children $2m$ and $2m+1$. A finite or infinite sequence of integers $\{n_i\}$ is an initial branch if $n_1 = 1$ and $\phi(n_{i+1}) = n_i$. Another common notation is for a binary tree uses the Cantor set $\Gamma = 2^\omega$.

The subtree rooted at m, S_m is the collection of integers that are descendants of m under ϕ. This has the same structure as the complete tree under a similarly function $\Phi = \Phi_m$ defined inductively by $\Phi(1) = m$ and $\Phi(2n) = 2\Phi(n)$, $\Phi(2n+1) = 2\Phi(n)+1$. In the Cantor set view, this is a dilation followed by a translation. We will call such Φ a tree translation.

The level of integer n, $\ell(n) = \lfloor \log 2n \rfloor$ is the number of generations to the integer 1, the root of the tree. A branch permutation β is permutation on $\mathbb{N}$ that preserves 1 and parenthood. A branch permutation clearly preserves levels while permuting the branches.

3. Tree basis

Definition 3.1. *A finite subset $F \subset \mathbb{N}$ is a* tree-subset *if $n \in F\backslash 1$ implies its predessor $\phi(n) \in F$. A basis (e_n) for X is a* tree basis *if there is a constant $M < \infty$ for that for all finite tree subsets F and $x = \sum \alpha_n e_n \in X$*

$$\|\sum\nolimits_{n\in F}\alpha_n e_n\| \le M\|\sum \alpha_n e_n\|$$

Remark. The space X can be renormed so that the constant M is one. The existence of a tree basis condition is strictly stronger than the existence of a basis Propostion 3.8 and strictly weaker than an unconditional basis Propostion (3.10). However, our first chore is to show that (in some sense) a tree basis is the only intermediate notion between Schauder and unconditional basis.

Definition 3.2. *A function $\phi : \mathbb{N}\backslash\{1\} \to \mathbb{N}$ is called a (scrub)* predecessor function *if $\phi(n) < n$ for all integers n.*

Remark. The condition $\phi(n) < n$ has two effects. Most importantly it requires well-foundedness, that is there are no sequences $(n_i)_{i=1}^\infty$ so that $\phi(n_i) = n_{i+1}$. Secondly, it implies there is a single initial integer, 1, that has no predecessor, and it is the root. This second condition is unimportant as multiple roots (even infinitely many roots are easily handled).

Definition 3.3. *Given a* precessor function *ϕ we define a* splitting or branching node *to be an integer n with more than one solution to $\phi(m) = n$. Again $\phi(m) = n$ implies n is the* parent *of m and m is a* child *of n. Inductively to notions of ancestor and descent are also defined as they are for trees. a set $\{n_i\} \subset \mathbb{N}$ is* independent *if $i \ne j$ implies n_i and n_j are* unrelated*, neither is a descendent of the other. Independent sequences are mutally incomparable.*

Definition 3.4. *Given a precessor function ϕ, we define a finite subset $F \subset \mathbb{N}$ is a* scrub subset *if $n \in F\backslash\{1\}$ implies $\phi(n) \in F$. A basis (e_n) for X is a scrub basis of there is a* *constant $M < \infty$ so that for all finite scrub subsets F and $x = \sum \alpha_n e_n \in X$.*

$$\|\sum_{n\in F} \alpha_n e_n\| \le M\|\sum \alpha_n e_n\|$$

Lemma 3.5. *If ϕ is a precessor function, F an infinite scrub subset and (e_n) a scrub basis, then the projection $P(\sum \alpha_n e_n) = \sum_{n\in F} \alpha_n e_n$ is bounded by the scrub basis constant M.*

Proof. Let $F_k = F \cap \{1,\dots k\}$. Since each F_k is a scrub subset of $\mathbb{N}$, the projections $P_k(\sum \alpha_n e_n) = \sum_{n\in F_k} \alpha_n e_n$ are uniformly bounded in norm by some M. If $\sum \alpha_n e_n$ has norm one, then

$$\|\sum_{\substack{n\in F\\ p\le n\le q}} \alpha_n e_n\| = P_q(\sum_{n=p}^{q} \alpha_n e_n) \le M\|\sum_{n=p}^{q} \alpha_n e_n\|$$

hence $P_k(\sum \alpha_n e_n) \to P(\sum \alpha_n e_n)$ □

Proposition 3.6. *If the predecessor function ϕ has in infinite independent set $M \subset \mathbb{N}$, then any scrub basis (e_n) has an unconditional basic sequence $(e_n)_{n\in M}$ which is naturally complimented by*

$$P(\sum \alpha_n e_n) = \sum_{n\in M} \alpha_n e_n$$

Proof. Let M be the infinite independent set, let F be the smallest scrub set containing M and let $G = F\backslash M$. G is also a scrub set and $P = P_F - P_G$ where P_F and P_G are projections given by Lemma 3.5

To see $(e_n)_{n\in M}$ is unconditional note for any finite subset $H \subset M$. The projection $P_H(\sum_{n\in m} \alpha_n e_n) = \sum_{n\in H} \alpha_n e_n$ is $P_{G\cup H} - P_G$ which is bounded by $2M$

□

Proposition 3.7. *If the basis (e_n) has unconditional subsequence $(e_n)_{n\in M}$, which is naturally complemented, then there is a permutation π so that $(e_{\pi(n)})$is a tree basis.*

Proof. Let $N = \mathbb{N}\backslash M$ Let $N = (n_i)$ and $M = (m_j)$ be listing of these sets as increasing subsequences of $\mathbb{N}$. Define $\pi(n_i) = 2^{i-1}$ so that $(e_{\pi(n_i)})$ is left most branch of the tree and $(e_{\pi(m_j)})$ is the rest. If F is a finite tree subset then so is $F \cap \pi(n_i) = F_L$ and $F_R = F - F_L$. The projection onto F is the sum of the projection on F_L, which is an initial segment of $\pi(n_i)$, and F_R, one of the unconditional projections. Thus $(e_{\pi(n)})$ is a tree basis □

Corollary 3.8. *X has a tree basis if and only if $X \approx U \oplus Y$, where Y and U both have basis, and U has an unconditional basis.*

Corollary 3.9. *A space with a tree-basis contains a subspace isomorphic to c_o, ℓ_1 or a complemented infinite dimensional reflexive space.*

Proposition 3.10. *The space J, James quasi-reflexive space has a tree basis.*

Proof. One common basis for J is the shrinking basis e_n with norm

$$\|\sum \alpha_n e_n\| = \sup(\sum_{i=1}^{k} (\alpha_{n(i+1)} - \alpha_{n(i)})^2)^{\frac{1}{2}}$$

where the sup is over finite sequences $n(1) < n(2) < \dots n(k) < n(k+1)$. the projection $P(\sum \alpha_n e_n) \to \sum_{n=1}^{\infty} (\alpha_{2n}+\alpha_{2n+1})(e_{2n}+e_{2n+1})/2$ is a norm one projection with range isometric to J. The projection $Q + I - P$ has range $[(e_{2n} - e_{2n+1})]$, and $(e_{2n} - e_{2n+1})_{n=1}^{\infty}$ is equivalent to the usual basis of Hilbert space. Thus the basis which alternates between these two basic sequences $e_1 + e_2$, $e_1 - e_2$, $e_3 + e_4$, $e_3 - e_4$, ... is a basis which satisfies the hypothesis of proposition 3.8 □

Remark. Since $J \approx J \oplus \ell_2$ this also follows from Corolary 3.8.
Remark. Is is well known that J cannot have an unconditional basis and hence having a tree basis is strictly weaker than having an unconditional basis.

Proposition 3.11. *If ϕ is a non-trivial predecessor function and X has a ϕ scrub basis e_n, then X has a tree basis.*

Proof. Suppose for ϕ there is an integer n whose set of children M is infinite. This is an independent set, so $(e_n)_{n \in M}$ is unconditional by Proposition 3.6 and X has a tree basis(b_n) Proposition 3.7. Otherwise ϕ has infinitely many splitting nodes. By the Infinity Lemma, there is an infinite branch (n_i) which contains infinitely many splitting nodes $(n(s(i)))_i$. For each i, there must be $m(i) \neq n(s(i)+1)$ but $\phi(m(i)) = n(s(i))$. It follows that $M = (m(i))$ is an independent set. Thus, as in the first case X has a tree basis. □

Remark. The "minimal" scrub ϕ is given by $\phi(n+1) = n$ which only requires the same projections as those for a Schauder basis. Since there spaces with a basis with no non-trivial decomposition into $Y \oplus Z$. These must be spaces without tree basis.

The next step would be $X \oplus X$, which has a ϕ scrub basis for ϕ given by $\phi(n) = \max\{1, n-2\}$. Which has exactly one split note 1. Clearly $X \oplus X$ is not isomporphic to X. There is an infinite family of trivial precessor functions realizable by the finite sums $X \oplus ... \oplus X$ which do not have tree basis.

4. Tree Translation Invariance

Given a tree basis (e_n) for X we will say X is *tree translation equivalent* (respectively *tree translation invariant*) if each transformation T of the form $T = T_m$

$$T(\sum \alpha_n e_n) = \sum \alpha_n e_{\Phi(n)}$$

where $\Phi = \Phi_m$ is a tree translation, is an isomorphism (respectively an isometry).

Example 4.1. *Any subsymmetric basis is tree translation invariant.*

Example 4.2. *The usual Schauder system for $\{f \in C : f(0) = f(1) = 0\}$ is tree translation invariant, but not unconditional.*

Example 4.3. *The Haar system in a rearrangement invariant function space X on $[0,1]$ (actually the co-dimension one subspace of functions f so that $\int f = 0$) is tree translation equivalent.*

Example 4.4. *Tsirelson space T is an example of a space that is tree translation equivalent but not tree translation invariant. Tree translation equivalence follows since the growth rate of the function $\Phi_m(n)$ function is bounded [4]. Attempts to renorm the space T to make it tree translation invariant using the usual construction fail as this will generate a norm equivalent to ℓ_1-norm.*

Example 4.5. *In [5], a superspace S of a Tsirelson spaces is constructed that is not isomorphic to its square. By the theorem below, S is not even tree translation equivalent. However one side of the equation holds as $\|\sum \alpha_n e_n\| \leq \|T_m(\sum \alpha_n e_n)\|$.*

Tree spaces with bases satisfying similar one sided dominance conditions were also constructed in [3].

Theorem 4.6. *If (e_n) is a tree translation equivalent basis for X, then*

(1) *X is isomorphic to its hyperplanes*
(2) *X is isoporphic to its square $X \oplus X$*
(3) *X is isomorphic to an unconditional decomposition (X_n) with each X_n naturally isomorphic to X*

Proof Let $\mathbb{K}$ be the scaler field.

(1) The isomorphism T_2 maps the complemented subspace $W = [e_2, e_4, e_8, ...]$ so W is isomorphic to $W \oplus \mathbb{K}$. Hence $X \oplus \mathbb{K} \approx W \oplus Z \oplus \mathbb{K} \approx W \oplus Z \approx X$.
(2) Let the isomorphisms T_2 and T_3 have ranges X_2 and X_3 respectively. Clearly $X \approx X_2 \oplus X_3 \oplus \mathbb{K} \approx X \oplus X \oplus \mathbb{K} \approx X \oplus X$ by part (1).
(3) Let $W_1 = X_2 \cup e_1]$ and $W_{n+1} = T_3(W_n)$. The (W_n) form a decomposition of X and T_3 provides a translation for this decomposition. Obviously $W_n \approx X \oplus \mathbb{K} \approx X$ by part (1).

Remark. Most of the known primary spaces (with exception of J [6]) are tree translation equivalent. It is not known if all symmetric sequence spaces are primary. The usual conditions to imply primary can be modeled after [7], [2] and [1]. To apply the Pelcynski decompostion method one needs two facts in addition to the theorem above. First we need a condition that says $X \approx Y \oplus Z$ implies either Y or Z has a complemented copy of X. For tree spaces, this could be done with the following complemented subtree condition below. Second we need a way to shift Y into the the unconditional decomposition $(X_n) \approx (Y_n \oplus Z_n)$ while holding the Z_n fixed. The usual proves require additional information on the unconditional decomposition for example the fact it is a ℓ_2 sum in the JT case.

Definition 4.7. *A subtree S of T is a subset of the integers so that the order inherited from T is order isomorphic to the order of a binary tree. A tree basis is said to have the complemented subtree condition if for each subtree S, the basis $(e_n)_{n \in S}$ is equivalent to (e_n) and the projection $P_S(\sum \alpha_n e_n) = \sum_{n \in S} \alpha_n e_n$ is bounded.*

Example 4.8. *The Tsirelson space T fails the complemented subtree condition as we can pick a subtree $S = (i(n))_n$ so that the rate of growth is too large for $(e_n)_{n \in S}$ to be equivalent to (e_n)* [4].

5. Branch invariant tree spaces

Definition 5.1. *A tree basis is branch invariant if for every branch permutation β the operator*

$$T_\beta(\sum \alpha_n e_n) = \sum \alpha_n e_{\beta(n)}$$

is an isometry.

If $\delta, \gamma \subset \mathbb{N}$ are a branches, then the basis $\{(e_n) n \in \delta\}$ and $\{(e_n) n \in \gamma\}$ are isometrically equivalent in a branch invariant space.

Example 5.2. *Rearrangement invariant spaces and symmetric spaces are examples of branch invariant spaces as is* JT. *Since the basis of JT is conditional, branch invariance doesn't imply unconditionality.*

Example 5.3. *The space C is tree translation invariant but not branach invariant. Indeed, if $n(i) = 2^{i-1}$, the $(e_{n(i)})$ is equivalent to usual basis of C, while if $m(i)$ is inductively defined by $m(i) = 1$, $m(2n+1) = 2m(2n)$ and $m(2n+2) = 2(m(2n+1)) + 1$ then $(e_{m(i)})$ is equivalent to the summing basis.*

Proposition 5.4. *A branach invariant tree basis, the projection*

$$P(\sum \alpha_i e_i) = \sum_{n=0}^{\infty} (\sum_{l=0}^{2^n-1} \alpha_{2^n+i})(\sum_{l=0}^{2^n-1} e_{x^n+i})/2^n$$

has norm one.

Proof. If $\sum \alpha_i e_i$ is non-zero only on when $\ell(i) \leq n$ then P is the average of 2^n branch permutations generated by the permutations on level n integers. □

Remark. If the basis is the standard Haar basis, the range of this projection is the closed linear span of the Rademacher functions.

References

[1] D. Alspach, P. Enflo and E. Odell *On the structure of separable $\mathcal{L}_p$ spaces* $(1 < p < \infty)$ Studia Math 60 (1977), 79–90.

[2] A.D. Andrew, *The Banach space JT is primary*, Pacific J. Math 108 (1983), 9–17.

[3] S.F. Bellenot, R Haydon and E. Odell, *Quasi-reflexive and tree spaces constructed in the spirit of RC James* Contem. Math 85 (1989), 19–43.

[4] S.F. Bellenot *The Banach space T and the fast growing hierarchy from logic*, Israel J. Math 47 (1984), 305–313.

[5] S.F. Bellenot *Tsirelson superspaces and lp,* J. Functional Analysis, 69 (1986), 207–228.

[6] P.G. Casazza *James' quasi-reflexive space is primary*, Israel J. Math 26 (1977), 294–305.

[7] P.G. Casazza and B. L. Lin, *Projections on Banach spaces with symmetric bases*, Studia Math 52 (1974), 189–193.

[8] W Gowers and B Maurey *The unconditional basic sequence problem*, J. Amer Math Soc 6 (1993), 851–874.

[9] W. Gowers *A Solution to Banach's hyperplane problem*, Ball London math Soc 26 (1994), 523–530.

[10] W.T. Gowers *An infinite Ramsey theorem and som Banach spaces dichoton=mies*, Amer of Math 156 (2002), 3 797–833.

[11] W.B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach space*, in [12] (2001), 1–84.

[12] W.B. Johnson and J. Lindenstrauss (Editors), *Handbook of the geometry of Banach spaces, Volume 1* Elsevier, Amsterdam, 2001.

[13] N.J. Kalton *A remark on Banach spaces isomorphic to their squares*, Contemp. Math 232 (1999), 211–217.

[14] J. Lindenstraus and L. Tzafriri, *Classical Banach spaces I: Sequence spaces* Springer-Verlag, Berlin and New York, 1977.

[15] J. Lindenstraus and L. Tzafriri, *Classical Banach spaces II: Function spaces*, Springer-Verlag, Berlin and New York, 1979.

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